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Research article

On Some Structural and Metric Properties of Suborbital Graphs Arising From Wreath Products of Classical Permutation Groups

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Abstract: This paper investigates structural, metric, and spectral properties of suborbital graphs arising from product actions of finite wreath products $G \wr S_n$, where $G \in \{A_m, C_m, D_{2m}, S_m\}$. We establish conditions under which these graphs are Cartesian powers of corresponding base suborbital graphs. Using permutation group theory and algebraic graph theory, we derive formulas for their degree, size, diameter, distance, triangles, and four-cycles. We also examine connectivity, clique number, bipartiteness, eigenvalues, spanning trees, and distance-based invariants. The results show that key properties of the resulting graphs are determined by the structure of their base suborbital graphs.

Mathematics Subject Classification: 05C25, 05C12, 05C50, 20B25, 20B15
Keywords: Suborbital graphs; Wreath products; Product actions; Cartesian products; Permutation groups; Graph spectra.

1 Introduction

Permutation group theory and algebraic graph theory provide important mathematical frameworks for investigating symmetry and structural properties of finite combinatorial objects. In permutation group theory, the study of group actions, orbits, stabilizers, and invariant relations provides a means of understanding the algebraic structure of finite groups and their actions. On the other hand, algebraic graph theory investigates how symmetry influences graph structures and properties such as connectivity, regularity, distance, diameter, and spectral characteristics. The foundational theories of finite permutation groups and their actions are developed in [6,7,13], while systematic treatments of graph-theoretic structures and invariants are presented in [5,8,12]. The interaction between these two mathematical areas provides a natural foundation for constructing and analysing graphs associated with permutation group actions.

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Let G be a finite transitive permutation group acting on a finite set X, with $|X| = m$. For a fixed $\alpha \in X$, the orbits of the point stabilizer G_α on X are called the suborbits of G, whereas the orbits of G on $X \times X$ are called its orbitals. Every non-diagonal orbital defines a directed graph with vertex set X, called a suborbital graph. Such a graph is undirected precisely when the corresponding orbital is self-paired. In particular, for an orbital

$$\mathcal{O} = (\alpha, \beta)^G,$$

its paired orbital is given by

$$\mathcal{O}^* = \{(y, x) : (x, y) \in \mathcal{O}\}.$$

Thus, $\mathcal{O}$ is self-paired if and only if $\mathcal{O} = \mathcal{O}^*$. The relationship between orbitals, suborbits, and invariant graph relations forms a fundamental basis for the construction and investigation of suborbital graphs [6,7,9].

Wreath products constitute an important class of permutation groups for extending these constructions to Cartesian product domains. Let $G \leq \mathrm{Sym}(X)$ be a transitive permutation group and let S_n denote the symmetric group of degree n. The wreath product in product action is

$$W = G \wr S_n = G^n \rtimes S_n,$$

acting on

$$\Omega = X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ factors}}.$$

The base group G^n acts independently on the coordinate positions, while the top group S_n permutes these positions. Consequently, the product action combines the individual permutation structures of the component groups with the additional symmetry arising from coordinate permutations. The structural properties of wreath products, their product actions, and their connections with Cartesian decompositions have been extensively investigated in [7,11]. Further studies have considered other algebraic properties of wreath-product actions, including derangements [4].

Recent investigations have also examined suborbital graphs arising from direct-product and product-action permutation groups. Aliyu et al. [1] studied suborbital graphs associated with direct products of symmetric groups acting on Cartesian products of finite sets. Their investigation contributes to the understanding of how direct-product permutation actions induce graph structures on product domains. Similarly, Morang'a et al. [10] investigated suborbital graphs arising from the actions of alternating and cyclic groups on Cartesian products, considering properties including self-pairedness, regularity, connectivity, and girth. In related work, Aliyu et al. [2] examined rank, subdegrees, and primitivity in product-action wreath products, while [3] investigated further combinatorial and group-theoretic properties of wreath-product actions. These

contributions provide relevant foundations for understanding the orbital structures of product actions and the associated combinatorial constructions.

Despite these developments, the structural and metric properties of suborbital graphs arising from wreath products of classical permutation groups have not been sufficiently developed within a unified framework. In particular, results obtained from direct-product actions do not automatically account for the additional coordinate permutations introduced by the symmetric top group in wreath-product actions. Although the underlying domain is a Cartesian power, this alone does not guarantee that every suborbital graph arising from the wreath product is a Cartesian power of a corresponding graph of the base group. The orbital structure must first be characterized to establish the precise conditions under which such a representation is valid. Furthermore, the extent to which graph-theoretic properties, particularly distance, diameter, local configurations, connectivity, and spectral parameters, are inherited from or determined by the corresponding base graph requires explicit investigation. Addressing these issues constitutes the principal motivation for the present study.

To establish the required relationship, let G be a finite transitive permutation group acting on X, and let $\mathcal{O} \subseteq X \times X$ be a selected non-diagonal orbital of G. For the construction of an undirected suborbital graph, we require $\mathcal{O}$ to be self-paired. More generally, let $R \subseteq X \times X$ be a symmetric, G-invariant relation that is disjoint from the diagonal, allowing R to be either a single self-paired orbital or a union of paired orbitals. The associated base graph is defined by

$$\Delta = (X, E(\Delta)),$$

where

$$E(\Delta) = \{\{x, y\} : (x, y) \in R\}.$$

For $n \geq 2$, define a relation $R^{(n)}$ on $\Omega = X^n$ by declaring that $(u, v) \in R^{(n)}$ if and only if there exists exactly one coordinate $i \in \{1, 2, \dots, n\}$ for which

$$(u_i, v_i) \in R,$$

while

$$u_j = v_j, \qquad j \neq i.$$

Thus, two vertices are adjacent precisely when they differ in exactly one coordinate and their entries in that coordinate are adjacent in the base graph. The resulting graph Γ consequently has the Cartesian-product representation

$$\Gamma \cong \underbrace{\Delta \square \Delta \square \cdots \square \Delta}_{n \text{ factors}} = \Delta^{\square n}.$$

When R is a single non-diagonal orbital of G, the induced relation $R^{(n)}$ is a single orbital of $G \wr S_n$. Moreover, the corresponding wreath-product orbital is self-paired if and only if the selected base orbital is self-paired. If R is a union of orbitals, then $R^{(n)}$ is a $G \wr S_n$-invariant union of orbitals. This distinction provides the algebraic condition underlying the Cartesian representation employed in this study.

The investigation considers the classical permutation groups

$$G \in \{A_m, C_m, D_{2m}, S_m\},$$

where A_m, C_m, D_{2m}, and S_m denote the alternating, cyclic, dihedral, and symmetric groups, respectively, in their specified transitive actions on X. The selection of the base relation depends on the orbital structure of the particular group action. For example, in the natural action of A_m, with $m \geq 4$, the non-diagonal orbital gives the complete graph K_m. In the natural action of a dihedral group on the vertices of a regular polygon, the orbital associated with adjacent vertices gives a cycle graph. Similarly, for a regular cyclic action, the undirected cycle relation is obtained from the symmetric union of the orbitals corresponding to the two opposite directed steps. These examples illustrate the importance of identifying the appropriate orbital and establishing its self-pairedness before constructing the associated undirected suborbital graph.

The principal objective of this study is to establish how the structural, metric, and spectral properties of suborbital graphs arising from product actions of $G \wr S_n$ depend on the corresponding properties of their base suborbital graphs. First, a criterion for self-pairedness is established, and the corresponding orbital under the wreath-product action is characterized. This provides the foundation for establishing the Cartesian-power representation

$$\Gamma \cong \Delta^{\square n}.$$

The representation enables the investigation of the resulting graph through the properties of its base graph and the interaction of its coordinate factors.

The metric structure is investigated by determining the distance between arbitrary vertices

$$u = (u_1, u_2, \dots, u_n), \qquad v = (v_1, v_2, \dots, v_n)$$

of Γ. In particular, it is established that

$$d_\Gamma(u, v) = \sum_{i=1}^{n} d_\Delta(u_i, v_i).$$

Consequently, the diameter of the wreath-product suborbital graph is given by

$$\operatorname{diam}(\Gamma) = n \operatorname{diam}(\Delta).$$

These results demonstrate how coordinate distances combine to determine the global metric structure of the resulting graph.

Furthermore, the study investigates local structural properties, particularly triangles and common neighbours of adjacent vertices. It establishes that the number of triangles containing a fixed vertex in Γ is

$$T_\Gamma = nT_\Delta,$$

where T_Δ and T_Γ denote the respective triangle numbers of the base graph and the wreath-product graph. In contrast, the number of common neighbours of adjacent vertices is inherited from the base graph, yielding

$$\lambda_\Gamma = \lambda_\Delta,$$

whenever the base graph has a constant adjacent common-neighbour number. The degree and size of the resulting graph are also determined. Specifically, if Δ is a k_Δ-regular graph on m vertices, then

$$|V(\Gamma)| = m^n, \qquad \deg(\Gamma) = nk_\Delta,$$

and

$$|E(\Gamma)| = n|E(\Delta)|m^{n-1} = \frac{nk_\Delta m^n}{2}.$$

Further structural investigations establish that connectivity is preserved under Cartesian powers and that the clique number satisfies

$$\omega(\Gamma) = \omega(\Delta).$$

The number of distinct unoriented four-cycles is also determined by

$$c_4(\Gamma) = nm^{n-1}c_4(\Delta) + \binom{n}{2}m^{n-2}|E(\Delta)|^2.$$

This formula accounts for four-cycles arising within individual coordinate factors and those formed through interactions between two distinct coordinate positions. These results provide further information about how local and global structural characteristics of the base graph are reflected in the wreath-product suborbital graph.

The study is further extended to spectral and distance-based invariants. By employing the Kronecker-sum representation of the adjacency matrix of a Cartesian product, the adjacency eigenvalues of Γ are expressed as sums of n adjacency eigenvalues of Δ. An analogous decomposition is established for the Laplacian eigenvalues. These results provide characterizations of the spectral radius for regular base graphs, the algebraic connectivity for connected base graphs, and the number of spanning trees through the Matrix–Tree Theorem. The distance formula is also used to derive the Wiener index and average distance of the resulting graph:

$$W(\Gamma) = nm^{2n-2}W(\Delta),$$

and

$$\overline{d}(\Gamma) = \frac{2n\,W(\Delta)}{m^2(m^n-1)}.$$

In addition, the study establishes that Γ is bipartite if and only if Δ is bipartite and obtains a spectral expression for the number of closed walks of any prescribed positive length. The spectral and graph-invariant techniques employed are consistent with established methods in algebraic graph theory [5,8,12].

The principal contribution of this study is the establishment of a systematic relationship between the orbital structures of classical permutation groups acting through wreath products and the structural, metric, and spectral properties of their associated suborbital graphs. In contrast to investigations that consider only the underlying Cartesian domain, this study identifies the precise orbital conditions required for the resulting suborbital graph to admit a Cartesian-power representation. It further derives explicit expressions for important graph invariants and establishes how these parameters behave as the number of coordinate factors increases. In particular, the results distinguish between properties that scale with the number of coordinates, those inherited directly from the base graph, and those determined by combinations of its spectral parameters. Consequently, the study provides a unified mathematical framework for investigating the relationship between permutation-group actions, orbital structures, and the combinatorial and spectral characteristics of suborbital graphs arising from wreath products.

2 Results

In this section, we establish additional structural properties of suborbital graphs arising from wreath product actions. The results extend the known properties of regularity, vertex-transitivity and Hamming graph structures. Let

$$G \in \{A_m, C_m, D_{2m}, S_m\}$$

be a transitive permutation group on a set X with

$$|X| = m.$$

Let

$$W = G \wr S_n$$

act on

$$\Omega = X^n$$

in product action.

Let Γ denote the first non-trivial suborbital graph of this action.

Definition 1 (Base Suborbital Graph)**:** *Let G be a transitive permutation group acting on a finite set X. The first non-trivial suborbital graph of G is the graph*

$$\Delta = (X, E_G),$$

where

$$\{x, y\} \in E_G$$

whenever the ordered pair (x, y) belongs to a non-diagonal orbital of G. The valency and distance in this graph are denoted by

$$k_G$$

and

$$d_\Delta(x, y),$$

respectively.

Definition 2 (Selected suborbital graph)**:** *Let $G \leq \mathrm{Sym}(X)$ be transitive on a finite set X, and fix $\alpha \in X$. Let*

$$G_\alpha = \{g \in G : \alpha^g = \alpha\}$$

be the stabilizer of α. Choose

$$\beta \in X \setminus \{\alpha\}$$

and let

$$\mathcal{O} = (\alpha, \beta)^G = \{(\alpha^g, \beta^g) : g \in G\}$$

be the corresponding non-diagonal orbital of G.

The paired orbital of $\mathcal{O}$ is

$$\mathcal{O}^* = \{(y, x) : (x, y) \in \mathcal{O}\}.$$

The orbital $\mathcal{O}$ is called self-paired if

$$\mathcal{O} = \mathcal{O}^*.$$

Whenever $\mathcal{O}$ is self-paired, the associated undirected suborbital graph is denoted by

$$\Delta_{\mathcal{O}} = (X, E_{\mathcal{O}}),$$

where

$$V(\Delta_{\mathcal{O}}) = X$$

and

$$E_{\mathcal{O}} = \big\{\{x, y\} : (x, y) \in \mathcal{O}\big\}.$$

The orbital $\mathcal{O}$ is called the selected non-diagonal orbital, and $\Delta_{\mathcal{O}}$ is called the selected suborbital graph of G.

Lemma 1 (Criterion for self-pairedness)**:** *Let G be transitive on X, let $\alpha \in X$, and let $\beta \in X \setminus \{\alpha\}$. Put*

$$\mathcal{O} = (\alpha, \beta)^G.$$

Then the following statements are equivalent:

1. *$\mathcal{O}$ is self-paired;*
2. *$(\beta, \alpha) \in (\alpha, \beta)^G$;*
3. *there exists $g \in G$ such that*

   $$\alpha^g = \beta, \qquad \beta^g = \alpha;$$

4. *there exists $g \in G$ such that*

   $$\alpha^g = \beta \quad \text{and} \quad \beta^g = \alpha.$$

Equivalently, the selected suborbit β^{G_α} is paired with itself.

Proof: By definition,

$$\mathcal{O} = (\alpha, \beta)^G.$$

The paired orbital is

$$\mathcal{O}^* = (\beta, \alpha)^G.$$

Since G acts transitively on $X \times X$ within each orbital, two orbitals are either equal or disjoint. Hence

$$\mathcal{O} = \mathcal{O}^*$$

if and only if

$(\beta, \alpha) \in \mathcal{O}$.

This proves the equivalence of (1) and (2).

By the definition of an orbital,

$(\beta, \alpha) \in (\alpha, \beta)^G$

if and only if there exists $g \in G$ such that

$(\alpha^g, \beta^g) = (\beta, \alpha)$.

Thus (2) and (3) are equivalent. Statements (3) and (4) are identical.

Finally, the orbital $\mathcal{O}$ corresponds to the suborbit β^{G_α}. Its paired suborbit is the suborbit corresponding to $\mathcal{O}^*$. Therefore, $\mathcal{O}$ is self-paired precisely when the selected suborbit is paired with itself. □

Proposition 1 (Self-paired orbital in product action)**:** *Let G be transitive on X, and let*

$W = G \wr S_n$

act on $\Omega = X^n$ in product action. Fix $\alpha, \beta \in X$, with $\alpha \neq \beta$, and let

$\mathcal{O} = (\alpha, \beta)^G$

be a non-diagonal orbital of G. Define

$\widehat{\alpha} = (\alpha, \dots, \alpha), \qquad \widehat{\beta} = (\beta, \alpha, \dots, \alpha).$

Let

$\mathcal{O}_W = (\widehat{\alpha}, \widehat{\beta})^W$.

Then $\mathcal{O}_W$ consists precisely of the ordered pairs $(u, v) \in X^n \times X^n$ for which there is exactly one coordinate i satisfying

$(u_i, v_i) \in \mathcal{O}$,

while

$u_j = v_j \qquad (j \neq i)$.

Moreover,

$$\mathcal{O}_W = \mathcal{O}_W^* \quad \Longleftrightarrow \quad \mathcal{O} = \mathcal{O}^*.$$

Consequently, the undirected suborbital graph associated with $\mathcal{O}_W$ is isomorphic to

$$\Delta_{\mathcal{O}}^{\square n}$$

if and only if the selected base orbital $\mathcal{O}$ is self-paired.

Proof: The base group G^n acts independently on the coordinates. Thus, under its action, the first coordinate pair (α, β) ranges over $\mathcal{O}$, whereas each remaining diagonal pair (α, α) ranges over the diagonal orbital

$$\mathcal{D} = \{(x, x) : x \in X\}.$$

The top group S_n permutes the coordinate positions. Hence the W-orbit of $(\widehat{\alpha}, \widehat{\beta})$ consists exactly of the ordered pairs having one coordinate pair in $\mathcal{O}$ and diagonal pairs in all remaining coordinates.

Taking the paired orbital reverses the exceptional coordinate pair. Therefore, the paired orbital has its exceptional coordinate pair in $\mathcal{O}^*$ and diagonal pairs elsewhere. It follows that

$$\mathcal{O}_W = \mathcal{O}_W^*$$

if and only if

$$\mathcal{O} = \mathcal{O}^*.$$

If $\mathcal{O}$ is self-paired, adjacency in the associated undirected graph is therefore equivalent to changing exactly one coordinate along an edge of $\Delta_{\mathcal{O}}$. This is precisely the Cartesian-product adjacency relation. Hence

$$\Gamma \cong \Delta_{\mathcal{O}}^{\square n}.$$

Conversely, if $\mathcal{O}$ is not self-paired, $\mathcal{O}_W$ is not self-paired, and the selected orbital does not define an undirected suborbital graph. This completes the proof. □

Remark 1: *A transitive permutation group may possess several non-diagonal orbitals, and these orbitals need not all be self-paired. Consequently, the expression "first non-trivial suborbital graph" is not used without an explicit ordering of the non-diagonal orbitals. Throughout this paper, we select a specific non-diagonal orbital*

$$\mathcal{O} = (\alpha, \beta)^G$$

and impose the self-pairedness condition

$\mathcal{O} = \mathcal{O}^*$.

All undirected suborbital graphs considered in the subsequent results are constructed from such selected self-paired orbitals.

Proposition 2 (Cartesian-product structure)**:** *Let $G \leq \mathrm{Sym}(X)$ be transitive on a finite set X, and let*

$W = G \wr S_n = G^n \rtimes S_n$

act on $\Omega = X^n$ in product action. Let $\mathcal{R} \subseteq X \times X$ be a symmetric, G-invariant relation satisfying

$\mathcal{R} \cap \{(x, x) : x \in X\} = \emptyset$.

Let $\Delta = (X, E_\Delta)$ be the graph defined by

$E_\Delta = \{\{x, y\} : (x, y) \in \mathcal{R}\}$.

Define $\mathcal{R}^{(n)}$ on Ω by

$(u, v) \in \mathcal{R}^{(n)}$

if and only if there is exactly one index i such that

$(u_i, v_i) \in \mathcal{R}, \qquad u_j = v_j \quad (j \neq i)$.

Then the graph $\Gamma = (\Omega, E_\Gamma)$ defined by $\mathcal{R}^{(n)}$ is isomorphic to the n-fold Cartesian power of Δ. In particular,

$$\boxed{\Gamma \cong \Delta^{\square n}.}$$

If $\mathcal{R}$ is a single non-diagonal self-paired orbital of G, then $\mathcal{R}^{(n)}$ is a single self-paired orbital of W. If $\mathcal{R}$ is a union of non-diagonal orbitals, then $\mathcal{R}^{(n)}$ is a W-invariant union of orbitals.

Proof: Write an element of W as

$g = (g_1, \ldots, g_n)\sigma, \qquad g_i \in G, \quad \sigma \in S_n$.

Under the product action, the base group G^n acts independently on the coordinates, while S_n permutes the coordinate positions.

First, we show that $\mathcal{R}^{(n)}$ is W-invariant. Suppose that $(u, v) \in \mathcal{R}^{(n)}$. By definition, there exists a unique coordinate i such that

$$(u_i, v_i) \in \mathcal{R},$$

and $u_j = v_j$ for all $j \neq i$.

For an arbitrary element $g \in W$, the transformed coordinates are obtained by applying the corresponding coordinate permutations and permuting their positions. Since $\mathcal{R}$ is G-invariant, the exceptional coordinate pair remains in $\mathcal{R}$ after the action of G^n. Moreover, equal coordinate entries remain equal under the same coordinate permutation. The permutation group S_n merely moves the exceptional coordinate to another position. Hence the transformed ordered pair again has exactly one coordinate pair in $\mathcal{R}$ and agrees in all other coordinates. Therefore,

$$(u, v) \in \mathcal{R}^{(n)} \Longrightarrow (u^g, v^g) \in \mathcal{R}^{(n)}.$$

Thus $\mathcal{R}^{(n)}$ is W-invariant.

Next, by the definition of $\mathcal{R}^{(n)}$, two vertices

$$u = (u_1, \dots, u_n), \qquad v = (v_1, \dots, v_n)$$

are adjacent in Γ if and only if there is exactly one coordinate i such that

$$\{u_i, v_i\} \in E_\Delta$$

and

$$u_j = v_j \qquad \text{for all } j \neq i.$$

But this is precisely the definition of adjacency in the Cartesian product of n copies of Δ. Consequently, the identity map on X^n is a graph isomorphism

$$\Gamma \longrightarrow \Delta^{\square n}.$$

Finally, suppose that $\mathcal{R}$ is a single non-diagonal orbital of G. Choose $(a, b) \in \mathcal{R}$ and consider

$$\alpha = (a, a, \dots, a), \qquad \beta = (b, a, \dots, a).$$

The W-orbit of (α, β) consists exactly of the ordered pairs having one coordinate pair in $\mathcal{R}$ and equal entries in every other coordinate. Indeed, the coordinate action of G^n moves (a, b) throughout $\mathcal{R}$ and moves (a, a) throughout the diagonal orbital, while S_n moves the exceptional coordinate to any position. Therefore,

$$(\alpha, \beta)^W = \mathcal{R}^{(n)}.$$

Thus $\mathcal{R}^{(n)}$ is a single orbital of W. Since $\mathcal{R}$ is self-paired, reversing the ordered pairs leaves $\mathcal{R}$ invariant. Hence $\mathcal{R}^{(n)}$ is also self-paired.

If $\mathcal{R}$ is a union of orbitals, the same invariance argument shows that $\mathcal{R}^{(n)}$ is a union of W-orbitals, although it need not be a single orbital. This completes the proof. □

Lemma 2 (Distance in Wreath Product Suborbital Graph)**:** *Let*

$$W = G \wr S_n$$

act on

$$\Omega = X^n$$

in the product action. If

$$u = (u_1, \dots, u_n), \qquad v = (v_1, \dots, v_n)$$

are vertices of the corresponding suborbital graph Γ*, then*

$$d_\Gamma(u, v) = \sum_{i=1}^{n} d_\Delta(u_i, v_i).$$

Proof: The product action allows the coordinates to be changed independently. For each coordinate i, moving from u_i to v_i in the base graph requires

$$d_\Delta(u_i, v_i)$$

steps.

Combining these paths coordinate by coordinate gives a path in Γ of length

$$\sum_{i=1}^{n} d_\Delta(u_i, v_i).$$

Since every edge of Γ changes only one coordinate, a shorter path cannot exist. Hence

$$d_\Gamma(u, v) = \sum_{i=1}^{n} d_\Delta(u_i, v_i).$$

□

Example 1: *Let*

$G = C_{10}$

and

$W = C_{10} \wr S_4$.

The base suborbital graph is

C_{10}.

Take

$u = (0, 0, 0, 0)$,

and

$v = (5, 5, 5, 5)$.

Since

$d_\Delta(0, 5) = 5$,

we have

$d_\Gamma(u, v) = 5 + 5 + 5 + 5 = 20$.

Hence the distance formula is verified.

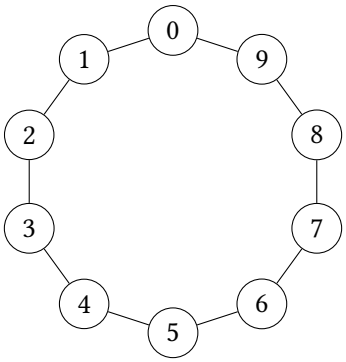


Figure 1: Base graph C_{10}.

Theorem 1 (Diameter of General Wreath Product Suborbital Graph)**:** *Let*

$G \in \{A_m, C_m, D_{2m}, S_m\}$

be a transitive permutation group acting on a finite set X *with*

$|X| = m$.

Let

$W = G \wr S_n$

act on

$\Omega = X^n$

in the product action, and let Γ *be the first non-trivial suborbital graph of this action. Suppose that the base suborbital graph of* G *has diameter* $\mathrm{diam}(\Delta)$. *Then the diameter of* Γ *is*

$\mathrm{diam}(\Gamma) = n\,\mathrm{diam}(\Delta)$.

Proof: Let

$u = (u_1, u_2, \ldots, u_n), \qquad v = (v_1, v_2, \ldots, v_n)$

be arbitrary vertices of Γ.

By the product action of the wreath product, changes in different coordinates occur independently. Hence, the distance between two vertices is the sum of the distances in the corresponding coordinates of the base suborbital graph:

$$d_\Gamma(u, v) = \sum_{i=1}^{n} d_\Delta(u_i, v_i).$$

Since the diameter of the base graph is $\operatorname{diam}(\Delta)$, we have

$$d_\Delta(u_i, v_i) \leq \operatorname{diam}(\Delta)$$

for every coordinate i. Therefore,

$$d_\Gamma(u, v) \leq \sum_{i=1}^{n} \operatorname{diam}(\Delta) = n \operatorname{diam}(\Delta).$$

Thus,

$$\operatorname{diam}(\Gamma) \leq n \operatorname{diam}(\Delta).$$

Now choose vertices $x, y \in X$ such that

$$d_\Delta(x, y) = \operatorname{diam}(\Delta).$$

Define

$$u = (x, x, \dots, x),$$

and

$$v = (y, y, \dots, y).$$

Then every coordinate contributes the maximum possible distance:

$$d_\Delta(u_i, v_i) = \operatorname{diam}(\Delta).$$

Hence,

$$d_\Gamma(u, v) = \sum_{i=1}^{n} \operatorname{diam}(\Delta) = n \operatorname{diam}(\Delta).$$

Therefore the upper bound is attained, and consequently,

$$\boxed{\operatorname{diam}(\Gamma) = n \operatorname{diam}(\Delta)}.$$

□

Example 2: *Consider*

$G = S_3$

acting on

$X = \{1, 2, 3\}$.

The first suborbital graph is the complete graph

K_3.

Since every pair of vertices is adjacent,

$\mathrm{diam}(\Delta) = 1$.

Now consider

$W = S_3 \wr S_2$

acting on

$\Omega = X^2$.

Take

$u = (1, 1), \qquad v = (2, 3).$

Since the base graph is K_3,

$d_\Delta(1, 2) = 1, \qquad d_\Delta(1, 3) = 1.$

By Lemma 2

$d_\Gamma(u, v) = d_\Delta(1, 2) + d_\Delta(1, 3) = 1 + 1 = 2.$

Thus the distance equals the sum of coordinate distances.Since

$\mathrm{diam}(\Delta) = 1,$

Theorem1 gives

$\mathrm{diam}(\Gamma) = 2\,\mathrm{diam}(\Delta) = 2.$

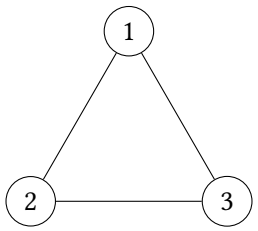


Figure: Base graph K_3.

Example 3: *Let*

$G = C_5.$

The base graph is the cycle

$C_5.$

The vertices are

$V(C_5) = \{0, 1, 2, 3, 4\}.$

The edges are

$E(C_5) = \{01, 12, 23, 34, 40\}.$

Therefore,

$|V| = 5,$

$|E| = 5,$

and

$k = 2.$

The diameter is

$$\text{diam}(\Delta) = \left\lfloor \frac{5}{2} \right\rfloor = 2.$$

For

$C_5 \wr S_3,$

we have

$|\Omega| = 5^3 = 125.$

The diameter becomes

$diam(\Gamma) = 3(2) = 6.$

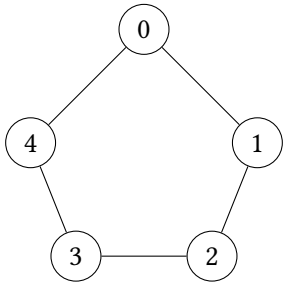


Figure 2: Base suborbital graph for C_5: cycle graph

Verification:

$$\boxed{diam(C_5 \wr S_3) = 6}$$

Example 4: *Let*

$G = C_9$

and consider

$C_9 \wr S_5$.

The diameter of the base graph is

$$\mathrm{diam}(\Delta) = \left\lfloor \frac{9}{2} \right\rfloor = 4.$$

The vertex set is

$\Omega = X^5$,

hence

$|\Omega| = 9^5 = 59049$.

By Theorem 3.1,

$\mathrm{diam}(\Gamma) = n\,\mathrm{diam}(\Delta)$.

Therefore,

$\mathrm{diam}(\Gamma) = 5(4) = 20$.

Thus the wreath product graph has diameter 20.

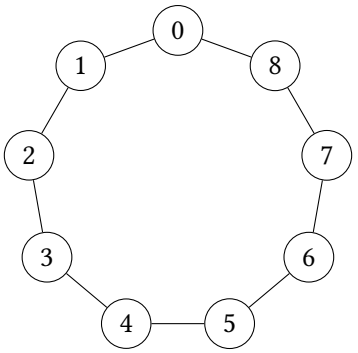


Figure 3: Base graph C_9.

Definition 3 (Triangle Number)**:** *Let* Γ *be a graph and let* $v \in V(\Gamma)$.

The triangle number of v *is defined as*

$T_\Gamma(v) = |\{\textit{triangles containing } v\}|$.

If Γ *is vertex-transitive, this value is constant and is denoted by*

T_Γ.

Lemma 3 (Coordinate Structure of Triangles)**:** *Let* Γ *be a wreath product suborbital graph.*

Every triangle in Γ *is obtained by changing a single coordinate and corresponds to a triangle in the base suborbital graph.*

Proof: An edge of Γ differs in exactly one coordinate. For three vertices to form a triangle, all three connecting edges must involve the same coordinate; otherwise two vertices would differ in more than one coordinate and could not be adjacent.

Therefore each triangle of Γ comes from a triangle in one coordinate of the base graph.

□

Example 5: *Let*

$G = S_5$.

The base graph is

K_5.

Every triangle in the wreath product graph changes only one coordinate.
Take

$W = S_5 \wr S_3.$

For the vertex

$u = (1, 1, 1),$

a triangle can be formed by changing the first, second or third coordinate.
Hence every triangle is inherited from the base graph.
Therefore Lemma 3 holds.

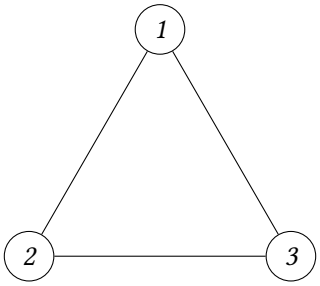


Figure 4: A triangle in the base graph K_5.

Theorem 2 (Triangle Number of Wreath Product Suborbital Graph)**:** *Let*

$G \in \{A_m, C_m, D_{2m}, S_m\}$

be a transitive permutation group acting on a finite set X, and let

$W = G \wr S_n$

act on

$\Omega = X^n$

in the product action. Let Γ be the first non-trivial suborbital graph of this action.
Suppose that the base suborbital graph of G has triangle number T_Δ, where T_Δ denotes the number of triangles containing any fixed vertex.
Then every vertex of Γ lies in exactly

$T_\Gamma = nT_\Delta$

triangles.

Proof: Let

$u = (u_1, u_2, \dots, u_n)$

be a vertex of Γ.

An edge in the wreath product suborbital graph is obtained by changing exactly one coordinate according to an edge in the corresponding base suborbital graph. Therefore, a triangle containing u can only be formed by changing the same coordinate throughout the triangle.

Fix a coordinate i. The possible triangles obtained by changing only the i-th coordinate correspond exactly to the triangles containing u_i in the base graph.

By assumption, the base graph contains

T_Δ

triangles through every vertex. Hence, the i-th coordinate contributes

T_Δ

triangles.

Since there are n independent coordinates, the total number of triangles containing u is

$$T_\Gamma = \underbrace{T_\Delta + T_\Delta + \cdots + T_\Delta}_{n \text{ times}}.$$

Therefore,

$$\boxed{T_\Gamma = nT_\Delta}.$$

Hence the theorem follows. □

Example 6: *let*

$G = S_3$.

The base graph is K_3.
Every vertex belongs to exactly one triangle.
Hence

$T_\Delta = 1$.

Consider

$S_3 \wr S_2$.

Take

$u = (1, 1)$.

Changing only the first coordinate gives

$(1, 1), (2, 1), (3, 1),$

which form a triangle.
Changing only the second coordinate gives

$(1, 1), (1, 2), (1, 3),$

which form another triangle.
Therefore

$T_\Gamma = 2$.

This agrees with Theorem 2:

$T_\Gamma = nT_\Delta = 2(1) = 2$.

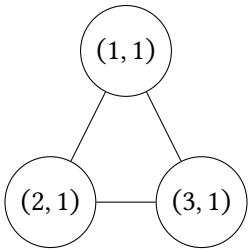


Figure: Triangle obtained from the first coordinate.

Example 7: *Let*

$G = S_8$.

The base graph is

K_8.

The number of triangles through a vertex is

$$T_\Delta = \binom{7}{2} = 21.$$

Consider

$S_8 \wr S_4$.

Theorem 2 gives

$T_\Gamma = nT_\Delta$.

Therefore,

$T_\Gamma = 4(21) = 84$.

Hence each vertex lies in exactly

84

triangles.

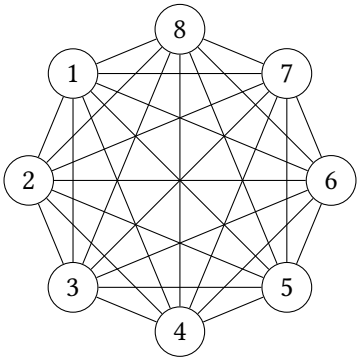


Figure 5: Complete graph K_8.

Definition 4 (Adjacent Common-Neighbour Number)**:** *Let* Γ *be a graph.*

For two adjacent vertices $u, v \in V(\Gamma)$*, define*

$$\lambda_\Gamma(u,v) = |N(u) \cap N(v)|.$$

If this number is the same for every adjacent pair of vertices, then Γ *is said to have adjacent common-neighbour number*

λ_Γ.

Lemma 4 (Common Neighbours in Wreath Product Graphs)**:** *Let* u *and* v *be adjacent vertices of the wreath product suborbital graph* Γ.

Assume they differ only in coordinate j*. Then every common neighbour of* u *and* v *is determined by a common neighbour of* u_j *and* v_j *in the base graph.*

Proof: Since u and v differ only in coordinate j, a vertex w adjacent to both must satisfy

$$w_i = u_i = v_i, \qquad i \neq j.$$

The only possible variation occurs in the j-th coordinate.

Hence w_j must be a common neighbour of u_j and v_j in the base suborbital graph.

Therefore the number of common neighbours in Γ is exactly the number of common neighbours in the base graph.

□

Example 8: *Let*

$G = S_6$.

The base graph is

K_6.

Take adjacent vertices

1, 2.

Their common neighbours are

3, 4, 5, 6.

Hence

$\lambda_\Delta = 4$.

Therefore, in the wreath product graph,

$\lambda_\Gamma = 4$.

This verifies Lemma 4.

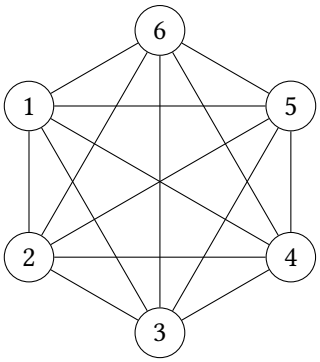


Figure 6: Common neighbours in K_6.

Theorem 3 (Common Neighbours of Wreath Product Suborbital Graph)**:** *Let*

$G \in \{A_m, C_m, D_{2m}, S_m\}$

be a transitive permutation group acting on a finite set X, and let

$W = G \wr S_n$

act on

$\Omega = X^n$

in the product action. Let Γ be the first non-trivial suborbital graph arising from this action. Assume that the base suborbital graph of G has adjacent common-neighbour number λ_Δ. Then for every pair of adjacent vertices in Γ, the number of common neighbours is

$\lambda_\Gamma = \lambda_\Delta$.

Proof: Let

$u = (u_1, u_2, \dots, u_n)$

and

$v = (v_1, v_2, \dots, v_n)$

be two adjacent vertices of Γ.

Since adjacency in the wreath product suborbital graph is inherited from the base graph, the vertices u and v differ in exactly one coordinate. Assume that this occurs in coordinate j; that is,

$$u_i = v_i \qquad \text{for all } i \neq j,$$

while

$$\{u_j, v_j\}$$

is an edge of the base suborbital graph.

Let

$$w = (w_1, w_2, \dots, w_n)$$

be a common neighbour of u and v.

For every coordinate $i \neq j$, the vertex w must satisfy

$$w_i = u_i = v_i,$$

because no change is possible in those coordinates.

Therefore, the only freedom occurs in the j-th coordinate. In this coordinate, w_j must be a common neighbour of u_j and v_j in the base graph.

By the definition of λ_Δ, there are exactly

$$\lambda_\Delta$$

possible choices for w_j.

Hence the number of common neighbours of u and v in Γ is

$$\lambda_\Gamma = \lambda_\Delta.$$

Therefore,

$$\boxed{\lambda_\Gamma = \lambda_\Delta}.$$

□

Example 9: *Let*

$G = S_4$.

Its first suborbital graph is

K_4.

Choose adjacent vertices

1, 2.

The common neighbours are

3, 4.

Hence

$\lambda_\Delta = 2$.

Now consider

$S_4 \wr S_2$.

Take

$u = (1, 1), \qquad v = (2, 1).$

The vertices differ only in the first coordinate.
A common neighbour must therefore be of the form

$(x, 1)$

where x is a common neighbour of 1 and 2 in K_4.
Thus the common neighbours are

$$(3,1), \qquad (4,1).$$

Therefore

$$\lambda_\Gamma = 2.$$

This confirms Theorem 3.3:

$$\lambda_\Gamma = \lambda_\Delta = 2.$$

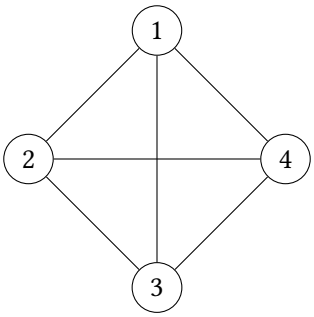


Figure: K_4 with common neighbours of 1 and 2 equal to 3 and 4.

Example 10: *Let*

$$G = S_{10}.$$

The base graph is

$$K_{10}.$$

For adjacent vertices,

$$\lambda_\Delta = 10 - 2 = 8.$$

Now consider

$S_{10} \wr S_5$.

By Theorem 3,

$\lambda_\Gamma = \lambda_\Delta$.

Hence,

$\boxed{\lambda_\Gamma = 8}$.

Therefore every adjacent pair in the wreath product graph has exactly eight common neighbours.

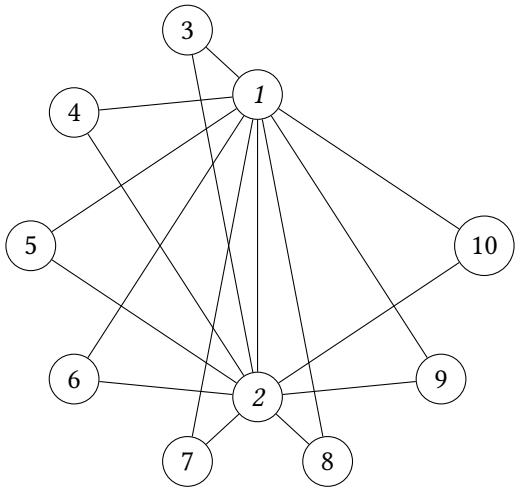


Figure 7: Common-neighbour structure of K_{10}.

Theorem 4 (Degree and size of the wreath-product graph)**:** *Let Δ be a k_Δ-regular graph on m vertices, and let $\Gamma = \Delta^{\square n}$. Then Γ is nk_Δ-regular. Moreover,*

$$|V(\Gamma)| = m^n$$

and

$$|E(\Gamma)| = n|E(\Delta)|m^{n-1}.$$

In particular, if Δ is k_Δ-regular, then

$$|E(\Gamma)| = \frac{nk_\Delta m^n}{2}.$$

Proof: By the definition of the Cartesian product,

$V(\Gamma) = X^n$.

Consequently,

$|V(\Gamma)| = |X|^n = m^n$.

Let

$u = (u_1, u_2, \dots, u_n) \in V(\Gamma)$.

A neighbour of u is obtained by selecting exactly one coordinate i and replacing u_i by a neighbour of u_i in Δ, while keeping all other coordinates fixed.

Since Δ is k_Δ-regular, there are k_Δ possible changes in each coordinate. The coordinate choices are disjoint, and therefore

$$\deg_\Gamma(u) = \sum_{i=1}^{n} k_\Delta = nk_\Delta.$$

Thus Γ is nk_Δ-regular.

For each coordinate, an edge of Δ can be selected in $|E(\Delta)|$ ways, while the remaining $n-1$ coordinates can be chosen in m^{n-1} ways. Since there are n possible coordinates in which an edge can occur, we obtain

$|E(\Gamma)| = n|E(\Delta)|m^{n-1}$.

Finally, by the handshaking lemma,

$2|E(\Gamma)| = |V(\Gamma)|nk_\Delta$.

Hence

$$|E(\Gamma)| = \frac{nk_\Delta m^n}{2}.$$

□

Theorem 5 (Connectivity of wreath-product suborbital graphs)**:** *Let Δ be a finite graph with at least one vertex, and let $\Gamma = \Delta^{\square n}$, where $n \geq 1$. Then*

Γ *is connected* $\iff$ Δ *is connected.*

Proof: Suppose first that Δ is connected. Let

$$u = (u_1, \dots, u_n), \qquad v = (v_1, \dots, v_n)$$

be arbitrary vertices of Γ.

Since Δ is connected, for every i there exists a path in Δ joining u_i to v_i. Changing the coordinates successively along these paths produces a path in Γ from u to v. Hence Γ is connected.

Conversely, suppose that Δ is disconnected. Then there exist vertices $x, y \in X$ belonging to different connected components of Δ. Fix arbitrary elements $a_2, \dots, a_n \in X$ and put

$$u = (x, a_2, \dots, a_n), \qquad v = (y, a_2, \dots, a_n).$$

Every edge of Γ changes exactly one coordinate along an edge of Δ. Consequently, a path in Γ from u to v would induce a path in Δ from x to y, which is impossible.

Therefore Γ is disconnected. This establishes the equivalence. □

Theorem 6 (Clique number of the Cartesian power)**:** *Let Δ be a finite simple graph and let*

$$\Gamma = \Delta^{\square n}, \qquad n \geq 1.$$

Then

$$\boxed{\omega(\Gamma) = \omega(\Delta).}$$

Proof: Let $u = (u_1, \dots, u_n)$ be a vertex of Γ, and let C be a clique containing u.

For every $v \in C \setminus \{u\}$, the vertices u and v are adjacent. Thus they differ in exactly one coordinate.

Suppose that there exist $v, w \in C \setminus \{u\}$ such that v differs from u in coordinate i, whereas w differs from u in coordinate j, with $i \neq j$. Then v and w differ in two coordinates and therefore cannot be adjacent in the Cartesian product. This contradicts the assumption that C is a clique.

Consequently, all vertices of C differ from u in one and the same coordinate. Hence C is contained in a coordinate copy of Δ, and

$$|C| \leq \omega(\Delta).$$

Therefore

$$\omega(\Gamma) \leq \omega(\Delta).$$

On the other hand, a maximum clique of Δ can be embedded in any one coordinate of Γ by fixing the remaining $n - 1$ coordinates. This gives

$$\omega(\Gamma) \geq \omega(\Delta).$$

Combining the inequalities yields

$$\omega(\Gamma) = \omega(\Delta).$$

□

Theorem 7 (Exact number of four-cycles)**:** *Let Δ be a finite simple graph on m vertices and let $\Gamma = \Delta^{\square n}$, where $n \geq 2$. Then the number of distinct unoriented four-cycles in Γ is*

$$\boxed{c_4(\Gamma) = nm^{n-1}c_4(\Delta) + \binom{n}{2} m^{n-2}|E(\Delta)|^2.}$$

Proof: Every four-cycle in Γ is of one of two types.

First, all four edges may change the same coordinate. For each of the n coordinate positions, there are $c_4(\Delta)$ possible four-cycles in that coordinate. The remaining $n-1$ coordinates can be fixed in m^{n-1} ways. Therefore, the number of four-cycles of this type is

$$nm^{n-1}c_4(\Delta).$$

Second, the four edges may alternate between two distinct coordinate positions. There are

$$\binom{n}{2}$$

ways to choose these two coordinates. For each selected coordinate, an edge of Δ can be chosen in $|E(\Delta)|$ ways. The remaining $n-2$ coordinates can be fixed in m^{n-2} ways.

Each such pair of edges produces exactly one unoriented four-cycle, forming a Cartesian rectangle. Hence the number of four-cycles of this type is

$$\binom{n}{2} m^{n-2}|E(\Delta)|^2.$$

These two types are disjoint and exhaust all four-cycles of the Cartesian product, because a closed walk of length four can use either one coordinate or two coordinates, with the latter necessarily alternating.

Adding the two contributions gives

$$c_4(\Gamma) = nm^{n-1}c_4(\Delta) + \binom{n}{2} m^{n-2}|E(\Delta)|^2.$$

This completes the proof. □

Theorem 8 (Spectral characterization)**:** *Let the adjacency eigenvalues of* Δ *be*

$\lambda_1, \lambda_2, \ldots, \lambda_m$.

Then the adjacency eigenvalues of Γ *are precisely*

$$\boxed{\Lambda_{i_1,\ldots,i_n} = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_n},}$$

where

$1 \leq i_1, i_2, \ldots, i_n \leq m$.

The eigenvalues are counted with their algebraic multiplicities.

Proof: Let A be the adjacency matrix of Δ. By the definition of the Cartesian product, the adjacency matrix of Γ is

$$A_\Gamma = \sum_{j=1}^{n} I_m^{\otimes(j-1)} \otimes A \otimes I_m^{\otimes(n-j)}.$$

Let v_i be an eigenvector of A corresponding to λ_i. Thus

$$Av_i = \lambda_i v_i.$$

Consider the tensor product

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}.$$

Applying A_Γ gives

$$\begin{aligned} &A_\Gamma(v_{i_1} \otimes \cdots \otimes v_{i_n}) \\ &\quad = \sum_{j=1}^{n} v_{i_1} \otimes \cdots \otimes Av_{i_j} \otimes \cdots \otimes v_{i_n} \\ &\quad = \left(\sum_{j=1}^{n} \lambda_{i_j} \right) (v_{i_1} \otimes \cdots \otimes v_{i_n}). \end{aligned}$$

The tensor products of the eigenvectors form a basis of the product space. Therefore, all adjacency eigenvalues of Γ are given by the stated sums. □

Corollary 1 (Spectral radius)**:** *If* Δ *is connected and* k*-regular, then*

$$\boxed{\rho(\Gamma) = nk.}$$

Proof: The largest adjacency eigenvalue of Δ is k. By the spectral characterization, the largest eigenvalue of Γ is

$$k + k + \cdots + k = nk.$$

□

Let

$$L(\Delta) = kI_m - A$$

be the Laplacian matrix of Δ.

Theorem 9 (Laplacian eigenvalue decomposition)**:** *Suppose that the Laplacian eigenvalues of* Δ *are*

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_m.$$

Then the Laplacian eigenvalues of Γ *are*

$$\boxed{\mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_n},}$$

where $1 \leq i_j \leq m$.

Proof: Since Γ is nk-regular,

$$L(\Gamma) = nkI_{m^n} - A_\Gamma.$$

Substituting the adjacency matrix representation gives

$$L(\Gamma) = \sum_{j=1}^{n} I_m^{\otimes(j-1)} \otimes L(\Delta) \otimes I_m^{\otimes(n-j)}.$$

The tensor-product eigenvectors of the base Laplacian are consequently eigenvectors of $L(\Gamma)$, with eigenvalues equal to the sums of the corresponding base eigenvalues. □

Theorem 10 (Algebraic connectivity invariance)**:** *Let Δ be connected. If $\mu_2(\Delta)$ denotes its second-smallest Laplacian eigenvalue, then*

$$\boxed{\mu_2(\Gamma) = \mu_2(\Delta).}$$

Consequently, the algebraic connectivity of the wreath-product suborbital graph is independent of the number of coordinate factors.

Proof: Every Laplacian eigenvalue of Γ is a sum of n nonnegative Laplacian eigenvalues of Δ.

The smallest positive such sum is obtained by choosing μ_2 in one coordinate and zero in all other coordinates. Therefore,

$$\mu_2(\Gamma) = \mu_2(\Delta).$$

□

Let $\tau(H)$ denote the number of spanning trees of a connected graph H.

Theorem 11 (Spanning-tree formula)**:** *Let the Laplacian eigenvalues of Δ be*

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_m.$$

Then the number of spanning trees of $\Gamma = \Delta^{\square n}$ is

$$\boxed{\tau(\Gamma) = \frac{1}{m^n} \prod_{\substack{1 \leq i_1, \ldots, i_n \leq m \\ (i_1, \ldots, i_n) \neq (1, \ldots, 1)}} \left(\sum_{j=1}^{n} \mu_{i_j} \right).}$$

Proof: By the Laplacian spectral theorem established above, the eigenvalues of $L(\Gamma)$ are

$$\sum_{j=1}^{n} \mu_{i_j}.$$

Since Δ is connected, Γ is connected and has m^n vertices.

The only zero Laplacian eigenvalue corresponds to the tuple $(1, \ldots, 1)$. By the Matrix-Tree Theorem, the number of spanning trees equals the product of the nonzero Laplacian eigenvalues divided by the number of vertices.

Therefore,

$$\tau(\Gamma) = \frac{1}{m^n} \prod_{(i_1, \ldots, i_n) \neq (1, \ldots, 1)} \left(\sum_{j=1}^{n} \mu_{i_j} \right).$$

□

The Wiener index of a connected graph H is defined by

$$W(H) = \sum_{\{u,v\}\subseteq V(H)} d_H(u,v).$$

Theorem 12 (Wiener index of wreath-product graphs)**:** *Let Δ be a connected graph on m vertices. Then*

$$\boxed{W(\Gamma) = nm^{2n-2}W(\Delta).}$$

Proof: By the coordinate distance formula,

$$d_\Gamma(u,v) = \sum_{i=1}^{n} d_\Delta(u_i,v_i).$$

Consequently,

$$W(\Gamma) = \sum_{\{u,v\}\subseteq X^n} \sum_{i=1}^{n} d_\Delta(u_i,v_i).$$

Interchanging the order of summation,

$$W(\Gamma) = \sum_{i=1}^{n} \sum_{\{u,v\}\subseteq X^n} d_\Delta(u_i,v_i).$$

For each fixed coordinate, every unordered pair $\{x,y\} \subseteq X$ occurs in exactly m^{2n-2} unordered pairs of product vertices with those entries in that coordinate. Hence

$$\sum_{\{u,v\}\subseteq X^n} d_\Delta(u_i,v_i) = m^{2n-2}W(\Delta).$$

There are n coordinates. Thus

$$W(\Gamma) = nm^{2n-2}W(\Delta).$$

□

Corollary 2 (Mean distance)**:** *The average distance between two distinct vertices of Γ is*

$$\boxed{\overline{d}(\Gamma) = \frac{2nW(\Delta)}{m^2(m^n-1)}.}$$

Proof: Since Γ has m^n vertices, the number of unordered pairs of distinct vertices is

$$\binom{m^n}{2}.$$

Therefore,

$$\overline{d}(\Gamma) = \frac{W(\Gamma)}{\binom{m^n}{2}}.$$

Substituting the Wiener index formula gives

$$\overline{d}(\Gamma) = \frac{nm^{2n-2}W(\Delta)}{\frac{m^n(m^n-1)}{2}},$$

which simplifies to the required expression. □

Theorem 13 (Bipartiteness of the Cartesian power)**:** *Let Δ be a finite simple graph and let $n \geq 1$. Then*

$$\boxed{\Gamma = \Delta^{\square n} \text{ is bipartite} \iff \Delta \text{ is bipartite.}}$$

Proof: Suppose that Δ is bipartite with bipartition

$$X = X_0 \cup X_1.$$

Define a partition of X^n by

$$\Omega_0 = \left\{ (x_1, \ldots, x_n) \,:\, \sum_{i=1}^{n} \varepsilon(x_i) \equiv 0 \pmod 2 \right\},$$

and

$$\Omega_1 = \left\{ (x_1, \ldots, x_n) \,:\, \sum_{i=1}^{n} \varepsilon(x_i) \equiv 1 \pmod 2 \right\},$$

where

$$\varepsilon(x) = \begin{cases} 0, & x \in X_0, \\ 1, & x \in X_1. \end{cases}$$

Every edge of Γ changes exactly one coordinate along an edge of Δ. Such a change switches the parity of the coordinate sum. Hence every edge joins a vertex of Ω_0 to a vertex of Ω_1.

Thus Γ is bipartite.

Conversely, fix arbitrary entries in all coordinates except the first. The vertices obtained by varying the first coordinate induce a copy of Δ. If Γ is bipartite, every induced subgraph is bipartite. Hence Δ is bipartite. □

Theorem 14 (Closed-walk enumeration)**:** *Let the adjacency eigenvalues of* Δ *be*

$\lambda_1, \ldots, \lambda_m$.

For every positive integer r*, the total number of closed walks of length* r *in* Γ *is*

$$\boxed{\operatorname{tr}(A_\Gamma^r) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \left(\lambda_{i_1} + \cdots + \lambda_{i_n}\right)^r.}$$

Proof: The number of closed walks of length r in a graph is equal to the trace of the rth power of its adjacency matrix.

By the adjacency spectral characterization, the eigenvalues of A_Γ are

$$\Lambda_{i_1,\ldots,i_n} = \sum_{j=1}^{n} \lambda_{i_j}.$$

The trace of a matrix power is the sum of the corresponding eigenvalue powers. Therefore,

$$\operatorname{tr}(A_\Gamma^r) = \sum_{i_1,\ldots,i_n} \Lambda_{i_1,\ldots,i_n}^r.$$

Substitution yields the required formula. □

3 Conclusions

In this paper, we investigated structural and metric properties of suborbital graphs arising from wreath product actions $G \wr S_n$. We established formulas for distances, diameters, triangle numbers, and adjacent common-neighbour parameters, showing how these properties are determined by the corresponding base suborbital graph. In particular, the distance and diameter depend on the coordinate structure, while triangle numbers scale with the number of coordinates and adjacent common-neighbour numbers are preserved. Future research may extend these results to more general wreath products, investigate higher-order cycles and other graph invariants, and explore spectral and computational properties of the resulting graphs.

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