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Research article

On Some Inequalities For Convex Functions In Non-Newtonian Calculus

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Abstract: Fundamental principles in mathematics encompass differentiation and integration, as well as various modes of addition and subtraction. Notably, Robert Katz and Micheal Grossman introduced a novel calculus framework known as Non-Newtonian calculus between 1967 and 1970. In this innovative approach, multiplication and division replace addition and subtraction respectively. α-arithmetic and geometric fields were employed as well as convex functions. This paper explores and established some inequalities for convex functions in non-Newtonian calculus. The paper further established affine convexity in Non-Newtonian Calculus.

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1 Introduction

Non-Newtonian calculi were developed by Grossman and Katz for the first time between 1967 and 1970 [1], presenting alternatives to Newton's (1643-1727) and Leibniz's (1646-1716) classical calculus. It is stated similarly in [2], that two American mathematicians, Grossman and Katz, originally discovered non-Newtonian calculus in the years between 1967 and 1970 to offer an alternate method to the standard (classical) calculus of Isaac Newton and Gottfried Wilhelm Leibniz.
It offers tools for differentiation and integration based on non-Newtonian operations rather than classical operations [1].
According to [3], multiplicative calculus uses a multiplicative mechanism, where the non-Newtonian derivative of an exponential function is one (1) (constant). Multiplicative derivative of a constant function equals one (1). The foundations of non-newtonian calculus, which modifies the calculi started by Gottfried Wilhelm Leibnitz and Isaac Newton in the 17th century, were laid by Grossman and Katz in 1972.
There exist a consistent relationship between the relevant operators in this calculus and the classical

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calculus. It is thus, recommended utilizing variants of the type indicated in (1) as follows

$$f(.) = p(.).\xi^{\ln q(.)} \tag{1}$$

in non-Newtonian calculus rather than the conventional variations of type (2)

$$f(.) = p(.) + \xi q(.) \tag{2}$$

which Lagrange (1736-1813) established and which is currently used in all current formulations of the calculus of variations.
While classical calculus effectively models phenomena governed by linear and additive rates of change, numerous complex systems in economics, biology, and engineering exhibit exponential or multiplicative behaviors that render standard analytical tools insufficient. Non-Newtonian calculus - encompassing alternative arithmetic structures such as geometric and α-arithmetic calculi - provides a robust paradigm to handle such systems natively without logarithmic transformations. Although foundational elements of Non-Newtonian calculus have been explored, a systematic investigation of convexity inequalities within these generalized fields remains scarce. This study bridges this gap by establishing novel inequalities for convex functions under Non-Newtonian arithmetic, offering enhanced analytical flexibility for modeling multiplicative and nonlinear processes.

2 Materials and methods

Here, we delve into the discussion of concepts such as the α-generator and geometric field, the classical and Non-Newtonian Arithmetic, and Convex functions as well as some definitions.

2.1 α-generator and geometric real field

With reference to [4], and many other literature cited therein, a variety of arithmetic operations and their generators are used in non-Newtonian calculi.
A generator is an injective (one-to-one) function with the domain $\mathbb{R}$ and range $X \subset \mathbb{R}$. Every generator produces precisely an arithmetic, and the reverse also holds. Each arithmetic is generated by one generator [5]. Geometric arithmetic is produced by the exponential function, whereas classical arithmetic is generated by the identity function.
As a generator, the function exp from $\mathbb{R}$ to the set $\mathbb{R}^+$ of positive reals is considered, thus;

$$\begin{gathered}\alpha : \mathbb{R} \to \mathbb{R}^+ \\ \alpha^{-1} : \mathbb{R}^+ \to \mathbb{R} \\ \alpha(m) = e^m = n \text{ and} \\ \alpha^{-1}(n) = \ln n = m.\end{gathered}$$

If any of the following four operations are specified for a bijection $\alpha : M \to N \subseteq \mathbb{R}$, referred to as a generator, then the bijection $\alpha : M \to N \subseteq \mathbb{R}$ defines an arithmetic:

$$\begin{aligned}
&m_\alpha \dot{+} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) + \alpha^{-1}(n_\alpha)],\\
&m_\alpha \dot{-} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) - \alpha^{-1}(n_\alpha)],\\
&m_\alpha \dot{\times} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) \times \alpha^{-1}(n_\alpha)],\\
&m_\alpha \dot{/} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) / \alpha^{-1}(n_\alpha)],\\
&m_\alpha \dot{\leq} n_\alpha \iff \alpha^{-1}(m_\alpha) \leq \alpha^{-1}(n_\alpha).
\end{aligned} \tag{3}$$

Let $I(m_\alpha) = m_\alpha$ for every $m_\alpha \in \mathbb{R}$, then I is referred to as an identity function whose inverse is itself. The identity function generates classical arithmetic and the exponential function generates geometric arithmetic.

2.2 Classical Arithmetic

Choosing α-generator I, the identity function $\alpha(m_\alpha) = m_\alpha$ for every $m_\alpha \in \mathbb{R} \implies \alpha^{-1}(m_\alpha) = m_\alpha$, then from (3), it is obvious α-arithmetic turns out to be classical(standard) arithmetic from which the traditional calculus is developed.

$$\begin{aligned}
&m_\alpha \dot{+} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) + \alpha^{-1}(n_\alpha)] = \alpha[m_\alpha + n_\alpha] = m_\alpha + n_\alpha,\\
&m_\alpha \dot{-} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) - \alpha^{-1}(n_\alpha)] = \alpha[m_\alpha - n_\alpha] = m_\alpha - n_\alpha,\\
&m_\alpha \dot{\times} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) \times \alpha^{-1}(n_\alpha)] = \alpha[m_\alpha \times n_\alpha] = m_\alpha \times n_\alpha,\\
&m_\alpha \dot{/} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) \div \alpha^{-1}(n_\alpha)] = \alpha[m_\alpha \div n_\alpha] = \frac{m_\alpha}{n_\alpha}, n_\alpha \neq 0.\\
&m_\alpha \dot{\leq} n_\alpha \iff \alpha^{-1}(m_\alpha) \leq \alpha^{-1}(n_\alpha) \iff m_\alpha \leq n_\alpha.
\end{aligned} \tag{4}$$

For $\alpha(m_\alpha) = I$, the summation is defined as

$$\begin{aligned}
{}_\alpha \sum_{i=1}^{k} m_{\alpha i} &= \alpha\left\{ \sum_{i=1}^{k} \alpha^{-1}(m_{\alpha i}) \right\}\\
&= \alpha\{\alpha^{-1}(m_{\alpha 1}) + ... + \alpha^{-1}(m_{\alpha k})\}
\end{aligned} \tag{5}$$

for all $m_{\alpha i} \in \mathbb{R}^+$.
If we fix α-generator as exp, thus, given the collection of strictly larger real numbers, $\alpha(m_\alpha) = e^{m_\alpha}$, $\alpha^{-1}(m_\alpha) = \ln(m_\alpha)$ and $M \in \mathbb{R}^+$, we obtain one option of the non-Newtonian calculi known as the geometric calculus [6].

2.3 Non-Newtonian Arithmetic

Define $\alpha\,(m_\alpha) = e^{m_\alpha}$ for $m_\alpha \in \mathbb{R}$ as an α-generator then $\alpha^{-1}(m_\alpha) = \ln(m_\alpha)$, and $M \in \mathbb{R}^+$. From (3), we obtain the following operations:

$$
\begin{aligned}
&m_\alpha \dot{+} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) + \alpha^{-1}(n_\alpha)] = e^{(\ln m_\alpha + \ln n_\alpha)} = m_\alpha \cdot n_\alpha \\
&m_\alpha \dot{-} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) - \alpha^{-1}(n_\alpha)] = e^{(\ln m_\alpha - \ln n_\alpha)} = \frac{m_\alpha}{n_\alpha}, n_\alpha \neq 0 \\
&m_\alpha \dot{\times} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) \times \alpha^{-1}(n_\alpha)] = e^{(\ln m_\alpha \times \ln n_\alpha)} = m_\alpha^{\ln n_\alpha} = n_\alpha^{\ln m_\alpha} \\
&m_\alpha \dot{/} n_\alpha = \alpha[\alpha^{-1}(m_\alpha) / \alpha^{-1}(n_\alpha)] = e^{(\ln m_\alpha \div \ln n_\alpha)} = m_\alpha^{\frac{1}{\ln n_\alpha}}, n_\alpha \neq 1. \\
&m_\alpha \dot{\leq} n_\alpha \iff \alpha^{-1}(m_\alpha) \leq \alpha^{-1}(n_\alpha) \iff \ln(m_\alpha) \leq \ln(n_\alpha).
\end{aligned}
\tag{6}
$$

The set of geometric integers $\mathbb{Z}(G)$, real numbers $\mathbb{R}(G)$, and complex numbers, $\mathbb{C}(G)$ according to [4] are defined respectively as follows:

$$
\begin{aligned}
&\mathbb{Z}(G) = e^k \,:\, k \in \mathbb{Z} \\
&\mathbb{R}(G) = e^k \,:\, k \in \mathbb{R} = (0, \infty) \\
&\mathbb{C}(G) = e^q \,:\, q \in \mathbb{C} = \mathbb{C}\backslash\{0\}.
\end{aligned}
$$

The geometric positive real numbers and geometric negative real numbers are each defined as

$$
\begin{aligned}
&\mathbb{R}^+(G) = \{k \in \mathbb{R}(G) \,:\, k > 1\} \\
&\mathbb{R}^-(G) = \{k \in \mathbb{R}(G) \,:\, k < 1\}.
\end{aligned}
$$

Definition 1: *Let $\alpha(q) = \dot{q}\ \forall\ q \in \mathbb{Z}$. If for $s \in \mathbb{R}_\alpha$, $s \dot{+} \dot{0} = s$ and $s \dot{\times} \dot{1} = s$, then by α-arithmetic, $\alpha(0)$ and $\alpha(1)$ numbers are respectively called identity and unit elements.*

Definition 2: *Let $\dot{-}\dot{l} = \dot{0} \dot{-} \dot{l} = \alpha(-l)\ \forall\ l \in \mathbb{Z}$. Set of α-integers is defined and denoted by $\mathbb{Z}_\alpha$ as*

$$\mathbb{Z}_\alpha = \{..., \dot{-2}, \dot{-1}, \dot{0}, \dot{1}, \dot{2}, ...\} = \{..., \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), ...\}.$$

$$\mathbb{Z}_\alpha = \{\dot{l} \,:\, \dot{l} = \alpha(l), l \in \mathbb{Z}\}.$$

Definition 3: *For any $x \in \mathbb{R}_\alpha$, α-square is $x_k^2 = x_k \dot{\times} x_k$, and is defined as*

$$x_k^2 = x_k \dot{\times} x_k = \alpha\{[\alpha^{-1}(x_k)]^2\}$$

We define the function $q_p : \mathbb{R} \to \mathbb{R}_q \subseteq \mathbb{R}$ and it's inverse q_p^{-1} as

$$q_p(x_\propto) = \begin{cases} x_\propto^{\frac{1}{p}}, & x_\propto > 0 \\ 0, & x_\propto = 0 \\ -(-x_\propto)^{\frac{1}{p}}, & x_\propto < 0 \end{cases} \tag{7}$$

And

$$q_p^{-1}(x_\propto) = \begin{cases} x_\propto^{p}, & x_\propto > 0 \\ 0, & x_\propto = 0 \\ -(-x_\propto)^{p}, & x_\propto < 0 \end{cases} \tag{8}$$

For $p \in \mathbb{R}\backslash\{0\}$. In the unique cases for $p = 2$ and $p = -1$, q_p-arithmetic is respectively quadratic and harmonic arithmetic.

2.4 Non-Newtonian Algebraic Curves

The distance between two real numbers x_k and x_q is defined in the non-Newtonian calculus by

$$\begin{aligned} [[x_k \dot{-} x_q]] &= \alpha[\alpha^{-1}(x_k) - \alpha^{-1}(x_q)] \\ &= \alpha[\alpha^{-1}(x_q) - \alpha^{-1}(x_k)] \\ &= [[x_q \dot{-} x_k]] \\ &= \frac{x_k}{x_q}, x_q \neq 0 \\ &= \frac{x_q}{x_k}, x_k \neq 0. \end{aligned} \tag{9}$$

In the classical calculus, the distance between two points x_k and x_q is defined as

$$\begin{aligned} |x_q \dot{-} x_k| &= \alpha[\alpha^{-1}(x_q) - \alpha^{-1}(x_k)] \\ &= \alpha[\alpha^{-1}(x_k) - \alpha^{-1}(x_q)] \\ &= |x_k \dot{-} x_q| \\ &= |x_k - x_q|. \end{aligned} \tag{10}$$

Given the coordinates of two points $A(x_k, y_k)$ and $B(x_q, y_q)$, then the non-Newtonian slope of the line segment denoted by ${}^*m$ is defined as

$$\begin{aligned} {}^*m &= (y_q \dot{-} y_k) \dot{/} (x_q \dot{-} x_k) \\ &= \left(\frac{y_q}{y_k}\right) \dot{/} \left(\frac{x_q}{x_k}\right) \\ &= \left(\frac{y_q}{y_k}\right)^{\frac{1}{\ln\left(\frac{x_q}{x_k}\right)}}. \end{aligned} \tag{11}$$

The equation of a straight line can be described as

$$\begin{aligned} y &=^* m \dot{\times} x \dot{+} c \\ &=^* m^{\ln x} \dot{+} c \\ &=^* m^{lnx} c. \end{aligned} \tag{12}$$

2.5 Convex functions

α-convex set: Let L_α be an α-linear space and $K \subseteq L_\alpha$. A set is α-convex if for every $r, s \in K$,

$$X_\alpha = \{z = \theta_1 r + \theta_2 s, 0 \leq \theta_1, \theta_2 \leq 1\} \subseteq K.$$

Let I be an interval in $\mathbb{R}$. Then a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval I if

$$f(\varkappa x_i + (1 - \varkappa) x_j) \leq \varkappa f(x_i) + (1 - \varkappa) f(x_j) \tag{13}$$

holds for all $x_i, x_j \in I$ and $\varkappa \in [0, 1]$. f is strictly convex if the inequality is strict, and concave if the inequality is inverted. Let I_α be an interval in $\mathbb{R}_\alpha$. Then $f : I_\alpha \to \mathbb{R}_\alpha$ is said to define non-Newtonian convexity if

$$f\left(\tau_1 \dot{\times} x \dot{+} \tau_2 \dot{\times} y\right) \dot{\leq} \tau_1 \dot{\times} f(x) \dot{+} \tau_2 \dot{\times} f(y) \tag{14}$$

for all $\tau_1, \tau_2 \in [1, e]$ [7].
If a set $\mathcal{A} \subseteq \mathcal{X}$ contains all of the combinations of points $a_1, a_2 \in \mathcal{A}$ and coefficients $\zeta, \xi \in \mathbb{R}$ such that the sum $\zeta + \xi = 1$, then it is said to be affine.
If h is affine function defined for $h : \mathcal{A} \to \mathbb{R}$, then the equality

$$h(\zeta a_1 + \xi a_2) = \zeta h(a_1) + \xi h(a_2) \tag{15}$$

holds for all binomial affine combinations $\zeta a_1 + \xi a_2$ of points $a_1, a_2 \in \mathcal{A}$.

3 Results

The results for this study is now discussed here. The section focused on applying non-Newtonian calculus to some inequalities for convex functions, and also established affine convexity in non-Newtonian calculus.

3.1 Sum of numbers in Non-Newtonian Calculus

We begin this section by first expressing the summation $\sum_{i=1}^{k} m_i$ in equation (5) for some $m_i \in \mathbb{R}^+$ in terms of non-Newtonian calculus as follows:

$$\begin{aligned}\sum_{i=1}^{k} m_i &\equiv \alpha\left[\sum_{i=1}^{k} \alpha^{-1}(m_i)\right] \\ &\equiv \alpha[\alpha^{-1}(m_1) + \alpha^{-1}(m_2) + ... + \alpha^{-1}(m_k)] \\ &\equiv \alpha[\ln m_1 + \ln m_2 + ... + \ln m_k] \\ &\equiv e^{[\ln m_1 + \ln m_2 + ... + \ln m_k]} \\ &\equiv m_1.m_2...m_k \\ &\equiv \prod_{i=1}^{k} m_i.\end{aligned} \tag{16}$$

3.2 Some Inequalities in non-Newtonian Calculus

Lemma 1: *Let I be an interval in $\mathbb{R}^+$ and $h : I \to \mathbb{R}^+$ be a convex function. For $m, n \in I$, the inequality*

$$\frac{h(m)^{\ln \mu}}{h(n)^{\ln \mu}} \leq \left(\frac{h(m)}{h(n)}\right)^{\ln \mu} \tag{17}$$

holds for all $\mu \in [1, e]$.

Proof: By applying the non-Newtonian arithmetic, we have

$$\frac{h(m)^{\ln \mu}}{h(n)^{\ln \mu}} = h(\mu \dot{\times} m \dot{+} (1 \dot{-} \mu) \dot{\times} n) \tag{18}$$

By convexity, we obtain

$$\frac{h(m)^{\ln \mu}}{h(n)^{\ln \mu}} \leq \mu \dot{\times} h(m) \dot{+} (1 \dot{-} \mu) \dot{\times} h(n) \tag{19}$$

Applying (6), we obtain the following;

$$\begin{aligned}\frac{h(m)^{\ln\mu}}{h(n)^{\ln\mu}} &\leq \alpha\left[\alpha^{-1}(\mu)\times\alpha^{-1}(h(m))-\alpha^{-1}(\mu)\times\alpha^{-1}(h(n))\right]\\ &\leq \alpha[\ln\mu\ln h(m)-\ln\mu\ln h(n)]\\ &\leq e^{[\ln\mu\ln h(m)-\ln\mu\ln h(n)]}\\ &\leq e^{[\ln h(m)-\ln h(n)]\ln\mu}\\ &\leq \left(e^{[\ln h(m)-\ln h(n)]}\right)^{\ln\mu}\\ &\leq \left(\frac{e^{\ln h(m)}}{e^{\ln h(n)}}\right)^{\ln\mu}\end{aligned} \tag{20}$$

Thus

$$\frac{h(m)^{\ln\mu}}{h(n)^{\ln\mu}} \leq \left(\frac{h(m)}{h(n)}\right)^{\ln\mu} \tag{21}$$

As required. □

Lemma 2: *Let I be an interval in $\mathbb{R}^+$ and $h : I \to \mathbb{R}^+$ be a convex function. Then*

$$\left[h\left(\frac{m}{n}\right)^{\ln\mu}\right] \leq \left(\frac{h(m)}{h(n)}\right)^{\ln\mu} \tag{22}$$

holds for all $m, n \in I$ and $\mu \in [1, e]$.

Proof: Since h is convex and $m \dot{\leq} n \iff \ln(m) \leq \ln(n)$, then

$$\begin{aligned}\left[h\left(\frac{m}{n}\right)^{\ln\mu}\right] &= h(\mu\dot{\times}m\dot{+}(1\dot{-}\mu)\dot{\times}n)\\ &\leq \mu\dot{\times}h(m)\dot{+}(1\dot{-}\mu)\dot{\times}h(n)\end{aligned} \tag{23}$$

Applying equation (6) yields

$$\begin{aligned}\left[h\left(\frac{m}{n}\right)^{\ln\mu}\right] &\leq e^{[\ln\mu\ln h(m)+(\ln 1-\ln\mu)\ln h(n)]}\\ &\leq e^{[\ln\mu\ln h(m)-\ln\mu\ln h(n)]}\\ &\leq \frac{\left(e^{\ln h(m)}\right)^{\ln\mu}}{\left(e^{\ln h(n)}\right)^{\ln\mu}}\\ &\leq \frac{h(m)^{\ln\mu}}{h(n)^{\ln\mu}}\end{aligned} \tag{24}$$

It then follows from inequality (22) that

$$\left[h\left(\frac{m}{n}\right)^{\ln\mu}\right] \leq \left(\frac{h(m)}{h(n)}\right)^{\ln\mu}. \tag{25}$$

□

Theorem 1: *Let h be a convex function on an interval $I \subset \mathbb{R}^+$. Then*

$$h\left(\prod_{q=1}^{n} x_q^{\ln w_q}\right) \leq \prod_{q=1}^{n} h(x_q)^{\ln w_q} \tag{26}$$

where $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ with $x_q \in I$, and w_q $(1 \leq q \leq n)$ are positive weights associated with x_q such that $\sum_{q=1}^{n} w_q = 1$.

Proof: Since h is convex, then

$$\begin{aligned}h\left(\prod_{q=1}^{n} x_q^{\ln w_q}\right) &= h\left(x_1^{\ln w_1}.x_2^{\ln w_2}...x_n^{\ln w_n}\right)\\ &= h\left(\sum_{q=1}^{n} x_q \dot{\times} w_q\right)\end{aligned} \tag{27}$$

Applying (16) gives

$$h\left(\prod_{q=1}^{n} x_q^{\ln w_q}\right) \leq w_q \dot{\times} \sum_{q=1}^{n} h(x_q) \tag{28}$$

and therefore applying (6) yields

$$h\left(\prod_{q=1}^{n} x_q^{\ln w_q}\right) \leq \prod_{q=1}^{n} h(x_q)^{\ln w_q}. \tag{29}$$

□

Theorem 2: *Let h be a convex function on an interval $I \subset \mathbb{R}^+$. For $x_1, x_k, x_n \in I$ with $1 \leq k \leq n$, the following inequality holds:*

$$h\left(\frac{x_1 x_n}{x_k}\right) \leq \frac{h(x_1)h(x_n)}{h(x_k)} \tag{30}$$

for $1 \leq \mu \leq e$.

Proof: Let $y_k = \frac{x_1 x_n}{x_k}$. Then $x_1 x_n = x_k y_k$, there exist μ such that

$$x_k = \left(\frac{x_1}{x_n}\right)^{\ln \mu}, \tag{31}$$

and

$$y_k = \left(\frac{x_n}{x_1}\right)^{\ln \mu}. \tag{32}$$

By applying (22) yields

$$h(y_k) \leq \left(\frac{h(x_n)}{h(x_1)}\right)^{\ln \mu} \tag{33}$$

Also,

$$\begin{aligned} y_k &= \frac{x_1 x_n}{x_k} \\ &= \frac{x_1 x_n}{\left(\frac{x_1}{x_n}\right)^{\ln \mu}} \end{aligned} \tag{34}$$

implies that

$$\begin{aligned} h(y_k) &\leq \frac{h(x_1)h(x_n)}{h\left(\frac{x_1}{x_n}\right)^{\ln \mu}} \\ &\leq \frac{h(x_1)h(x_n)}{h(x_k)}. \end{aligned} \tag{35}$$

Hence replacing the L.H.S of (35) by equation (34) yields

$$h\left(\frac{x_1x_n}{x_k}\right) \leq \frac{h(x_1)h(x_n)}{h(x_k)}. \tag{36}$$

As required. □

Remark 1: *Equality holds in* (30) *when* $k = 1$ *or* $k = n$.

Theorem 3: *Let* h *be a convex function on an interval* $I \subset \mathbb{R}^+$. *For* $x_k \in I$ $(1 \leq k \leq n)$ *with* $x_1 \leq \cdots \leq x_n$, *and positive weights* w_k *such that* $\sum_{k=1}^{n} w_k = 1$, *the following inequality holds:*

$$h\left(\frac{x_1x_n}{\prod_{k=1}^{n} x_k^{\ln w_k}}\right) \leq \frac{h(x_1)h(x_n)}{\prod_{k=1}^{n} h(x_k)^{\ln w_k}}. \tag{37}$$

Proof:

$$h\left(\frac{x_1x_n}{\prod_{k=1}^{n} x_k^{\ln w_k}}\right) = h\left(x_1 \dot{+} x_n \dot{-} \sum_{k=1}^{n} w_k \dot{\times} x_k\right) \tag{38}$$

By convexity, we have

$$h\left(\frac{x_1x_n}{\prod_{k=1}^{n} x_k^{\ln w_k}}\right) \leq h(x_1) \dot{+} h(x_n) \dot{-} h\left(\sum_{k=1}^{n} w_k \dot{\times} x_k\right) \tag{39}$$

Applying equations (16) and (6) appropriately produces

$$\begin{aligned} h\left(\frac{x_1x_n}{\prod_{k=1}^{n} x_k^{\ln w_k}}\right) &= h\left(\prod_{k=1}^{n}\left(\frac{x_1x_n}{x_k}\right)^{\ln w_k}\right) \\ &\leq \prod_{k=1}^{n} h\left(\frac{x_1x_n}{x_k}\right)^{\ln w_k} \\ &\leq \prod_{k=1}^{n}\left(\frac{h(x_1)h(x_n)}{h(x_k)}\right)^{\ln w_k} \\ &\leq \frac{h(x_1)h(x_n)}{\prod_{k=1}^{n} h(x_k)^{\ln w_k}}. \end{aligned} \tag{40}$$

And hence

$$h\left(\frac{x_1x_n}{\prod_{k=1}^{n} x_k^{\ln w_k}}\right) \leq \frac{h(x_1)h(x_n)}{\prod_{k=1}^{n} h(x_k)^{\ln w_k}}. \tag{41}$$

As required. □

3.3 Affine Convexity

Lemma 3: *Let $f : \mathcal{A} \to \mathbb{R}^+$ be an affine convex function. Then the inequality*

$$f\left(a_1^{\ln \alpha_1} a_2^{\ln \alpha_2}\right) \leq f(a_1)^{\ln \alpha_1} f(a_2)^{\ln \alpha_2} \tag{42}$$

holds for all binomial affine combinations $a_1^{\ln \alpha_1} a_2^{\ln \alpha_2}$ of points $a_1, a_2 \in \mathcal{A}$.

Proof: Since $f : \mathcal{A} \to \mathbb{R}^+$ is affine $\forall\ \alpha_1 a_1 \dot{+} \alpha_2 a_2$ of points $a_1, a_2 \in \mathcal{A}$, then

$$\begin{aligned} f\left(a_1^{\ln \alpha_1} a_2^{\ln \alpha_2}\right) &= f(a_1 \dot{\times} \alpha_1 \dot{+} a_2 \dot{\times} \alpha_2) \\ &= f(\alpha_1 \dot{\times} a_1 \dot{+} \alpha_2 \dot{\times} a_2) \end{aligned} \tag{43}$$

Applying convexity yields

$$f\left(a_1^{\ln \alpha_1} a_2^{\ln \alpha_2}\right) \leq \alpha_1 \dot{\times} f(a_1) \dot{+} \alpha_2 \dot{\times} f(a_2) \tag{44}$$

By further applying equation (6) gives

$$f\left(a_1^{\ln \alpha_1} a_2^{\ln \alpha_2}\right) \leq \left(e^{\ln f(a_1)}\right)^{\ln \alpha_1} . \left(e^{\ln f(a_2)}\right)^{\ln \alpha_2} \tag{45}$$

Hence,

$$f\left(a_1^{\ln \alpha_1} a_2^{\ln \alpha_2}\right) \leq f(a_1)^{\ln \alpha_1} f(a_2)^{\ln \alpha_2}. \tag{46}$$

□

Remark 2: *The function f is multiplicative affine if equality holds in* (42).

Every point $x \in \mathbb{R}^+$ offered by the special affine combination

$$x = r^{\ln\left(\frac{t}{r}\right)} t^{\ln\left(\frac{x}{r}\right)} \tag{47}$$

is convex if and only if the point x is a part of the closed interval $\text{conv}\{r, t\}$ if $I \in \mathbb{R}^+$ are distinct points for $(r, t) \in I$.
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ represent a convex function, and $f_{\{r,t\}}^{line} : \mathbb{R}^+ \to \mathbb{R}^+$ represent a function of the secant line

traversing the graph of points $M(r, f(r))$ and $N(t, f(t))$.
We get the secant equation (48) by applying the affinity of the $f^{line}_{\{r,t\}}$ to equation (47).

$$f^{line}_{\{r,t\}}(x_*) = f(r)^{\left(\frac{t}{x_*}\right)} f(t)^{\left(\frac{x_*}{r}\right)}. \tag{48}$$

The inequality

$$f(x_*) \leq f^{line}_{\{r,t\}}(x_*), \text{if } x_* \in \text{conv}\{r, t\}, \tag{49}$$

and the converse inequality

$$f(x_*) \geq f^{line}_{\{r,t\}}(x_*) \text{ if } x_* \notin \text{conv}\{r, t\}\backslash\{r, t\}. \tag{50}$$

are obtained by combining the convexity of f and the affinity of $f^{line}_{\{r,t\}}$.

Theorem 4: *Let $\theta, \phi, \psi \in [1, e]$ be coefficients such that $\frac{\theta\phi}{\psi} = 1$. Let $r, s, t \in \mathbb{R}^+$ such that $t \in \text{conv}\{r, s\}$. Then the affine combination $\frac{\theta^{\ln r}\phi^{\ln s}}{\psi^{\ln t}} \in \text{conv}\{r, s\}$, and every convex function $f : I \to \mathbb{R}^+$ on an interval $I \subset \mathbb{R}^+$ containing $\text{conv}\{r, s\}$ satisfies the inequality*

$$f\left(\frac{r^{\ln\theta} s^{\ln\phi}}{t^{\ln\psi}}\right) \leq \frac{f(r)^{\ln\theta} f(s)^{\ln\phi}}{f(t)^{\ln\psi}}. \tag{51}$$

Proof: By the convex combination of $t = r^{\ln\kappa} s^{\ln\lambda}$, we have

$$\frac{r^{\ln\theta} s^{\ln\phi}}{t^{\ln\psi}} \leq r^{\ln\left(\frac{\theta\kappa}{(\theta\phi)^{\ln\kappa}}\right)} s^{\ln\left(\frac{\phi\lambda}{(\theta\phi)^{\ln\lambda}}\right)}. \tag{52}$$

The R.H.S of the inequality (52) is the convex combination of points r and s, and as a result is the combination $\frac{r^{\ln\theta} s^{\ln\phi}}{t^{\ln\psi}} \in \text{conv}\{r, s\}$.
Applying the convexity of f and affinity of $f^{line}_{\{r,t\}}$, we get

$$\begin{aligned} f\left(\frac{r^{\ln\theta} s^{\ln\phi}}{t^{\ln\psi}}\right) &\leq f^{line}_{\{r,t\}}\left(\frac{f(r)^{\ln\theta} f(s)^{\ln\phi}}{f(t)^{\ln\psi}}\right) \\ &= \frac{f(r)^{\ln\theta} f(s)^{\ln\phi}}{f^{line}_{\{r,t\}}(t)^{\ln\psi}} \\ &\leq \frac{f(r)^{\ln\theta} f(s)^{\ln\phi}}{f(t)^{\ln\psi}} \end{aligned} \tag{53}$$

and hence

$$f\left(\frac{r^{\ln\theta}s^{\ln\phi}}{t^{\ln\psi}}\right)\leq\frac{f(r)^{\ln\theta}f(s)^{\ln\phi}}{f(t)^{\ln\psi}}. \tag{54}$$

□

Corollary 1: *Let $\theta,\phi,\psi\in[1,e]$ and $\psi_i\in[1,e]$ be coefficients such that $\frac{\theta\phi}{\psi}=\prod_{i=1}^{n}\psi_i=e$. Let $r,s,t_i\in\mathbb{R}^+$ be points such that $t_i\in\operatorname{conv}\{r,s\}$ for all $i=1,\ldots,n$. Then the affine combination $\frac{r^{\ln\theta}s^{\ln\phi}}{\left(\prod_{i=1}^{n}t_i^{\ln\psi_i}\right)^{\ln\psi}}\in\operatorname{conv}\{r,s\}$, and every convex function $f:I\to\mathbb{R}^+$ on an interval $I\subset\mathbb{R}^+$ containing $\operatorname{conv}\{r,s\}$ satisfies the inequality*

$$f\left(\frac{r^{\ln\theta}s^{\ln\phi}}{\left(\prod_{i=1}^{n}f(t_i)^{\ln\psi_i}\right)^{\ln\psi}}\right)\leq\frac{f(r)^{\ln\theta}f(s)^{\ln\phi}}{\left(\prod_{i=1}^{n}f(t_i)^{\ln\psi_i}\right)^{\ln\psi}}. \tag{55}$$

Remark 3: *If $\theta=\phi=\psi=e$, inequality* (55) *reduces to inequality* (30).

Remark 4: *Let $\theta,\phi,\psi\in[1,\infty)$ be coefficients such that $\frac{\theta\phi}{\psi}=1$. Let $r,s,t\in\mathbb{R}^+$ be points such that $t\notin\operatorname{conv}\{r,s\}\setminus\{r,s\}$. Then every convex function $f:I\to\mathbb{R}^+$ defined on an interval $I\subset\mathbb{R}^+$ containing $\operatorname{conv}\{r,s,t\}$ satisfies the inequality*

$$f\left(\frac{r^{\ln\theta}s^{\ln\phi}}{t^{\ln\psi}}\right)\geq\frac{f(r)^{\ln\theta}f(s)^{\ln\phi}}{f(t)^{\ln\psi}}, \tag{56}$$

and the affine combination $\frac{r^{\ln\theta}s^{\ln\phi}}{t^{\ln\psi}}\notin\operatorname{conv}\{r,s\}\setminus\{r,s\}$.

Theorem 5: *Let $f:\mathbb{R}^+\to\mathbb{R}_\alpha$ be an s_α-convex function. For all $x,y\in[1,\infty)$ and $\lambda_1,\lambda_2\geq1$ with $\lambda_1^s\cdot\lambda_2^s=1$, which implies $\lambda_1=\lambda_2=1$, the inequality*

$$f\left(x^{\ln\lambda_1^s}y^{\ln\lambda_2^s}\right)\leq f(x)^{\ln\lambda_1^s}f(y)^{\ln\lambda_2^s} \tag{57}$$

holds for some fixed $s\in(1,e]$.

Proof: Since f is s_α-convex function, then

$$\begin{aligned}f\left(x^{\ln\lambda_1^s}y^{\ln\lambda_2^s}\right)&=f(\lambda_1^s\dot{\times}x\dot{+}\lambda_2^s\dot{\times}y)\\&\dot{\leq}\lambda_1^s\dot{\times}f(x)\dot{+}\lambda_2^s\dot{\times}f(y)\end{aligned} \tag{58}$$

Applying equation (6) to the R.H.S produces the required results and hence concludes the proof. □

Remark 5: *Taking $\lambda_1^s=\lambda_2^s=e$ yields the definition of a p_α-convex function.*

4 Conclusion

This study has systematically developed and proved several fundamental inequalities for convex functions within the framework of Non-Newtonian Calculus, focusing specifically on affine convex functions, geometric properties, and s_α-convexity. By leveraging alternative arithmetic structures, these findings successfully extend classical analytical bounds such as Jensen-type inequalities into generalized Non-Newtonian fields without requiring logarithmic transformations.
Building upon the theoretical groundwork established in this paper, several promising directions remain open for future study. These include:

- Extending these convexity inequalities to multi-variable non-Newtonian calculus and fractional-order Non-Newtonian operators.
- Investigating potential applications of these generalized inequalities in non-Newtonian optimization models, economic growth forecasting, and numerical algorithms.
- Exploring further properties of p_α-convex and other generalized convex classes in broader topological spaces.

Ultimately, the results presented offer enhanced analytical flexibility and contribute valuable tools for modeling nonlinear and multiplicative processes across applied mathematics, economics, and engineering.

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