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Research article

When does the sum of an arithmetic progression divide its odd power sums?

Mohammed Aassila

Académie de Strasbourg, F-67000 Strasbourg, France.

*Corresponding Author: Mohammed Aassila. Email: mohammed.aassila@ac-strasbourg.fr

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Abstract: Let $a_1, \dots, a_n$ be an integer arithmetic progression with common difference d and sum S, and let p be odd. It is classical that $1 + 2 + \cdots + n$ divides $1^p + 2^p + \cdots + n^p$, and that S divides $\sum a_i^3$ for every arithmetic progression; but S need not divide $\sum a_i^5$ (for instance, $9 \nmid 2^5 + 3^5 + 4^5$). We determine, for odd $p \geq 3$, the exact denominator of $\sum a_i^p / S$: it is the product of the odd primes q with $q \mid n$, $q \mid a_1 + a_n$, $q \nmid d$, $(q-1) \mid (p-1)$ and $q \nmid p$. Consequently, $S \mid \sum a_i^p$ for every integer progression of length n if and only if every prime dividing $\gcd(n, D_{p-1})$ divides $2p$, where D_{p-1} is the denominator of the Bernoulli number B_{p-1}; equivalently, $n^2 \mid 1^p + 3^p + \cdots + (2n-1)^p$. The exponents p for which $S \mid \sum a_i^p$ for all progressions of all lengths are exactly the odd p with $D_{p-1} \mid 2p$; the only prime among them is 3; nevertheless there are infinitely many. As secondary results we record the centred expansion of $\sum a_i^p$, the cubic identity $\sum a_i^3 = S\left(a_1 a_n + \binom{n}{2} d^2\right)$ for every length n, the determination of all progressions satisfying Nicomachus' identity $\sum a_i^3 = S^2$, and a self-contained proof of the even-exponent analogue, which is a consequence of a theorem of Moree.

Mathematics Subject Classification: 11B25, 11A07, 11B68
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1 Introduction

Throughout, $n \geq 1$ and

$$a_i = a + (i-1)d \qquad (1 \leq i \leq n), \qquad a, d \in \mathbb{Z},$$

is an integer arithmetic progression with sum, mean and power sums

$$S = \sum_{i=1}^{n} a_i, \qquad c = \frac{S}{n} = \frac{a_1 + a_n}{2}, \qquad P_p = \sum_{i=1}^{n} a_i^p.$$

Note that $c \in \frac{1}{2}\mathbb{Z}$ and $2c = a_1 + a_n \in \mathbb{Z}$. For divisibility we use the convention $0 \mid 0$, so that a statement "$S \mid P_p$" is true when $S = P_p = 0$; this case occurs for progressions symmetric about 0.

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The question

For odd p, pairing a_i with a_{n+1-i} shows that $a_1 + a_n$ divides $2P_p$, and it is classical that $1 + 2 + \cdots + n$ divides $1^p + \cdots + n^p$; the latter is part of the Carlitz–von Staudt theorem [1–3]. Moreover, for odd p the power sum of an arithmetic progression is, as a polynomial, divisible by its sum: Chen, Fu and Zhang [4, Theorem 2.1] observed that Faulhaber's theorem extends to arithmetic progressions, P_p being a polynomial without constant term in $2S$, with *rational* coefficients depending on the progression (see also Proposition 1 below). They note in particular that $1^p + 3^p + \cdots + (2n-1)^p$ is a polynomial in n^2 [4, §1]; condition (ii) of Theorem B below determines exactly when it is divisible by n^2 in $\mathbb{Z}$. For $p = 3$ the quotient P_3/S is always an integer (Corollary 1), but already

$$2^5 + 3^5 + 4^5 = 1299, \qquad S = 9, \qquad 9 \nmid 1299.$$

We answer completely the question *when S divides P_p*.

For an even integer $k \geq 2$ put

$$D_k = \prod_{\substack{q \text{ prime} \\ (q-1) \mid k}} q, \qquad D_0 = 1.$$

By the von Staudt–Clausen theorem [5, Ch. 15], D_k is the denominator of the Bernoulli number B_k; we only use the product definition. Note that $6 \mid D_k$ for every even $k \geq 2$. As usual D_k is only defined for even k; in Theorem B and Corollary C it appears as D_{p-1} with p odd, so that $p - 1$ is even and D_{p-1} is defined (with $D_{p-1} = D_0 = 1$ when $p = 1$).

Theorem A (Exact denominator)**:** *Let $p \geq 3$ be odd and let $a_1, \ldots, a_n$ be an integer arithmetic progression with difference d and sum $S \neq 0$. Let B be the set of* odd *primes q such that*

$$q \mid n, \qquad q \mid a_1 + a_n, \qquad q \nmid d, \qquad (q-1) \mid (p-1), \qquad q \nmid p.$$

Then the reduced denominator of P_p/S equals $\prod_{q \in B} q$. In particular $S \mid P_p$ if and only if $B = \emptyset$.

Theorem B (Divisibility for all progressions of a given length)**:** *Let $p \geq 1$ be odd and $n \geq 1$. The following are equivalent:*

(i) $S \mid P_p$ *for every integer arithmetic progression of length n;*

(ii) $n^2 \mid 1^p + 3^p + 5^p + \cdots + (2n-1)^p$*;*

(iii) *every prime dividing* $\gcd(n, D_{p-1})$ *divides $2p$; equivalently, when $p \geq 3$, there is no odd prime q with $q \mid n$, $(q-1) \mid (p-1)$ and $q \nmid p$.*

Call an odd exponent p *universal* if $S \mid P_p$ for every integer arithmetic progression of every length.

Corollary C (Universal exponents)**:** *An odd integer p is universal if and only if $D_{p-1} \mid 2p$. Every universal $p > 1$ is divisible by 3, so $p = 3$ is the only universal prime; on the other hand $p = 2s + 1$ is universal for every prime $s \equiv 1 \pmod 3$, so there are infinitely many universal exponents:* $1, 3, 15, 27, 39, 63, 75, 87, 99, \ldots$

Relation to the literature

The Carlitz–von Staudt theorem [1] describes $1^p + \cdots + (n-1)^p$ modulo n, and modulo $n(n-1)/2$ for odd p (see [3, Theorem 3] for a precise statement and a short proof, and [6–8] for related divisibility results for power sums of consecutive integers). Moree [9, Theorem 1] extended it to arithmetic progressions: if $\gcd(a, d) = 1$, then

$$\frac{1}{n}\sum_{j=0}^{n-1}(a+jd)^p \equiv \begin{cases} 0 \pmod{\tfrac{1}{2}}, & p > 1 \text{ odd}, \\ -\displaystyle\sum_{\substack{q \mid n,\ q \nmid d \\ (q-1) \mid p}} \frac{1}{q} \pmod 1, & p \text{ even}. \end{cases} \tag{1}$$

For odd p, (1) concerns divisibility by the number n of terms (up to a factor 2), and pairing gives divisibility by c (up to a factor 2); but $S = nc$, and divisibility by the *product* is a different matter as soon as n and $a_1 + a_n$ have a common prime factor. Theorem A shows that this interaction is the only obstruction, and identifies it exactly. To our knowledge, Theorems A and B and Corollary C are new; the even-exponent analogue (Theorem 1) is a direct consequence of (1), and we include it, with a self-contained proof, for comparison.

Table 1 summarizes what the known results give for odd p and what is new here. None of the earlier statements implies Theorem A: for the progression $2, 3, 4$ and $p = 5$, Moree's congruence (1) gives $3 \mid 2P_5$, pairing gives $3 \mid 2P_5$ (here $c = 3$), and [4, Theorem 2.1] gives $P_5 \in S \cdot \mathbb{Q}$; yet $9 \nmid P_5 = 1299$. Theorem A predicts exactly this failure and its size (one factor 3). Thus the contribution of the present paper is fourfold: (a) the exact reduced denominator of P_p/S, not only a criterion for its integrality; (b) the fact that the defect is always squarefree, each obstructing prime contributing exactly one factor, whatever the power of q dividing n or $a_1 + a_n$; (c) the uniform criterion of Theorem B, with a single extremal progression $1, 3, \ldots, 2n-1$; (d) the description of universal exponents in Corollary C, which has no counterpart for consecutive integers, where $S \mid P_p$ holds for every odd p.

The question was suggested by recent work of Mude, Gitunga and Ndung'u [10] (see also [11,12]) on identities of the form $\sum a_i^3 = (\sum a_i)L$ with $L \in \mathbb{Z}$. Their formula (3) gives $\sum a_i^3 = S\big(a_1^2 + (n-1)a_1 d + \binom{n}{2} d^2\big)$ for a progression of length n, and its specialization to odd n is their centred identity $\sum a_i^3 = S\big(m^2 + r(r+1)d^2\big)$, $n = 2r+1$, $m = a_{r+1}$. In Section 5 we place these identities in the centred-moment framework, record the even-length centred form, and use it to determine all progressions satisfying Nicomachus' identity.

Organization

Section 2 collects the (classical) centred expansion of P_p. Section 3 proves Theorems A and B; the 2-adic and odd-prime parts are isolated in Propositions 2 and 3. Section 4 proves Corollary C. Section 5 treats cubes and Nicomachus' identity, Section 6 even exponents, and Section 7 reports a brief numerical check.

Source	What it gives for odd p	What it does not give
Carlitz–von Staudt [1,3]	$\binom{n+1}{2} \mid 1^p + \cdots + n^p$	arbitrary first term and difference
Moree [9, Theorem 1]	$2P_p/n \in \mathbb{Z}$ when $\gcd(a,d)=1$	divisibility by $S = nc$
Pairing $a_i \leftrightarrow a_{n+1-i}$	$(a_1 + a_n) \mid 2P_p$	divisibility by $S = nc$
Chen–Fu–Zhang [4, Th. 2.1]	P_p is a polynomial in $2S$ with rational coefficients	integrality of P_p/S
Theorem A (this paper)	exact reduced denominator of P_p/S	—
Theorem B, Corollary C (this paper)	all progressions of length n; all lengths	—

Table 1: Earlier results on odd power sums of arithmetic progressions, and the results of this paper.

Notation

For a prime q, v_q denotes the q-adic valuation on $\mathbb{Q}$ and $\mathbb{Z}_{(q)} = \{x \in \mathbb{Q} : v_q(x) \geq 0\}$. For a statement $\mathcal{A}$, $[\mathcal{A}]$ is 1 if $\mathcal{A}$ holds and 0 otherwise. We write

$$x_i = a_i - c = \frac{y_i}{2}\, d, \qquad y_i = 2i - n - 1 \in \mathbb{Z} \qquad (1 \leq i \leq n),$$

so that $a_i = c + x_i$ and $x_{n+1-i} = -x_i$. In Section 3 we also use $\delta = d/2$, so that $x_i = \delta y_i$. All symbols are standard; for the reader's convenience they are collected in Table 2.

Symbol	Meaning
$a_i = a + (i-1)d$	terms of the progression ($1 \leq i \leq n$), first term a, difference d
S, P_p	$S = \sum_i a_i$ (sum), $P_p = \sum_i a_i^p$ (power sum)
c	mean $S/n = (a_1 + a_n)/2 \in \frac{1}{2}\mathbb{Z}$
x_i, y_i, δ	$x_i = a_i - c = \delta y_i$, $y_i = 2i - n - 1$, $\delta = d/2$
$\Lambda_s(n)$	$\sum_{i=1}^{n} (y_i/2)^s$, so that $\sum_i x_i^s = \Lambda_s(n) d^s$
H	the sum defined in (3), with $P_p = c(nc^{p-1} + H)$
D_k	product of the primes q with $(q-1) \mid k$; denominator of B_k
B	set of obstructing primes in Theorem A
q, e, Q	a prime, $e = v_q(n)$, $Q = q^e$
v_q, $\mathbb{Z}_{(q)}$	q-adic valuation; rationals with $v_q \geq 0$
$[\mathcal{A}]$	1 if $\mathcal{A}$ holds, 0 otherwise
$a \mid b$, $a \nmid b$	a divides b, a does not divide b (with $0 \mid 0$)

Table 2: Notation.

2 Centred power sums

For $s \geq 0$ put $\Lambda_s(n) = \sum_{i=1}^{n} (y_i/2)^s$, so that $\sum_i x_i^s = \Lambda_s(n) d^s$. The following facts are classical; we include short proofs. Here $B_k(x)$ are the Bernoulli polynomials [13, §9.1].

Lemma 1: *For $n \geq 1$ and $s \geq 0$: $\Lambda_s(n) = 0$ if s is odd, and $\Lambda_s(n) = \frac{2}{s+1} B_{s+1}\left(\frac{n+1}{2}\right)$ if s is even. In particular*

$$\Lambda_0 = n, \qquad \Lambda_2 = \frac{n(n^2-1)}{12}, \qquad \Lambda_4 = \frac{n(n^2-1)(3n^2-7)}{240}.$$

Proof: Since $y_{n+1-i} = -y_i$, odd sums vanish. As $\frac{y_i}{2} = \frac{1-n}{2} + (i-1)$, the formula $\sum_{j=0}^{n-1}(x+j)^s = \frac{B_{s+1}(x+n) - B_{s+1}(x)}{s+1}$ with $x = \frac{1-n}{2} = 1 - \frac{n+1}{2}$ and $B_{s+1}(1-x) = -B_{s+1}(x)$ (s even) gives the second claim; the explicit values follow from B_1, B_3, B_5. □

Proposition 1 (Centred expansion)**:** *For every $p \geq 0$,*

$$P_p = \sum_{k=0}^{\lfloor p/2 \rfloor} \binom{p}{2k} c^{p-2k} d^{2k} \Lambda_{2k}(n). \tag{2}$$

If p is odd, every term contains the factor c, and

$$P_p = c\left(nc^{p-1} + H\right), \qquad H = \sum_{k=1}^{(p-1)/2} \binom{p}{2k} c^{p-1-2k} \sum_{i=1}^{n} x_i^{2k}. \tag{3}$$

Proof: Expand $(c + x_i)^p$ and sum over i, using Lemma 1. For odd p, $p - 2k \geq 1$ for all k, and the term $k = 0$ equals nc^p. □

The proof only uses $x_{n+1-i} = -x_i$; thus (3) holds for every finite multiset symmetric about its mean.

3 Proof of Theorems A and B

3.1 Reduction to local statements

Let $p \geq 3$ be odd and $S \neq 0$ (so $c \neq 0$). By (3),

$$\frac{P_p}{S} = c^{p-1} + \frac{H}{n}. \tag{4}$$

Since $P_p/S \in \mathbb{Q}$, it is an integer if and only if $P_p/S \in \mathbb{Z}_{(q)}$ for every prime q, and its reduced denominator is $\prod_q q^{\max(0, -v_q(P_p/S))}$. Theorem A therefore follows from the next two propositions.

Proposition 2 (2-integrality)**:** *For odd $p \geq 3$ and every integer progression with $S \neq 0$, $P_p/S \in \mathbb{Z}_{(2)}$.*

Proposition 3 (Odd primes)**:** *Let $p \geq 3$ be odd, q an odd prime, and consider an integer progression with $S \neq 0$. Then $v_q(P_p/S) \geq 0$ unless $q \in B$, and $v_q(P_p/S) = -1$ if $q \in B$, where B is the set defined in Theorem A.*

3.2 Auxiliary congruences

The following facts are standard (compare [9, Lemmas 1–3], [3, Lemmas 2–3], [6, Theorem 1 and Corollary 1]); we include proofs to keep the argument self-contained.

Lemma 2 (Equidistribution)**:** *Let $Q \geq 1$ and $n = Qn'$ with $n' \geq 1$. If $z_1, \dots, z_n$ is an integer arithmetic progression whose difference g satisfies* $\gcd(g, Q) = 1$*, then every residue class modulo Q contains exactly n' of the z_i.*

Proof: We have $z_{i+Q} = z_i + Qg \equiv z_i \pmod{Q}$, and $z_1, \dots, z_Q$ are pairwise incongruent modulo Q because $z_i - z_j = (i-j)g$ with $0 < |i-j| < Q$ and g invertible modulo Q. Hence each of the n' consecutive blocks $z_{tQ+1}, \dots, z_{tQ+Q}$ is a complete residue system modulo Q. □

Lemma 3 (Power sums modulo odd prime powers)**:** *Let q be an odd prime, $e \geq 1$ and $j \geq 1$. Then*

$$\sum_{y=0}^{q^e-1} y^j \equiv -q^{e-1}\,[(q-1) \mid j] \pmod{q^e}.$$

Proof: For $e = 1$: if $(q-1) \mid j$ each of the $q-1$ nonzero terms is $\equiv 1$; otherwise, for a primitive root g, $g^j \not\equiv 1$ and $y \mapsto gy$ permutes the nonzero residues, so $(g^j - 1)\sum y^j \equiv 0$. Let $e \geq 2$ and assume the lemma for $e-1$. Writing $y = s + qt$ ($0 \leq s < q$, $0 \leq t < q^{e-1}$),

$$\sum_{y=0}^{q^e-1} y^j = \sum_{l=0}^{j} \binom{j}{l} q^l \Big(\sum_{s=0}^{q-1} s^{j-l}\Big)\Big(\sum_{t=0}^{q^{e-1}-1} t^l\Big).$$

The term $l = 0$ is $q^{e-1}\sum_{s<q} s^j \equiv -q^{e-1}[(q-1) \mid j] \pmod{q^e}$ by the case $e = 1$. For $l = 1$, $\sum_{t<q^{e-1}} t = \frac{q^{e-1}(q^{e-1}-1)}{2}$ has valuation $e-1$, so the term is divisible by q^e. For $l \geq 2$, the induction hypothesis gives $q^{e-2} \mid \sum_{t<q^{e-1}} t^l$, so the term is divisible by q^{l+e-2}, hence by q^e. □

Lemma 4 (Hermite)**:** *Let $p \geq 1$ be odd and q an odd prime. Then* $\displaystyle\sum_{1\leq j\leq p-1,\ (q-1)\mid j} \binom{p}{j} \equiv 0 \pmod{q}$.

Proof: This is the case $m = p$ odd of the Hermite–Bachmann congruence [14,15], [6, Corollary 1]. Directly: in $\mathbb{F}_q$, $\sum_{x\in\mathbb{F}_q^\times} x^j = -[(q-1) \mid j]$ for $j \geq 0$, so $\sum_{x\in\mathbb{F}_q^\times}(1+x)^p = -\sum_{0\leq j\leq p,\,(q-1)\mid j}\binom{p}{j}$. The left side equals $\sum_{y\in\mathbb{F}_q} y^p - 1 = -1$, since $(q-1) \nmid p$ ($q-1$ even, p odd). Removing the term $j = 0$, and noting that $j = p$ does not occur, gives the claim. □

Lemma 5 (2-adic power sums)**:**

1. *For $e \geq 0$ and even $s \geq 0$, $\sum u^s \equiv 2^e \pmod{2^{e+1}}$, the sum over the odd u with $0 < u < 2^{e+1}$.*
2. *For $e \geq 1$ and odd p, $\sum_{x=0}^{2^e-1} x^p \equiv 0 \pmod{2^{e-1}}$.*
3. *For $e \geq 1$ and even $p \geq 2$, $\sum_{x=0}^{2^e-1} x^p \equiv 2^{e-1} \pmod{2^e}$.*

Proof: (1) Call the sum $L(e)$; $L(0) = 1$. For $e \geq 1$ pair u with $2^{e+1} - u$ (distinct, as $u \neq 2^e$); since s is even, $(2^{e+1}-u)^s \equiv u^s \pmod{2^{e+1}}$, so $L(e) \equiv 2L(e-1) \pmod{2^{e+1}}$, and induction gives the claim. (2) For $0 < x < 2^e$, $x \neq 2^{e-1}$, pair x with $2^e - x$: $x^p + (2^e - x)^p \equiv 0 \pmod{2^e}$ as p is odd. The remaining terms

0 and $2^{(e-1)p}$ are divisible by 2^{e-1}. (3) Induction on e, the case $e = 1$ being $0 + 1 = 1$. For $e \geq 2$ the odd x contribute $L(e-1) \equiv 2^{e-1} \pmod{2^e}$ by (1), and the even $x = 2t$ contribute $2^p \sum_{t<2^{e-1}} t^p$, divisible by 2^{p+e-2} by induction, hence by 2^e. □

3.3 Proof of Proposition 2

Let $e = v_2(n)$.

Case $e = 0$. Then $n-1$ is even, so $c = a + \frac{n-1}{2} d \in \mathbb{Z}$ and $x_i = \frac{y_i}{2} d \in \mathbb{Z}$ (y_i is even). Hence $H \in \mathbb{Z}$, and (4) lies in $\mathbb{Z}_{(2)}$ because n is odd.

Case $e \geq 1$, d odd. Then $a_1 + a_n = 2a + (n-1)d$ is odd, so $v_2(S) = v_2\big(\frac{n}{2}(a_1 + a_n)\big) = e - 1$. The a_i form a progression with odd difference, so by Lemma 2 with $Q = 2^e$ they meet every residue class modulo 2^e exactly $n/2^e$ times; as $a_i^p \bmod 2^e$ depends only on $a_i \bmod 2^e$, Lemma 5(2) gives $P_p \equiv \frac{n}{2^e} \sum_{x=0}^{2^e-1} x^p \equiv 0 \pmod{2^{e-1}}$. Thus $v_2(P_p) \geq e - 1 = v_2(S)$.

Case $e \geq 1$, d even. Then $c = a + \frac{n-1}{2} d \in \mathbb{Z}$ and $\delta = d/2 \in \mathbb{Z}$, and $\sum_i x_i^{2k} = \delta^{2k} \sum_i y_i^{2k}$. The integers $y_i = 2i - n - 1$ are odd and form a progression with difference 2; modulo 2^{e+1} there are 2^e odd residue classes, and $y_{i+2^e} = y_i + 2^{e+1}$, while $y_1, \dots, y_{2^e}$ are pairwise incongruent modulo 2^{e+1} (their differences are $2(i-j)$ with $0 < |i-j| < 2^e$). Hence the y_i meet every odd class modulo 2^{e+1} exactly $n/2^e$ times, and Lemma 5(1) gives $\sum_i y_i^{2k} \equiv \frac{n}{2^e} \cdot 2^e = n \equiv 0 \pmod{2^e}$. Thus $2^e \mid H$, i.e. $H/n \in \mathbb{Z}_{(2)}$, and (4) lies in $\mathbb{Z}_{(2)}$. □

3.4 Proof of Proposition 3

Let q be an odd prime. Since 2 is invertible in $\mathbb{Z}_{(q)}$, the numbers c, $\delta = d/2$ and $x_i = \delta y_i$ lie in $\mathbb{Z}_{(q)}$; hence $c^{p-1} \in \mathbb{Z}_{(q)}$ and $H \in \mathbb{Z}_{(q)}$. Moreover

$$q \mid c \text{ in } \mathbb{Z}_{(q)} \iff q \mid a_1 + a_n, \qquad q \mid \delta \text{ in } \mathbb{Z}_{(q)} \iff q \mid d. \tag{5}$$

Step 1 (reduction). Let $e = v_q(n)$. If $e = 0$, then $1/n \in \mathbb{Z}_{(q)}$ and (4) lies in $\mathbb{Z}_{(q)}$; also $q \notin B$. Assume $e \geq 1$ and put $Q = q^e$. Since $c^{p-1} \in \mathbb{Z}_{(q)}$, (4) gives

$$\frac{P_p}{S} \in \mathbb{Z}_{(q)} \iff \frac{H}{n} \in \mathbb{Z}_{(q)} \iff v_q(H) \geq e, \tag{6}$$

and if $v_q(H) < e$ then $v_q(P_p/S) = v_q(H) - e$.

Step 2 (equidistribution). Write $H = \sum_{i=1}^{n} h(y_i)$ with

$$h(y) = \sum_{k=1}^{(p-1)/2} \binom{p}{2k} c^{\,p-1-2k} \delta^{2k} y^{2k} \in \mathbb{Z}_{(q)}[y].$$

The integers $y_1, \dots, y_n$ form a progression with difference 2, and $\gcd(2, Q) = 1$; by Lemma 2 each residue class modulo Q contains exactly n/Q of them. Since h has coefficients in $\mathbb{Z}_{(q)}$, $h(y) \bmod Q\mathbb{Z}_{(q)}$ depends only on $y \bmod Q$. Therefore

$$H \equiv \frac{n}{Q} \sum_{y=0}^{Q-1} h(y) = \frac{n}{Q} \sum_{k=1}^{(p-1)/2} \binom{p}{2k} c^{\,p-1-2k} \delta^{2k} \sum_{y=0}^{Q-1} y^{2k} \pmod{Q\mathbb{Z}_{(q)}}.$$

Step 3 (the obstruction polynomial). By Lemma 3 (with $j = 2k \geq 2$), $\sum_{y<Q} y^{2k} \equiv -q^{e-1}[(q-1) \mid 2k] \pmod{Q}$. Hence

$$H \equiv -\frac{n}{Q}\, q^{e-1}\, G \pmod{Q\mathbb{Z}_{(q)}}, \qquad G = G(c, \delta) = \sum_{j \in J} \binom{p}{j} c^{\,p-1-j} \delta^{\,j}, \tag{7}$$

where $J = \{j : 1 \leq j \leq p-1,\ (q-1) \mid j\}$ (the elements of J are even, so they are exactly the exponents $2k$ that survive). As n/Q is a unit in $\mathbb{Z}_{(q)}$, (7) shows: if $q \mid G$ then $v_q(H) \geq e$; if $q \nmid G$ then $v_q(H) = e - 1$. In detail, (7) means $H = -\frac{n}{Q}\, q^{e-1} G + Qu$ for some $u \in \mathbb{Z}_{(q)}$. If $q \mid G$, both terms have valuation at least e, hence so does H. If $q \nmid G$, the first term has valuation exactly $e - 1$ (as n/Q and G are units of $\mathbb{Z}_{(q)}$), while the second has valuation at least e; by the strict triangle inequality $v_q(x + y) = \min(v_q(x), v_q(y))$ when $v_q(x) \neq v_q(y)$, we get $v_q(H) = e - 1$. In that case $v_q(H/n) = -1$, and since $c^{p-1} \in \mathbb{Z}_{(q)}$, the same inequality applied to (4) gives $v_q(P_p/S) = -1$: the denominator of P_p/S contains q exactly once, however large $e = v_q(n)$ and $v_q(a_1 + a_n)$ may be.

Step 4 (evaluation of G modulo q). We compute in $\mathbb{F}_q$, using $z^{q-1} = 1$ for $z \in \mathbb{F}_q^{\times}$ and $(q-1) \mid j$ for $j \in J$.

- If $\delta \equiv 0$: every term contains δ^j with $j \geq 1$, so $G \equiv 0$.
- If $\delta \not\equiv 0$ and $c \not\equiv 0$: $c^{p-1-j}\delta^j = c^{p-1}c^{-j}\delta^j = c^{p-1}$, so $G \equiv c^{p-1} \sum_{j \in J} \binom{p}{j} \equiv 0$ by Lemma 4.
- If $\delta \not\equiv 0$ and $c \equiv 0$: every term with $j < p-1$ contains a positive power of c and vanishes; the term $j = p-1$ (where $c^0 = 1$) is present if and only if $(q-1) \mid (p-1)$, and then equals $\binom{p}{p-1}\delta^{p-1} \equiv p$. Thus $G \equiv p\,[(q-1) \mid (p-1)]$.

Consequently $q \nmid G$ if and only if $q \mid c$, $q \nmid \delta$, $(q-1) \mid (p-1)$ and $q \nmid p$. The decisive term is the constant term $\binom{p}{p-1}\delta^{p-1} = p\,\delta^{p-1}$ of G as a polynomial in c; it exists precisely when $(q-1) \mid (p-1)$, and the condition $q \nmid p$ is exactly what prevents it from vanishing modulo q.

Conclusion. By (5), together with $q \mid n$, the conditions of Step 4 say exactly that $q \in B$. By Steps 1 and 3, $v_q(P_p/S) \geq 0$ if $q \notin B$, and $v_q(P_p/S) = (e-1) - e = -1$ if $q \in B$. □

3.5 Proofs of Theorems A and B

Proof of Theorem A: By Propositions 2 and 3, $v_q(P_p/S) \geq 0$ for $q = 2$ and for odd $q \notin B$, and $v_q(P_p/S) = -1$ for $q \in B$. □

Proof of Theorem B: For $p = 1$ all three conditions hold ($P_1 = S$, $D_0 = 1$, $n^2 = 1 + 3 + \cdots + (2n-1)$); the second formulation of (iii) is only used for $p \geq 3$, since every odd q satisfies $(q-1) \mid 0$. Let $p \geq 3$. The two formulations of (iii) agree since D_{p-1} is the product of the primes q with $(q-1) \mid (p-1)$ and

$2 \mid 2p$. Progressions with $S = 0$ have $c = 0$, hence $P_p = 0$ by (3), so $S \mid P_p$ holds for them by the convention $0 \mid 0$, and they can be ignored. (i)⇒(ii): the progression $1, 3, \dots, 2n-1$ has sum n^2. (iii)⇒(i): under (iii) no odd prime satisfies the last three conditions defining B, so $B = \emptyset$ for every progression of length n, and Theorem A applies. (ii)⇒(iii): let q be an odd prime with $q \mid n$, $(q-1) \mid (p-1)$, $q \nmid p$. For $1, 3, \dots, 2n-1$ we have $a_1 + a_n = 2n \equiv 0$ and $d = 2 \not\equiv 0 \pmod{q}$, so $q \in B$, and Theorem A gives $n^2 \nmid 1^p + 3^p + \cdots + (2n-1)^p$. □

Example 1: *Let $p = 5$, so $D_4 = 30$ and the only odd prime q with $(q-1) \mid 4$, $q \nmid 5$ is $q = 3$. For $n = 3$: the progression $2, 3, 4$ has $a_1 + a_n = 6$, $d = 1$, so $B = \{3\}$, and indeed $1299/9 = 433/3$. The progression $3, 4, 5$ has $a_1 + a_n = 8$, so $B = \emptyset$, and $3^5 + 4^5 + 5^5 = 4392 = 12 \cdot 366$. For $p = 9$ and $n = 3$ we have $3 \mid p$, so $B = \emptyset$ always; e.g. $1^9 + 3^9 + 5^9 = 1972809 = 9 \cdot 219201$.*

Remark 1: *For $a_i = i$ we have $a_1 + a_n = n + 1$, coprime to n, so $B = \emptyset$: this recovers the classical fact that $1 + \cdots + n$ divides $1^p + \cdots + n^p$ for odd p. The extremal progression in Theorem B is $1, 3, \dots, 2n-1$, whose mean $c = n$ shares every prime factor with n.*

4 Universal exponents

Proof of Corollary C: By Theorem B, p is universal iff condition (iii) holds for every n; taking $n = D_{p-1}$, this means that every prime factor of the squarefree number D_{p-1} divides $2p$, i.e. $D_{p-1} \mid 2p$ (and conversely this implies (iii) for all n). For $p \geq 3$ we have $3 \mid D_{p-1}$, so universality forces $3 \mid p$; the only such prime is 3, which is universal since $D_2 = 6$. Let $s \equiv 1 \pmod 3$ be prime and $p = 2s + 1$. The divisors of $p - 1 = 2s$ are $1, 2, s, 2s$, so the primes q with $(q-1) \mid (p-1)$ lie in $\{2, 3, s+1, 2s+1\}$; but $s + 1 > 2$ is even and $2s + 1 > 3$ is divisible by 3. Hence $D_{p-1} = 6$, which divides $2p$ because $3 \mid p$. By Dirichlet's theorem there are infinitely many such s. □

Table 3 illustrates Theorem B for small odd exponents. Each row is read as follows. For $p = 13$, $p - 1 = 12$ has the divisors $1, 2, 3, 4, 6, 12$, so the primes q with $(q-1) \mid 12$ are $2, 3, 5, 7, 13$ and $D_{12} = 2730$; the prime 13 divides p and 2 is never an obstruction, so the obstructing primes are $3, 5, 7$. Hence, for $p = 13$, $S \mid P_{13}$ for every progression of length n if and only if $\gcd(n, 105) = 1$; for instance it fails for $n = 5$, e.g. $5^2 \nmid 1^{13} + 3^{13} + 5^{13} + 7^{13} + 9^{13}$. For $p = 9$ the prime 3 divides p and is therefore harmless, which is why only 5 remains.

odd p	D_{p-1}	primes q whose presence in n can destroy $S \mid P_p$
3	6	none (universal)
5, 7, 11	30, 42, 66	3
9	30	5
13	2730	3, 5, 7
15	6	none (universal)

Table 3: Theorem B for small odd exponents. The last column lists the odd primes q with $(q-1) \mid (p-1)$ and $q \nmid p$: these are exactly the primes whose presence in n can obstruct the divisibility $S \mid P_p$.

5 Cubes and Nicomachus' identity

For $p = 3$, (2) and $\Lambda_2 = n(n^2-1)/12$ give an elementary closed form.

Corollary 1: *For every arithmetic progression $a_1, \dots, a_n$ (with real or complex terms) and $1 \le j \le n$,*

$$\sum_{i=1}^{n} a_i^3 = S\Big(c^2 + \frac{n^2-1}{4}d^2\Big) = S\Big(a_1 a_n + \binom{n}{2} d^2\Big)$$
$$= S\Big(a_j a_{n+1-j} + \frac{(n^2-1) + (n+1-2j)^2}{4} d^2\Big).$$

In particular, with $r = \lfloor n/2 \rfloor$, $\sum a_i^3 = S\big(a_{r+1}^2 + r(r+1)d^2\big)$ if $n = 2r+1$ and $\sum a_i^3 = S\big(a_r a_{r+1} + r^2 d^2\big)$ if $n = 2r$. For integer progressions the factor $a_1 a_n + \binom{n}{2} d^2$ is an integer.

Proof: The first equality is (2) for $p = 3$. The others follow from $a_j a_{n+1-j} = c^2 - \frac{(n+1-2j)^2}{4} d^2$, with $j = 1$, $j = r+1$ ($n = 2r+1$) and $j = r$ ($n = 2r$). □

The case $n = 2r+1$ is the centred identity of [10], and the second form is formula (3) of [10], since $a_1 a_n = a_1^2 + (n-1)a_1 d$. The factor S has a simple explanation, valid beyond arithmetic progressions: if $x_1, \dots, x_n$ are real numbers with sum S, mean c, variance σ^2 and third central moment μ_3, then expanding $\sum (c + (x_i - c))^3$ gives

$$\sum_{i=1}^{n} x_i^3 = S\,(c^2 + 3\sigma^2) + n\mu_3,$$

so $\sum x_i^3 = S(c^2 + 3\sigma^2)$ exactly when $\mu_3 = 0$, e.g. for every multiset symmetric about its mean; for a progression $\sigma^2 = \frac{n^2-1}{12} d^2$.

Corollary 2 (Nicomachus' identity)**:** *Let $n \ge 2$ and let $a_1, \dots, a_n$ be a real arithmetic progression. Then $\sum a_i^3 = S^2$ if and only if $c = 0$, or $(n^2-1)d^2 \le n^2$ and $c = \frac{1}{2}\big(n \pm \sqrt{n^2 - (n^2-1)d^2}\big)$. An integer progression of length $n \ge 2$ satisfies $\sum a_i^3 = S^2$ if and only if it is $1, \dots, n$ or $0, \dots, n-1$ (in either order), the constant progression $n, \dots, n$, or a progression with $S = 0$.*

Proof: By Corollary 1, $\sum a_i^3 - S^2 = S\big(c^2 - nc + \frac{n^2-1}{4} d^2\big)$, whence the real statement. For integers and $c \ne 0$ we need $d^2 \le \frac{n^2}{n^2-1} < 2$, so $d \in \{0, \pm 1\}$. If $d = 0$, then $c = n$. If $d = \pm 1$, then $c = \frac{n \pm 1}{2}$, giving $1, \dots, n$ or $0, \dots, n-1$. The converse is immediate. □

Remark 2: *Over $\mathbb{Q}$ the condition is that $n^2 - (n^2-1)d^2$ be a rational square t^2; parametrizing this conic from the point $(d, t) = (0, n)$ gives $d = \frac{2nu}{u^2+n^2-1}$, $t = \frac{n(n^2-1-u^2)}{u^2+n^2-1}$ ($u \in \mathbb{Q}$), hence infinitely many rational solutions; for instance $\big(\frac{2}{7}\big)^3 + \big(\frac{10}{7}\big)^3 = \big(\frac{12}{7}\big)^2$.*

6 Even exponents

For even p the sum S does not divide P_p in general ($1^2 + 2^2 = 5$), and the natural divisor is n.

Theorem 1 (cf. [9, Theorem 1])**:** *Let $p \geq 2$ be even and $n \geq 1$. Then $n \mid P_p$ for every integer arithmetic progression of length n if and only if* $\gcd(n, D_p) = 1$. *Moreover, if* $\gcd(n, D_p) > 1$, *then already $n \nmid 1^p + 2^p + \cdots + n^p$.*

This follows from (1): if $\gcd(n, D_p) = 1$ the right-hand side of (1) is an empty sum. Moree's theorem assumes $\gcd(a, d) = 1$, but this is no restriction: if $\gcd(a, d) = g > 1$, then $a_i = g\, a_i'$ with $a_i' = a/g + (i-1)d/g$ a progression with coprime first term and difference, so $P_p = g^p \sum_i a_i'^p$, and $n \mid \sum_i a_i'^p$ implies $n \mid P_p$. Finally, the choice $a = d = 1$ gives the converse. We give an independent proof with the tools of Section 3. Its notable feature, shared with Theorem B, is that a single explicit progression (here $1, \ldots, n$) detects every failure.

Proof: Let $q^e \parallel n$ with $e \geq 1$, and $Q = q^e$. As i runs over $1, \ldots, n$, it meets each class modulo Q exactly n/Q times, and $(a + (i-1)d)^p \bmod Q$ depends only on $i \bmod Q$; hence $P_p \equiv \frac{n}{Q} \sum_{y=0}^{Q-1} (a + dy)^p \pmod{Q}$.

q odd. By Lemma 3, $\sum_{y<Q}(a + dy)^p = Qa^p + \sum_{l\geq 1} \binom{p}{l} a^{p-l} d^l \sum_{y<Q} y^l \equiv -q^{e-1} G' \pmod{Q}$ with $G' = \sum_{1\leq l\leq p,\,(q-1)\mid l} \binom{p}{l} a^{p-l} d^l$, so $Q \mid P_p$ iff $q \mid G'$. As in Lemma 4, $\sum_{x\in\mathbb{F}_q^\times}(1+x)^p = \sum_{y\in\mathbb{F}_q} y^p - 1 = -[(q-1) \mid p] - 1$ gives $\sum_{1\leq l\leq p,\,(q-1)\mid l} \binom{p}{l} \equiv [(q-1) \mid p]$. Hence $G' \equiv 0$ if $d \equiv 0$; $G' \equiv [(q-1) \mid p]$ if $a \equiv 0 \not\equiv d$; and $G' \equiv a^p [(q-1) \mid p]$ if $a, d \not\equiv 0$. So $Q \mid P_p$ for all progressions iff $(q-1) \nmid p$, and if $(q-1) \mid p$ the choice $a = d = 1$ gives $Q \nmid 1^p + \cdots + n^p$.

$q = 2$. Since $2 \mid D_p$, we must show failure: for $a = d = 1$, Lemma 5(3) gives $1^p + \cdots + n^p \equiv \frac{n}{2^e}\, 2^{e-1} \not\equiv 0 \pmod{2^e}$.

Combining the local statements gives the theorem. □

7 Numerical verification

As an independent check (not used in the proofs), all statements were tested with exact integer and rational arithmetic, using the short Python program reproduced in Appendix I. The procedure is as follows.

- *Theorem A.* We enumerated all triples (a, d, n) with $|a| \leq 14$, $|d| \leq 9$ and $n \in \{1, \ldots, 49\} \cup \{63, 75, 81, 99, 125, 135\}$; the extra lengths ensure that $v_q(n)$ reaches 4 (for $q = 3$) and 3 (for $q = 5$). This gives 30 305 progressions; the 175 of them with $S = 0$ were excluded, since then $P_p = 0$ by (3) and Theorem A assumes $S \neq 0$. For each of the remaining 30 130 progressions and each odd p with $3 \leq p \leq 21$, the reduced denominator of P_p/S was computed and compared with $\prod_{q\in B} q$. This makes 301 300 cases in all.
- *Theorem B.* For all $n \leq 45$ and all odd $p \leq 31$ (720 pairs), conditions (ii) and (iii) were evaluated directly (the second formulation of (iii) for $p \geq 3$), and condition (i) was tested on all progressions of length n with $|a| \leq 14$, $|d| \leq 9$ and $S \neq 0$.
- *Theorem 1.* For all $n \leq 45$ and all even $p \leq 24$ (540 pairs), the condition $\gcd(n, D_p) = 1$ was compared with $n \mid P_p$ for all progressions with $|a| \leq 14$, $|d| \leq 9$, and with $n \mid 1^p + \cdots + n^p$.

- *Corollary C.* The universal exponents below 200 were listed from the criterion $D_{p-1} \mid 2p$; they are 1, 3, 15, 27, 39, 63, 75, 87, 99, 123, 135, 147, 159, 183, 195.

No discrepancy occurred. The whole computation takes a few seconds on a laptop.

8 Conclusions

The arithmetic content of this note is that, for odd p, the only obstruction to $S \mid \sum a_i^p$ is a prime q dividing both n and $a_1 + a_n$ but not d, with $(q-1) \mid (p-1)$ and $q \nmid p$, and that it then costs exactly one factor q. Some natural questions remain.

1. Determine the exact q-adic valuation of $\sum a_i^p / S$ when it is an integer, in the spirit of the 2-adic results of [7] for consecutive integers.
2. Extend Theorems A and B to arithmetic progressions in rings of integers of number fields.
3. Study divisibility of $\sum a_i^p$ by S^2; for consecutive integers and odd $p \geq 3$ this is related to the classical fact that $\sum_{i \leq n} i^p$ is a polynomial in $\binom{n+1}{2}$ divisible by its square.
4. Count the universal exponents. By Corollary C they form an infinite set of multiples of 3, containing $2s+1$ for every prime $s \equiv 1 \pmod 3$. Is the number of universal $p \leq x$ asymptotic to a constant times $x/\log x$, and is the constant computable?
5. Replace arithmetic progressions by polynomial sequences $a_i = f(i)$, $f \in \mathbb{Z}[x]$, or by the values of f along a progression. Proposition 1 only uses the symmetry $x_{n+1-i} = -x_i$, so the expansion (3) holds for every integer sequence symmetric about its mean; the question is which arithmetic input replaces Lemma 2 in that setting.
6. Weighted and alternating sums: determine when S divides $\sum (-1)^i a_i^p$ or $\sum w_i a_i^p$ for natural weights, and whether the obstruction is again a single factor q.
7. For even p, find the natural analogue of Theorem A: the exact denominator of P_p/n is given by (1) when $\gcd(a, d) = 1$, but a description of the exact denominator of P_p/S (for $S \neq 0$) is still missing.

Appendix I Program used in Section 7

The following program (Python 3, standard library only) performs all the checks of Section 7. It prints the number of cases tested and the number of discrepancies (zero in each case), followed by the list of universal exponents below 200.

```
from fractions import Fraction
from math import gcd, prod

def primes_of(n):                    # distinct prime factors of n >= 1
    r, q = [], 2
    while q * q <= n:
        if n % q == 0:
            r.append(q)
            while n % q == 0:
                n //= q
```

```
        q += 1
    return r + [n] if n > 1 else r

def D(k):                           # product of primes q with (q-1) | k
    return prod(q for q in range(2, k + 2)
                if primes_of(q) == [q] and k % (q - 1) == 0)

def P(a, d, n, p):                  # power sum of the progression
    return sum((a + i * d) ** p for i in range(n))

BOX = [(a, d) for a in range(-14, 15) for d in range(-9, 10)]

# Theorem A
N = list(range(1, 50)) + [63, 75, 81, 99, 125, 135]
cases = zero = bad = 0
for n in N:
    for a, d in BOX:
        S = n * a + n * (n - 1) // 2 * d
        if S == 0:
            zero += 1
            continue
        for p in range(3, 22, 2):
            B = 1
            for q in primes_of(n):
                if (q > 2 and (2 * a + (n - 1) * d) % q == 0
                        and d % q and (p - 1) % (q - 1) == 0
                        and p % q):
                    B *= q
            cases += 1
            bad += Fraction(P(a, d, n, p), S).denominator != B
print("Thm A:", cases, "cases;", zero, "with S=0 skipped;",
      bad, "discrepancies")

# Theorem B
bad = 0
for n in range(1, 46):
    for p in range(1, 32, 2):
        ii = P(1, 2, n, p) % (n * n) == 0
        iii = all(2 * p % q == 0 for q in primes_of(gcd(n, D(p - 1))))
        i = True
        for a, d in BOX:
            S = n * a + n * (n - 1) // 2 * d
            if S != 0 and P(a, d, n, p) % S != 0:
                i = False
        bad += not (i == ii == iii)
```

```
        if p >= 3:                # second formulation of (iii)
            iii2 = all(q == 2 or (p - 1) % (q - 1) or p % q == 0
                       for q in primes_of(n))
            bad += iii2 != iii
print("Thm B: 720 pairs;", bad, "discrepancies")

# Theorem 1 (even exponents)
bad = 0
for n in range(1, 46):
    for p in range(2, 25, 2):
        cond = gcd(n, D(p)) == 1
        allp = all(P(a, d, n, p) % n == 0 for a, d in BOX)
        cons = P(1, 1, n, p) % n == 0
        bad += not (cond == allp == cons)
print("Thm 1: 540 pairs;", bad, "discrepancies")

# Corollary C
U = [p for p in range(1, 200, 2)
     if all(2 * p % q == 0 for q in primes_of(D(p - 1)))]
print("universal p < 200:", U)
```

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References

1. L. Carlitz, The Staudt–Clausen theorem, *Math. Mag. 34* **(1961)**, 131–146.
2. P. Moree, Moser's mathemagical work on the equation $1^k + 2^k + \cdots + (m-1)^k = m^k$, *Rocky Mountain J. Math. 43* **(2013)**, 1707–1737.
3. P. Moree, A top hat for Moser's four mathemagical rabbits, *Amer. Math. Monthly 118* **(2011)**, 364–370.

4. W. Y. C. Chen, A. M. Fu, I. F. Zhang, Faulhaber's theorem on power sums, *Discrete Math. 309* **(2009)**, 2974–2981.
5. K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Graduate Texts in Mathematics 84, *Springer*, New York, **1990**.
6. K. MacMillan, J. Sondow, Proofs of power sum and binomial coefficient congruences via Pascal's identity, *Amer. Math. Monthly 118* **(2011)**, 549–551.
7. K. MacMillan, J. Sondow, Divisibility of power sums and the generalized Erdős–Moser equation, *Elem. Math. 67 (4)* **(2012)**, 182–186. https://doi.org/10.4171/EM/208
8. T.-X. He, P. J.-S. Shiue, Divisibility of the sums of the power of consecutive integers, *Notes Number Theory Discrete Math. 30 (3)* **(2024)**, 557–574.
9. P. Moree, On a theorem of Carlitz–von Staudt, *C. R. Math. Rep. Acad. Sci. Canada 16 (4)* **(1994)**, 166–170.
10. L. H. Mude, R. M. Gitunga, K. J. Ndung'u, On generalization of centred odd cube of length $2r + 1$, *Earthline J. Math. Sci. 16 (5)* **(2026)**, 1009–1016. https://doi.org/10.34198/ejms.16526.64.10091016
11. L. H. Mude, Z. K. Kayiita, K. J. Ndung'u, Some generalized formula for sums of cubes, *J. Adv. Math. Comput. Sci. 38 (8)* **(2023)**, 47–52.
12. B. S. Kimtai, L. H. Mude, On generalized sum of six, seven and nine cubes, *Science Mundi 3 (1)* **(2023)**, 135–142.
13. H. Cohen, *Number Theory, Volume II: Analytic and Modern Tools*, Graduate Texts in Mathematics 240, *Springer*, New York, **2007**.
14. C. Hermite, Extrait d'une lettre de M. Ch. Hermite à M. Borchardt, *J. Reine Angew. Math. 81* **(1876)**, 93–95.
15. P. Bachmann, *Niedere Zahlentheorie*, Part 2, *Teubner*, Leipzig, **1910**; reprinted by Chelsea, New York, 1968.

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