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Research article

On outer-independent fair Roman domination in graphs

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Abstract: Let G be an undirected graph and let $\rho : V(G) \to \{0,1,2\}$ be a Roman dominating function (RDF) on G. For each $i \in \{0,1,2\}$, let $V_i = \{x \in V(G) : \rho(x) = i\}$. Then we represent ρ as $\rho = (V_0, V_1, V_2)$. A function ρ is an outer-independent fair Roman dominating function (OiFRDF) on G if, for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(u,v) = 1$, and for every $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$, and V_0 is an independent vertex set on G. The *weight* of OiFRDF ρ is denoted by $\widetilde{\omega}_G^{iFR}(\rho)$, and is defined as $\widetilde{\omega}_G^{iFR}(\rho) = \sum_{z \in V(G)} \rho(z)$. The minimum weight of an OiFRDF ρ is denoted by $\widetilde{\gamma}_{iFR}(G)$ and is called the *outer-independent fair Roman domination number*, and is defined as $\widetilde{\gamma}_{iFR}(G) = min\{\widetilde{\omega}_G^{iFR}(\rho) : \rho \; is \; an$ OiFRDF $on \; G\}$. Every OiFRDF ρ satisfying the condition $\widetilde{\omega}_G^{iFR}(\rho) = \widetilde{\gamma}_{iFR}(G)$ is called a $\widetilde{\gamma}_{iFR}$-function on G. This study introduces a new restricted version of Roman domination in graphs called the outer-independent fair Roman dominating function and discusses some graph-theoretic properties.

Mathematics Subject Classification: 05C69
Keywords: Outer-independent domination, Fair domination, Roman domination, Outer-independent fair Roman domination

1 Introduction

Roman domination in graph theory is an interesting topic inspired by the Roman Emperor Constantine's defense strategy in the fourth century A.D. The vertices of a graph are considered locations and are labeled 0, 1, or 2, indicating the number of legions (Roman soldiers) that protect the whole region [10]. A vertex with label 0 must be adjacent to a location with label 2, and the objective is to optimize (minimize) the total number of legions. This parameter is often used in network security and facility location, particularly in computer science. Several mathematicians are contributing to the topic of Roman dominating functions, as seen in [2,3,7,11,15,16]. One of the interesting restricted parameters of Roman domination is the outer-independent Roman dominating function, which was introduced by [1]. Meanwhile, Caro et al. [5] pioneered a motivating parameter of domination in graphs called a fair dominating set, which gained interest among many graph theorists. Hence, the author combined the concept of the outer-independent Roman dominating function

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and the fair dominating set to come up with a new restricted parameter in domination theory called an outer-independent fair Roman domination in graphs. The new parameter describes the fairness of undefended locations (vertices) concerning security and the independence of unsecured locations.

2 Preliminaries

Let $G = (V(G), E(G))$ be a graph. Here, $V(G)$ is the set containing all the vertices and $E(G)$ is the set containing all the edges of G. A set $I \subseteq V(G)$ is called an *independent vertex set* if no two vertices in I are adjacent, that is, for every $x, y \in I$, $xy \notin E(G)$. If I is a singleton set, then it is called a *trivial independent vertex set* since a trivial graph is considered an independent graph. Let $F \subseteq V(G)$. Then F is called a fair vertex set on G provided that for all $x, y \in F$, $|N_G(x) \cap (V(G) \setminus F)| = |N_G(y) \cap (V(G) \setminus F)| > 0$. Let $D \subseteq V(G)$. Then D is called a *dominating set* of G if for all $v \in V(G) \setminus D$, there exists $u \in D$ such that $d_G(u, v) = 1$ [13]. The *domination number* of G is denoted by $\gamma(G)$ and is defined as the smallest cardinality of a set D on G. The dominating set D satisfying the restriction $|D| = \gamma(G)$ is called a γ-set on G. For some studies on domination in graphs, one may refer to: [4,6,9,12].

Let $I_d \subseteq V(G)$. Then I_d is called an *outer-independent dominating set* of G if for all $v \in V(G) \setminus I_d$, there exists $u \in I_d$ such that $d_G(u, v) = 1$ and the subgraph $\langle V(G) \setminus I_d \rangle$ is an empty graph [14]. The *outer-independent domination number* of G is the minimum cardinality of an I_d on G, which is denoted by $\widetilde{\gamma}_i(G)$. The outer-independent dominating set I_d with $|I_d| = \widetilde{\gamma}_i(G)$ is called $\widetilde{\gamma}_i$-set on G. Let $F_d \subseteq V(G)$. Then F_d is called a *fair dominating set* of G if for all $v \in V(G) \setminus F_d$, there exists $u \in F_d$ such that $d_G(u, v) = 1$ and the set $V(G) \setminus F$ is a fair vertex set on G [5]. The *fair domination number* of G is denoted by $fd(G)$ and defined as the minimum cardinality of a set F_d on G. The fair dominating set F_d satisfying the restriction $|F_d| = fd(G)$ is called FD-set on G. Let $I_f \subseteq V(G)$. Then I_f is called an *outer-independent fair dominating set* of G if for every $v \in V(G) \setminus I_f$, there exists $u \in I_f$ such that $d_G(u, v) = 1$, and the set $V(G) \setminus I_f$ is a fair vertex set and an independent vertex set on G. The *outer-independent fair domination number* of G is denoted by $\widetilde{\gamma}_{cF}(G)$ and defined as the minimum cardinality of a set I_f on G. The dominating set I_f satisfying the restriction $|I_f| = \widetilde{\gamma}_{iF}(G)$ is called $\widetilde{\gamma}_{iF}$-set on G.

Let $\rho : V(G) \to \{0, 1, 2\}$ be a function on a graph G and let $V_i = \{x \in V(G) : \rho(x) = i\}$, for each $i \in \{0, 1, 2\}$. Then we write $\rho = (V_0, V_1, V_2)$. A function $\rho = (V_0, V_1, V_2)$ is a *Roman dominating function* (RDF) on G provided that for each vertex $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 1$ [10]. The *weight* of function ρ on G is defined as $\omega_G^R(\rho) = \sum_{x \in V(G)} \rho(x)$, that is, $\omega_G^R(\rho) = |V_1| + 2|V_2|$. The *Roman domination number* of graph G is the minimum weight of an RDF ρ on G, that is, $\gamma_R(G) = min\{\omega_G^R(\rho) : \rho\, is\, an$ RDF $on\, G\}$ and is denoted by $\gamma_R(G)$. All RDF ρ on G satisfying the restriction $\omega_G^R(\rho) = \gamma_R(G)$ is a γ_R-function on G. A function ρ is an outer-independent fair Roman dominating function (OiFRDF) on G if for every $v \in V_0$ there exists $u \in V_2$ such that $d_G(u, v) = 1$, and for every $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$, and V_0 is an independent vertex set on G. The *weight* of OiFRDF ρ is denoted by $\widetilde{\omega}_G^{iFR}(\rho)$, and is defined as the sum $\widetilde{\omega}_G^{iFR}(\rho) = \sum_{z \in V(G)} \rho(z)$, that is, $\widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2|$. The minimum weight of an OiFRDF ρ denoted by $\widetilde{\gamma}_{iFR}(G)$ is called the *outer-independent fair Roman domination number*, and is defined as $\widetilde{\gamma}_{iFR}(G) = min\{\widetilde{\omega}_G^{iFR}(\rho) : \rho\, is\, an$ OiFRDF $on\, G\}$. Every OiFRDF ρ satisfying the condition $\widetilde{\omega}_G^{iFR}(\rho) = \widetilde{\gamma}_{iFR}(G)$ is called a $\widetilde{\gamma}_{iFR}$-function on G.

For some definitions in graph theory that are not introduced here, the readers may refer to the following references: [6,8,13]. In this paper, we introduced a new restricted version of Roman domination in graphs called the outer-independent fair Roman dominating function. In addition, some combinatorial properties of outer-independent fair Roman dominating functions were investigated.

3 Results

This section explores some significant results of an outer-independent fair Roman dominating function on a graph.

Theorem 1: *Let G be a graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. If $\widetilde{\gamma}_{iFR}(G) < n$, then $|V_0| = m \geq 2$, and for every $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$ and $\langle V_0 \rangle = \overline{K_{m\geq 2}}$.*

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Then ρ is an OiFRDF on G. Assume that $\widetilde{\gamma}_{iFR}(G) < n$. Then we have that $\widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| < |V(G)| = |V_0| + |V_1| + |V_2|$. This implies that $|V_2| < |V_0|$. Since $\widetilde{\gamma}_{iFR}(G) < n$, it follows that $V_0 \neq \emptyset$. Let $v \in V_0$. Since ρ is an OiFRDF on G, there exists $u \in V_2$ such that $uv \in E(G)$, and for every $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$ and V_0 is an independent vertex set on G. In that case, $|V_2| \geq 1$. Since $|V_2| < |V_0|$, it follows that $|V_0| = m \geq 2$. Therefore, we have that $\langle V_0 \rangle = \overline{K_{m\geq 2}}$. This completes the proof. □

Theorem 2: *Let G be a graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Then $|V_0| = |V_2|$ if and only if $\widetilde{\gamma}_{iFR}(G) = n$.*

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Assume that $|V_0| = |V_2|$. If $|V_0| = |V_2| = 0$, then $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\lambda) = |V_1| = |V(G)| = n$. If $|V_0| = |V_2| > 0$, then we have $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\lambda) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. Conversely, assume that $\widetilde{\gamma}_{iFR}(G) = n$. Seeking a contradiction. Suppose, for a contrary that $|V_0| \neq |V_2|$. Then either $|V_0| < |V_2|$ or $|V_0| > |V_2|$. In either case, we have that $\widetilde{\gamma}_{iFR}(G) \neq n$, a contradiction. Therefore, we conclude that $|V_0| = |V_2|$. This completes the proof. □

The following corollaries are direct consequences of Theorem 2.

Corollary 1: *Let G be a graph of order n and let $\rho = (V_0, V_1, V_2)$ be $\widetilde{\gamma}_{iFR}$-function on G. Then $\widetilde{\gamma}_{iFR}(G) < n$ if and only if $|V_2| < |V_0|$.*

Corollary 2: *Let G be a graph of order n and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. If $|V_0| = 1$, then $\widetilde{\gamma}_{iFR}(G) = n$.*

The following proposition is a direct consequence of Corollary 2.

Proposition 1: *Let $G = K_n$ (complete graph) where n is a positive integer. Then $\widetilde{\gamma}_{iFR}(G) = n$.*

Proof: Assume that $G = K_n$ where $n \geq 1$. Let $f = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iFR}$-function on G. Since for every $x, y \in V(G)$, $xy \in E(G)$, then it forced that V_0 is a trivial independent vertex set on G, that is, $|V_0| = 1$. By Corollary 2, we therefore have $\widetilde{\gamma}_{iFR}(G) = n$. This completes the proof. □

Theorem 3: *Let G be a graph. If $f = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iFR}$-function on G, then $V_1 \cup V_2$ is an outer-independent fair dominating set on G.*

Proof: Assume that $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iFR}$-function on G. Then ρ is an OiFRDF on G. In that case, it follows that for each $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 1$, and for every $x, y \in V_0$, $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$ and V_0 is an independent vertex set on G. Since $V_2 \subseteq V_1 \cup V_2$, it follows that $u \in V_1 \cup V_2$. Therefore, it suffices to conclude that $V_1 \cup V_2$ is an outer-independent fair dominating set on G. This proves the assertion. □

Theorem 4: *Let G be a graph and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Then $|V_0| = 0$ if and only if $|V_2| = 0$. In that case, $\widetilde{\gamma}_{iFR}(G) = n$.*

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Assume that $|V_0| = 0$. Seeking a contradiction. Suppose, for a contrary, that $|V_2| \neq 0$. Let $u \in V_2$. Then we can set $U_0 = \emptyset$, $U_1 = V_1 \cup \{u\}$, and $U_2 = V_2 \setminus \{u\}$. In that case, it is easy to check that $\rho = (U_0, U_1, U_2)$ is an OiFRDF on G. Hence, we get $\widetilde{\omega}_G^{iFR}(f) = |U_1| + 2|U_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \widetilde{\omega}_G^{iFR}(\rho) = \widetilde{\gamma}_{iFR}(G)$. This contradicts to the fact that ρ is a $\widetilde{\gamma}_{iFR}$-function on G. Therefore, it suffices to say that $V_2 = \emptyset$. Conversely, assume that $|V_2| = 0$. Since ρ is a $\widetilde{\gamma}_{iFR}$-function on G, it is forced that $|V_0| = 0$. Thus, we have that $|V_2| = |V_0| = 0$. In that case, by Theorem 2, it is concluded that $\widetilde{\gamma}_{iFR}(G) = n$. This completes the proof. □

The next result is immediate from Theorem 4.

Theorem 5: *Let G be a graph. Then $\widetilde{\gamma}_{iFR}(G) = 1$ if and only if $G = K_1$.*

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Assume that $\widetilde{\gamma}_{iFR}(G) = 1$. Seeking a contradiction. Suppose, for the contrary, that $G \neq K_1$. Then it means that $|V(G)| > 1$. Consider that $V_0 = \emptyset$. Then by Theorem 4, we get $V_2 = \emptyset$. Thus, it implies that $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| > 1$, a contradiction. Now, if $V_0 \neq \emptyset$, then $V_2 \neq \emptyset$ by Theorem 4. Hence, we have that $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| \geq 2$. Again, a contradiction. Therefore, $G = K_1$. Conversely, we assume that $G = K_1$. By Proposition 1, we therefore have $\widetilde{\gamma}_{cFR}(G) = 1$. This completes the proof. □

Theorem 6: *Let G be a graph of order $n \geq 2$. Then $\widetilde{\gamma}_{iFR}(G) = 2$ if and only if $G \in \{\overline{K_2}, K_1 + H\}$ where H is an empty graph of order $n - 1$.*

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G of order $n \geq 2$. Then ρ is an OiFRDF on G. Assume that $\widetilde{\gamma}_{iFR}(G) = 2$. Then we have that $|V_1| + 2|V_2| = 2$. This follows that $|V_2| \leq 1$. Suppose that $|V_2| = 0$. Then we have $V_0 = \emptyset$ by Theorem 4. Hence, we have $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 2$. It

implies that $G \in \{\overline{K_2}, K_2\}$. Accordingly, we have $G \in \{\overline{K_2}, K_1 + H\}$ where $H = K_1$. Now, suppose that $|V_2| = 1$. Since $\widetilde{\gamma}_{iFR}(G) = 2$, it follows that $|V_1| = 0$. Let $u \in V_2$. Then we have that $G = K_1 + \langle V(G) \setminus \{u\}\rangle$ where $V(K_1) = \{u\}$. Set $H = \langle V(G) \setminus \{u\}\rangle$ be an empty graph. Since for all $a, b \in V(H)$, $ab \notin E(G)$, and $V(H)$ is a fair vertex set on G, it implies that $\langle V_0\rangle = H = \langle V(G) \setminus \{u\}\rangle$. Therefore, it suffices to conclude that $G \in \{\overline{K_2}, K_1 + H\}$ where H is an empty graph of order $n - 1$. Conversely, we assume that $G \in \{\overline{K_2}, K_1 + H\}$ where H is an empty graph of order $n - 1$. If $G = \overline{K_2}$, then it is easy to check that $\widetilde{\gamma}_{iFR}(G) = 2$. Suppose that $G = K_1 + H$ where H is an empty graph of order $n - 1$. Set $V_2 = V(K_1)$. Since for every $a, b \in V(H)$, $|N_G(a) \cap V_2| = |N_G(b) \cap V_2| = 1$ and H is an independent vertex set on G, it implies that $V_0 = V(H)$. Moreover, it suffices to say that $V_1 = \emptyset$. By construction, it is clear that ρ is a $\widetilde{\gamma}_{iFR}$-function on G. Therefore, we end up with $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = 2|V(K_1)| = 2$. This completes the proof. □

The corollary below is a direct consequence of Theorem 6.

Corollary 3: *If $G = S_n$ (star graph) where $n \geq 2$, then $\widetilde{\gamma}_{iFR}(G) = 2$.*

Proof: Assume that $G = S_n$ where $n \geq 2$. Then it follows that $G = K_1 + \overline{K_n}$ where $n \geq 2$. By Theorem 6, we end up with $\widetilde{\gamma}_{iFR}(G) = 2$. This completes the proof. □

Theorem 7: *Let G be a graph and let $\rho = (V_0, V_1, V_2)$ be an OiFRDF on G for which $|V_1| = 0$. Then ρ is $\widetilde{\gamma}_{iFR}$-function on G if and only if V_2 is a $\widetilde{\gamma}_{iF}$-set on G.*

Proof: Let $\rho = (V_0, V_1 = \emptyset, V_2)$ be an OiFRDF on G. Assume that ρ is $\widetilde{\gamma}_{cFR}$-function on G. Seeking a contradiction. Suppose for a contrary that V_2 is not a $\widetilde{\gamma}_{iFR}$-set on G. Then there exists $\widetilde{\gamma}_{iF}$-set U_2 on G such that for every $x, y \in V(G) \setminus U_2$, $|N_G(x) \cap U_2| = |N_G(y) \cap U_2| > 0$ and $V(G) \setminus U_2$ is an independent vertex set on G. Let $W_0 = V(G) \setminus U_2$, $W_1 = \emptyset$, and $W_2 = U_2$. Then $\rho' = (W_0, W_1, W_2)$ is an RDF on G. let $v \in W_0$. Since U_2 is a $\widetilde{\gamma}_{iF}$-set on G, then there exists $u \in W_2 = U_2$ such that $d_G(u, v) = 1$ and for every $x, y \in W_0 = V(G) \setminus U_2$, $|N_G(x) \cap U_2| = |N_G(y) \cap U_2| > 0$, and W_0 is an independent vertex set on G. Hence, it follows that ρ' is an OiFRDF on G. Observe that, $\widetilde{\omega}_G^{iFR}(\rho') = |W_1| + 2|W_2| = 2|U_2| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{iFR}(\rho) = \widetilde{\gamma}_{iFR}(G)$. This is a contradiction to our assumption. Therefore, it suffices to say that V_2 is a $\widetilde{\gamma}_{iF}$-set on G. Conversely, assume that V_2 is a $\widetilde{\gamma}_{iF}$-set on G. Seeking a contradiction. Suppose for a contrary that $\rho = (V_0, V_1 = \emptyset, V_2)$ is not a $\widetilde{\gamma}_{iFR}$-function on G. Then there exists $\widetilde{\gamma}_{iFR}$-function $\rho'' = (M_0, M_1 = \emptyset, M_2)$ on G. Hence, we have $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho'') = |M_1| + 2|M_2| = 2|M_2| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{iFR}(\rho)$. This is a contradiction to the fact that V_2 is a $\widetilde{\gamma}_{iF}$-set on G. Therefore, we have that ρ is a $\widetilde{\gamma}_{iFR}$-function on G. This completes the proof. □

Theorem 8: *Let $G \neq K_1 + H$ where H is an empty graph and let $\rho = (V_0, V_1 = \emptyset, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Then $\widetilde{\gamma}_{iFR}(G) = 4$ if and only if there exists a dominating set $\{u, v\}$ such that $V(G) \setminus \{u, v\}$ is an independent and a fair vertex set on G.*

Proof: Let $\rho = (V_0, V_1 = \emptyset, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on $G \neq K_1 + H$ where H is an empty graph. Assume that $\widetilde{\gamma}_{iFR}(G) = 4$. Then we have $\widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = 4$. Since $V_1 = \emptyset$, it implies that $|V_2| = 2$. By Theorem 7, it follows that V_2 is a $\widetilde{\gamma}_{iF}$-set on G. This means that V_2 is an outer-independent fair dominating set on

G. Let $V_2 = \{u, v\}$. Then $\{u, v\}$ is a dominating set on G. Moreover, we have that $V_0 = V(G) \setminus \{u, v\}$ and so, $V(G) \setminus \{u, v\}$ is an independent and fair vertex set on G. Conversely, we assume that there exists a dominating set $\{u, v\}$ such that $V(G) \setminus \{u, v\}$ is an independent and a fair vertex set on G. Then $D = \{u, v\}$ is an outer-independent fair dominating set on G. Since $\rho = (V_0, V_1 = \emptyset, V_2)$ is a $\widetilde{\gamma}_{iFR}$-function on G, it implies that V_2 is a $\widetilde{\gamma}_{iF}$-set on G by Theorem 7. Hence, we obtain $2 = |D| \geq |V_2|$. If $|V_2| = 0$, then $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = 0$, a contradiction. Now, if $|V_2| = 1$, then by Theorem 6, we have that $G = K_1 + H$ where H is an empty graph, a contrary to our assumption. In that case, we obtain $|V_2| \geq 2$. Therefore, we end up with $|V_2| = 2$ and hence, $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = 2|V_2| = 4$. This completes the proof. □

Theorem 9: *Let G be a graph of order n. Then*

$$max\{\gamma_R(G), \widetilde{\gamma}_{iF}(G)\} \leq \widetilde{\gamma}_{iFR}(G) \leq n$$

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G with $|V(G)| = n \geq 1$. By Theorem 3, $V_1 \cup V_2$ is an outer-independent fair dominating set on G. Hence, we have that $\widetilde{\gamma}_{iF}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\gamma}_{iFR}(G)$. Thus, we get $\widetilde{\gamma}_{iF}(G) \leq \widetilde{\gamma}_{iFR}(G)$. Since all OiFRDF on G is an RDF on G, it implies that $\gamma_R(G) \leq \widetilde{\gamma}_{iFR}(G)$. Therefore, we obtain $max\{\widetilde{\gamma}_{iF}(G), \gamma_R(G)\} \leq \widetilde{\gamma}_{iFR}(G)$. Let $|V_0| = 0$. By Theorem 4, we get $|V_2| = 0$. Then it is clear that $\rho = (\emptyset, V_1 = V(G), \emptyset)$ is an OiFRDF on G. This follows that $\widetilde{\gamma}_{iFR}(G) \leq \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Thus, $\widetilde{\gamma}_{cFR}(G) \leq n$. This proves the assertion. □

The lower and upper bounds in Theorem 9 are sharp. To see this, we let $G = \overline{K_n}$ for all positive integers n. Then it follows that $\gamma_R(G) = \widetilde{\gamma}_{iF}(G) = \widetilde{\gamma}_{iFR}(G) = n$. The next result is immediate from Theorem 9.

Corollary 4: *Let G be a graph. If $\rho = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{iFR}$-function on G, then $0 \leq |V_2| \leq |V_0|$.*

Proof: Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. By Theorem 9, it follows that $\widetilde{\gamma}_{iFR}(G) \leq n$. Suppose that $\widetilde{\gamma}_{iFR}(G) = n$. By Theorem 2, we have that $0 \leq |V_2| = |V_0|$. Suppose that $\widetilde{\gamma}_{iFR}(G) < n$. By Corollary 1, it follows that $|V_2| < |V_0|$. And the conclusion follows. □

Theorem 10: *Let G be a graph. Then $\widetilde{\gamma}_{iFR}(G) \leq 2\widetilde{\gamma}_{iF}(G)$.*

Proof: Let I_f be a $\widetilde{\gamma}_{iF}$-set on G. Then $\widetilde{\gamma}_{iF}(G) = |I_f|$. Let $W_0 = V(G) \setminus I_f$, $W_1 = \emptyset$, and $W_2 = I_f$. Since for each $v \in W_0$, there exists $u \in W_2$ such that $uv \in E(G)$, and W_0 is an independent vertex set on G and W_0 is a fair vertex set on G, it follows that $\rho = (W_0, W_1, W_2)$ is an OiFRDF on G. Hence, we get $\widetilde{\gamma}_{iFR}(G) \leq \widetilde{\omega}_G^{iFR}(\rho) = |W_1| + 2|W_2| = 2|I_f| = 2\widetilde{\gamma}_{iF}(G)$. This proves the assertion. □

It is worth noting that the upper bound of outer-independent fair Roman domination number in Theorem 10 is sharp. To see this, we let $G = P_4 = [x_1, x_2, x_3, x_4]$ and let $\rho = (V_0, V_1, V_2)$ be a function on G. Then let $V_1 = \emptyset$ and $V_2 = \{x_2, x_3\}$. This follows that $V_0 = V(G) \setminus (V_1 \cup V_2) = \{x_1, x_4\}$. Clearly, ρ is a $\widetilde{\gamma}_{iFR}$-function on G. Since $V_1 = \emptyset$, by Theorem 4, it follows that V_2 is a $\widetilde{\gamma}_{iF}$-set on G. Therefore, we end up with $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = 2|\{x_2, x_3\}| = 4 = 2\widetilde{\gamma}_{iF}(G)$.

Proposition 2: *Let $G = P_n$ where $n \geq 1$. Then*

$$\widetilde{\gamma}_{iFR}(G) = \begin{cases} n, & \text{if } n \leq 2, \\ \frac{3n}{4}, & \text{if } n \equiv 0 \pmod 4, \\ \frac{3n+1}{4}, & \text{if } 1 \neq n \equiv 1 \pmod 4, \\ \frac{3n+2}{4}, & \text{if } 2 \neq n \equiv 2 \pmod 4, \\ \frac{3n-1}{4}, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof: Let $G = P_n$ where $n \geq 1$ and let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Let $n \leq 2$. Then $|V_0| = |V_2| = 0$ or 1. By Theorem 4, we have that $\widetilde{\gamma}_{iFR}(G) = n$. Let $G = P_n = [x_1, x_2, ..., x_n]$ where $n \geq 3$. Then consider the following:

Case 1: $n \equiv 0 \pmod 4$

Let k be a positive integer. Then $n = 4k$. Hence, $k = \frac{n}{4}$. Consider the set $S_i = \{x_{2+4(i-1)}\}$ and $T_i = \{x_{4+4(i-1)}\}$. Let $V_1 = \bigcup_{i=1}^{k} T_i = \bigcup_{i=1}^{\frac{n}{4}} \{x_{4+4(i-1)}\}$ and $V_2 = \bigcup_{i=1}^{k} S_i = \bigcup_{i=1}^{\frac{n}{4}} \{x_{2+4(i-1)}\}$. Hence, $V_0 = V(G) \setminus (V_1 \cup V_2)$. So, for every $v = x_p \in V_0$ there exists $u = x_q \in V_2$ such that $uv \in E(G)$ and for all $x, y \in V_0$, $xy \notin E(G)$, and $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$. This follows that $\rho = (V_0, V_1, V_2)$ is an OiFRDF on G. By construction, ρ is a $\widetilde{\gamma}_{iFR}$-function on G. Thus, we get $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{rhR}(\rho) = |V_1| + 2|V_2| = \left|\bigcup_{i=1}^{\frac{n}{4}} \{x_{4+4(i-1)}\}\right| + 2\left|\bigcup_{i=1}^{\frac{n}{4}} \{x_{2+4(i-1)}\right| = \frac{3n}{4}$.

Case 2: $n \equiv 1 \pmod 4$

Let k' be a positive integer. Then $n = 4k' + 1$. Thus, we get $k' = \frac{n-1}{4}$. Set the following: $S_i' = \{x_{2+4(i-1)}\}$ and $T_i' = \{x_{4+4(i-1)}\}$. Let $V_1' = \bigcup_{i=1}^{k} T_i' \cup \{x_n\} = \bigcup_{i=1}^{\frac{n}{4}} \{x_{4+4(i-1)}\} \cup \{x_n\}$ and $V_2' = \bigcup_{i=1}^{k} S_i' = \bigcup_{i=1}^{\frac{n}{4}} \{x_{2+4(i-1)}\}$. Hence, we get $V_0' = V(G) \setminus (V_1' \cup V_2')$. Thus, for every $v' = x_p \in V_0'$ there exists $u' = x_q \in V_2'$ such that $u'v' \in E(G)$ and for all $x', y' \in V_0'$, $x'y' \notin E(G)$, and $|N_G(x') \cap V_2'| = |N_G(y') \cap V_2'| > 0$. This implies that $\rho' = (V_0', V_1', V_2')$ is an OiFRDF on G. By construction, we have that ρ' is a $\widetilde{\gamma}_{iFR}$-function on G. Thus, we get $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{rhR}(\rho') = |V_1'| + 2|V_2'| = \left|\bigcup_{i=1}^{\frac{n-1}{4}} \{x_{4+4(i-1)}\} \cup \{x_n\}\right| + 2\left|\bigcup_{i=1}^{\frac{n-1}{4}} \{x_{2+4(i-1)}\right| = \frac{3n+1}{4}$.

Case 3: $n \equiv 2 \pmod 4$

Let k'' be a positive integer. Then $n = 4k'' + 2$. Thus, we have $k'' = \frac{n-2}{4}$. Set the following: $S_i'' = \{x_{2+4(i-1)}\}$ and $T_i'' = \{x_{4+4(i-1)}\}$. Then, we let $V_1'' = \bigcup_{i=1}^{k} T_i'' = \bigcup_{i=1}^{\frac{n-2}{4}} \{x_{4+4(i-1)}\}$ and $V_2'' = \bigcup_{i=1}^{k+1} S_i'' = \bigcup_{i=1}^{\frac{n+2}{4}} \{x_{2+4(i-1)}\}$. Hence, we have $V_0'' = V(G) \setminus (V_1'' \cup V_2'')$. In that case, for every $v'' = x_p \in V_0''$ there exists $u'' = x_q \in V_2''$ such that $u''v'' \in E(G)$ and for all $x'', y'' \in V_0''$, $x''y'' \notin E(G)$, and $|N_G(x'') \cap V_2''| = |N_G(y'') \cap V_2''| > 0$. This means that $\rho'' = (V_0'', V_1'', V_2'')$ is an OiFRDF on G. By construction, it follows that ρ'' is a $\widetilde{\gamma}_{iFR}$-function on G. Therefore, we get $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{rhR}(\rho'') = |V_1''| + 2|V_2''| = \left|\bigcup_{i=1}^{\frac{n-2}{4}} \{x_{4+4(i-1)}\}\right| + 2\left|\bigcup_{i=1}^{\frac{n+2}{4}} \{x_{2+4(i-1)}\right| = \frac{3n+2}{4}$.

Case 4: $n \equiv 3 \pmod 4$

Let k''' be a positive integer. Then $n = 4k''' + 3$. Thus, $k''' = \frac{n-3}{4}$. Set $S_i''' = \{x_{2+4(i-1)}\}$ and $T_i''' = \{x_{4+4(i-1)}\}$. Let $V_1''' = \bigcup_{i=1}^{k} T_i''' = \bigcup_{i=1}^{\frac{n-3}{4}} \{x_{4+4(i-1)}\}$ and $V_2''' = \bigcup_{i=1}^{k+1} S_i''' = \bigcup_{i=1}^{\frac{n+1}{4}} \{x_{2+4(i-1)}\}$. Then $V_0''' = V(G) \setminus (V_1''' \cup V_2''')$. Thus, for every $v''' = x_p \in V_0'''$ there exists $u''' = x_q \in V_2'''$ such that $u'''v''' \in E(G)$

and for all $x''', y''' \in V_0'''$, $x'''y''' \notin E(G)$, and $|N_G(x''') \cap V_2'''| = |N_G(y''') \cap V_2'''| > 0$. This follows that $\rho''' = (V_0''', V_1''', V_2''')$ is an OiFRDF on G. By construction, it implies that ρ''' is a $\widetilde{\gamma}_{iFR}$-function on G. Therefore, we end up with $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{rhR}(\rho''') = |V_1'''| + 2|V_2'''| = \left|\bigcup_{i=1}^{\frac{n-3}{4}}\{x_{4+4(i-1)}\}\right| + 2\left|\bigcup_{i=1}^{\frac{n+1}{4}}\{x_{2+4(i-1)}\right| = \frac{3n-1}{4}$.
This completes the proof. □

Proposition 3: *Let $G = C_n$ where $n \geq 3$. Then*

$$\widetilde{\gamma}_{iFR}(G) = \begin{cases} 3, & \text{if } n = 3, \\ \frac{3n}{4}, & \text{if } n \equiv 0 \pmod 4, \\ \frac{3n+1}{4}, & \text{if } n \equiv 1 \pmod 4, \\ \frac{3n+2}{4}, & \text{if } n \equiv 2 \pmod 4, \\ \frac{3n+3}{4}, & \text{if } 3 \neq n \equiv 3 \pmod 4. \end{cases}$$

Proof: Similar to Proposition 2. □

Theorem 11: *Let G be a graph of order n and let $\rho = (V_0, V_1 = \emptyset, V_2)$ be an OiFRDF on G. Then ρ is a $\widetilde{\gamma}_{iFR}$-function on G and $\widetilde{\gamma}_{iFR}(G) = \widetilde{\gamma}_{iF}(G)+1$ if and only if there exists $u \in V(G)$ such that $deg_G(u) = n-\widetilde{\gamma}_{iF}(G)$.*

Proof: Let $\rho = (V_0, V_1 = \emptyset, V_2)$ be an OiFRDF on graph G. Assume that ρ is a $\widetilde{\gamma}_{iFR}$-function on G and $\widetilde{\gamma}_{iFR}(G) = \widetilde{\gamma}_{iF}(G) + 1$. Then we have that $|V_1| + 2|V_2| = \widetilde{\gamma}_{iF}(G) + 1$. Since $V_1 = \emptyset$, it follows that V_2 is a $\widetilde{\gamma}_{iF}$-set on G by Theorem 7. Hence, we have $\widetilde{\gamma}_{iF}(G) = |V_2|$. In that case, $|V_2| + 1 = 2|V_2|$. Thus, we obtain $|V_2| = 1$. Let $V_2 = \{u\}$. Since V_2 is a $\widetilde{\gamma}_{iF}$-set on G, V_2 is a dominating set on G. Hence, it follows that $V(G) \subseteq N_G[u]$ and $V(G) \setminus \{u\}$ is an independent and a fair vertex set on G. So, it suffices to say that $deg_G(u) = N_G(u) = |V(G) \setminus \{u\}| = |V(G)| - |\{u\}| = n - |V_2| = n - \widetilde{\gamma}_{iF}(G)$. Therefore, we concluded that there exists a vertex $u \in V(G)$ such that $deg_G(u) = n - \widetilde{\gamma}_{iF}(G)$. As for the converse, we assume that there exists $u \in V(G)$ such that $deg_G(u) = n - \widetilde{\gamma}_{iF}(G)$. Let $V_2 = \{u\}$. Then $|V_2| = 1$. Now, since $V_1 = \emptyset$, it implies that $|V_2| = \widetilde{\gamma}_{iF}(G) = n - deg_G(u)$, that is, V_2 is a $\widetilde{\gamma}_{iF}$-set on G by Theorem 7. Hence, $V(G) \subseteq N_G[V_2]$. Since V_2 is an outer-independent fair dominating set on G, it follows that $V(G) \setminus V_2$ is an independent and a fair vertex set on G. Hence, we can let $V_0 = V(G) \setminus V_2$. Since V_2 is a $\widetilde{\gamma}_{iF}$-set on G, by Theorem 7, it implies that $\lambda = (V_0, V_1 = \emptyset, V_2)$ is a $\widetilde{\gamma}_{iFR}$-function on G. Therefore, it suffices to conclude that $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = |V_2| + |V_2| = \widetilde{\gamma}_{iF}(G) + 1$. This proves the assertion. □

The next theorem is a realization problem.

Theorem 12: *Let p, q and n be positive integers for which $1 \leq p \leq q \leq n$. Then there exists a graph G with $|V(G)| = n$ such that $\gamma_R(G) = p$ and $\widetilde{\gamma}_{iFR}(G) = q$.*

Proof: Let G be a graph of order n, and $\xi = (W_0, W_1, W_2)$ be a γ_R-function on G and $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. Then consider the following cases:

Case 1. $1 \leq p \leq q = n$
Let $G = K_1$. Then by Proposition 1, we have that $1 = \gamma_R(G) = \widetilde{\gamma}_{iFR}(G) = |V(G)| = n$. Let $\gamma_R(G) = p$ and $\widetilde{\gamma}_{cFR}(G) = q$. Then $1 = p = q = n$. Let $G = K_n$ where $n \geq 3$. Pick any vertex in $V(G)$, say u. Then let $W_2 = \{u\}$. Since G is a complete graph, it follows that $V(G) \subseteq N_G[W_2]$ and so, $W_0 = V(G) \setminus \{u\}$ and $W_1 = \emptyset$. Hence, we have that $\gamma_R(G) = \omega_G^R(\xi) = |W_1| + 2|W_2| = 2|\{u\}| = 2$. Moreover, since $G = K_n$, by Proposition 1, we obtain $\widetilde{\gamma}_{iFR}(G) = n$. Let $\gamma_R(G) = 2 = p$ and $\widetilde{\gamma}_{iFR}(G) = n = q$. Now, since $n \geq 3$, it implies that $1 < p < q = n$. Therefore, Case 1 holds.
Case 2. $1 < p < q < n$
Let $G = K_1 + P_{n-1}$ where $n \geq 3$. Then we set $W_2 = V(K_1)$. Since $V(G) \subseteq N_G[W_2]$, it implies that $W_0 = V(P_{n-1})$ and $W_1 = \emptyset$. Hence, we obtain $\gamma_R(G) = \omega_G^R(\xi) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Let $P_{n-1} = [x_1, x_2, ..., x_{n-1}]$ and $V_2 = V(K_1)$. Without loss of generality, we consider $n \equiv 0 \pmod 2$. Since $n - 1 \equiv 1 \pmod 2$ and $V(G) \subseteq N_G[V_2]$, we set $A = \{x_1, x_3, ..., x_{n-1}\}$ and $B = \{x_2, x_4, ..., x_{n-2}\}$. Since for every $x, y \in A$, $xy \notin E(G)$, and $|N_G(x) \cap V_2| = 1 = |N_G(y) \cap V_2|$, it follows that $V_0 = A$ and $V_1 = B$. Hence, it follows that $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = |\{x_2, x_4, ..., x_{n-2}\}| + 2|V(K_1)| = \frac{n-2}{2} + 2(1) = \frac{n+2}{2}$. Let $\gamma_R(G) = 2 = p$ and $\widetilde{\gamma}_{iFR}(G) = \frac{n+2}{2} = q$. Hence, we have that $1 < p < q < n$. The same conclusion is obtained if $n \equiv 1 \pmod 2$. Therefore, Case 2 is satisfied.
Case 3. $1 < p = q < n$
Consider $G = S_m$ where $m \geq 2$. Then $G = K_1 + \overline{K_m}$ where $m \geq 2$ and so, $|V(G)| = n = m + 1 \geq 3$. Let $W_2 = V(K_1)$. Then $V(G) \subseteq N_G[W_2]$. This follows that $W_0 = \overline{K_n}$ and $W_1 = \emptyset$. Thus, we obtain $\gamma_R(G) = \omega_G^R(\xi) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Moreover, since $G = S_m$, by Corollary 3, it implies that $\widetilde{\gamma}_{iFR}(G) = 2$. Let $p = \gamma_R(G) = 2 = q = \widetilde{\gamma}_{iFR}(G)$. Since $n \geq 3$, it suffices to conclude that $1 < p = q < n$. Therefore, Case 3 is satisfied.
This completes the proof. □

The following corollary is immediate from Theorem 12.

Corollary 5: *Let H be a graph. Then the difference $\widetilde{\gamma}_{iFR}(H) - \gamma_R(H)$ can be made arbitrarily large.*

Proof: Let m be a positive integer. By Theorem 12, there exists a graph H of order n such that $\widetilde{\gamma}_{iFR}(H) = m + 2 < n$ and $\gamma_R(H) = 2$. Thus, we get $\widetilde{\gamma}_{iFR}(H) - \gamma_R(H) = m$. Since m is a positive integer, it follows that the difference $\widetilde{\gamma}_{iFR}(H) - \gamma_R(H)$ can be made arbitrarily large. This proves the assertion. □

The following result characterizes the OiFRDF on a graph G containing several components.

Theorem 13: *Let $G = G_1 \cup G_2 \cup ... \cup G_k$ be a graph. Then $\rho = (V_0, V_1, V_2)$ is an OiFRDF on G if and only if $\rho|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is an OiFRDF on G_i for any $i \in \{1, 2, ..., k\}$.*

Proof: Suppose that $\rho = (V_0, V_1, V_2)$ is an OiFRDF on G. Then let $V_0^i = V_0 \cap V(G_i)$, $V_1^i = V_1 \cap V(G_i)$ and $V_2^i = V_2 \cap V(G_i)$, for each $i \in \{1, 2, ..., k\}$. This follows that $\rho|_{G_i} = (V_0^i, V_1^i, V_2^i)$ for each $i \in \{1, 2, ..., k\}$. Let $v \in V_0^r$ for some $r \in \{1, 2, ..., k\}$. Since $V_0^i \subseteq V_0$ for all $i \in \{1, 2, ..., k\}$, it implies that $v \in V_0$. Since ρ is an outer-independent fair Roman dominating function on G, it implies that there exists $u \in V_2$ such that

$uv \in E(G)$, and for every $x, y \in V_0$, $xy \notin E(G)$ and $|N_G(x) \cap V_2| = |N_G(y) \cap V_2| > 0$. Since $d_G(x, y) = \infty$ for all $x \in G_p$ and for all $y \in H_q$ where $p \neq q \in \{1, 2, ..., k\}$, it means that $u \in V_2^r$ for which $uv \in E(G_r)$, and for every $x, y \in V_0^r$, $xy \notin E(G_r)$ and $|N_G(x) \cap V_2^r| = |N_G(y) \cap V_2^r| > 0$. Thus, we have that $\rho|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is an OiFRDF on G_i for any $i \in \{1, 2, ..., k\}$. On the other hand, suppose that $\rho|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is an OiFRDF on G_i for all $i \in \{1, 2, ...k\}$. Then let $V_0 = \bigcup_{i=1}^k V_0^i$, $V_1 = \bigcup_{i=1}^k V_1^i$ and $V_2 = \bigcup_{i=1}^k V_2^i$. This implies that $\rho = (V_0, V_1, V_2)$ is an RDF on G. Let $v' \in V_0$. Then $v' \in V_0^s$ for some $s \in \{1, 2, ...k\}$. Since $\rho|_{G_i}$ is an OiFRDF on G_i for all $i \in \{1, 2, ...k\}$, it follows that there exists $u' \in V_2^s$ for which $u'v' \in E(G_s)$, and for every $x', y' \in V_0^s$, $x'y' \notin E(G_s)$ and $|N_G(x') \cap V_2^s| = |N_G(y') \cap V_2^s| > 0$. Now, since $V_0 = \bigcup_{i=1}^k V_0^i$ and $V_2 = \bigcup_{i=1}^k V_2^i$, it follows that $u' \in V_2$ such that $u'v' \in E(G)$, and for every $x', y' \in V_0$, $x'y' \notin E(G)$ and $|N_G(x') \cap V_2| = |N_G(y') \cap V_2| > 0$. Therefore, it suffices to conclude that ρ is an OiFRDF on G. This completes the proof. □

The next corollaries below are a consequence of Theorem 13 above.

Corollary 6: *If* $G = G_1 \cup G_2 \cup ... \cup G_k$, *then* $\widetilde{\gamma}_{iFR}(G) = \sum_{i=1}^{k} \widetilde{\gamma}_{iFR}(G_i)$.

Proof: Assume that $G = G_1 \cup G_2 \cup ... \cup G_k$. Let $\rho = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{iFR}$-function on G. For any $i \in \{1, 2, ..., k\}$, let $V_0^i = V_0 \cap V(G_i)$, $V_1^i = V_1 \cap V(G_i)$ and $V_2^i = V_2 \cap V(G_i)$. This follows that $\rho|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is an RDF on G_i for all $i \in \{1, 2, ..., k\}$. By Theorem 13, it implies that $\rho|_{G_i}$ is an OiFRDF on G_i for all $i \in \{1, 2, ..., k\}$. Thus, we get $\widetilde{\gamma}_{iFR}(G) = \widetilde{\omega}_G^{iFR}(\rho) = |V_1| + 2|V_2| = \sum_{i=1}^k |V_1^i| + 2\sum_{i=1}^k |V_2^i| = \sum_{i=1}^k \left(|V_1^i| + 2|V_2^i|\right) = \sum_{i=1}^k \widetilde{\omega}_G^{iFR}(\rho|_{G_i}) \geq \sum_{i=1}^k \widetilde{\gamma}_{iFR}(G_i)$. On the other hand, let $\rho|_{G_i} = (W_0^i, W_1^i, W_2^i)$ be an OiFRDF on G_i for all $i \in \{1, 2, ...k\}$. Consider the following: $W_0 = \bigcup_{i=1}^k W_0^i$, $W_1 = \bigcup_{i=1}^k W_1^i$ and $W_2 = \bigcup_{i=1}^k W_2^i$. This implies that $\rho = (W_0, W_1, W_2)$ is an RDF on G. Invoking Theorem 13, we have that ρ is an OiFRDF on G. Hence, we have $\sum_{i=1}^k \widetilde{\gamma}_{iFR}(G_i) = \sum_{i=1}^k \widetilde{\omega}_G^{iFR}(\rho|_{G_i}) = \sum_{i=1}^k \left(|W_1^i| + 2|W_2^i|\right) = \sum_{i=1}^k |W_1^i| + 2\sum_{i=1}^k |W_2^i| = |W_1| + 2|W_2| = \widetilde{\omega}_G^{iFR}(\rho) \geq \widetilde{\gamma}_{iFR}(G)$. Therefore, we end up with $\widetilde{\gamma}_{iFR}(G) = \sum_{i=1}^k \widetilde{\gamma}_{iFR}(G_i)$. This completes the proof. □

Corollary 7: *Let* $G = G_1 \cup G_2 \cup ... \cup G_k$ *be a graph of order n. If* $G_i = K_{n_i}$ *where* $n_i \in \mathbb{N}$ *for each* $i \in \{1, 2, ..., k\}$, *then* $\widetilde{\gamma}_{iFR}(G) = n$.

4 Conclusions

This study introduced a new restricted version of Roman domination in graphs: the outer-independent fair Roman dominating function. In addition, we investigated important graph-theoretic properties of outer-independent fair Roman dominating functions in certain classes of graphs. Some bounds on outer-independent fair Roman domination number, and the realization problem were given. Some characterizations were also provided, including a necessary and sufficient condition for an outer-independent fair Roman dominating function in a graph with several components. As for future studies, one may characterize the outer-independent fair Roman dominating function under some binary operations of two graphs, which may strengthen the current findings of the initial investigation.

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