

Research article

Sectional Semi-B-Near lattices: A Hierarchy, Filters, and Congruences

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Abstract: Near-Heyting algebras generalise Heyting algebras to distributive nearlattices, linking each principal upset's Heyting structure by a global identity. Sectional Hilbert nearlattices [23] drop this identity and instead build a filter and congruence theory from sectional residuation alone. We extend this programme to the Epstein–Horn B-algebra setting [11,24]. A *sectional semi-B-nearlattice* (SSBN-algebra) is a distributive nearlattice $(A, \vee, 1)$ whose implication $\Rightarrow$ restricts, on every principal upset $[a]$, to a semi-B-algebra satisfying the Birkhoff-center adjunction $x \wedge_a b \leq z \Leftrightarrow b \leq x \Rightarrow z$. SSBN-algebras sit in a hierarchy $\mathbf{NsB} \subseteq \mathbf{SSBN} \subsetneq \mathbf{DNSB}$ between near-semi-B-algebras and the strictly larger class of distributive nearlattice semi-B-algebras. Within this hierarchy, ordinary (non-sectional) semi-B-algebras and near-semi-B-algebras turn out to be incomparable classes, with neither containing the other. For downward-directed SSBN-algebras, we build a complete lattice of implicative semi-B-filters and characterise generated filters through a sectional meet; this repairs a natural implication-based formula that fails when the adjoined element is not a Birkhoff element. From this characterisation we obtain an unconditional prime extension theorem, a separation theorem, a prime-spectrum embedding, and a congruence-and-quotient theory for the semi-B-algebra reduct.

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1 Introduction

Heyting algebras furnish the algebraic semantics of intuitionistic propositional logic, and near-Heyting algebras [15,16] extend this structure to distributive nearlattices [6] by requiring each principal upset to be a Heyting algebra, linking the sections through the identity $(x \vee y) \Rightarrow y = x \Rightarrow y$. Sectional Hilbert



nearlattices [23] show that this linking identity is not needed: a complete filter and congruence theory follows from sectional residuation alone.

The present paper transplants that programme from Hilbert implication to the Epstein–Horn/Birkhoff-center setting, and the transplant is not routine. Hilbert implication has no analogue of the Birkhoff center, so every element plays a uniform role in generating a filter; here, only Birkhoff elements satisfy $y \leq x \Rightarrow y$ (property (b) of Section 2), and Example 1 shows that the natural implication-based description of a generated filter, which works unconditionally for Hilbert algebras, fails outright once a non-Birkhoff witness is adjoined. Proposition 7 replaces it with a meet-based description that has no counterpart in the Hilbert-algebra case, and this is what makes the prime extension theorem (Theorem 6) unconditional rather than requiring an extra join-compatibility hypothesis, as discussed further in Remark 8.

Both existing sectional approaches have limitations that the present setting is designed to remove. Near-Heyting algebras impose the global linking identity $(x \vee y) \Rightarrow y = x \Rightarrow y$ across all principal upsets at once, which is unavailable whenever the algebra carries only a local, Birkhoff-center-relative notion of implication rather than a full Heyting structure on each section. Sectional Hilbert nearlattices [23] drop this identity, but their filter and congruence theory rests on Hilbert implication, in which every element of a section plays the same role when a filter is generated. This uniformity is specific to the Hilbert setting: in the Epstein–Horn setting, only Birkhoff elements satisfy $y \leq x \Rightarrow y$ (property (b) of Section 2), so the Hilbert-algebra route to a generated filter cannot be transplanted verbatim, as Example 1 demonstrates explicitly. The present paper closes this gap by building the Birkhoff-center distinction between complemented and non-complemented elements into the sectional filter theory from the start (Proposition 7), rather than assuming it away by a Hilbert-style or global-linking hypothesis.

B-algebras in the sense of Epstein and Horn [11] arise as abstractions of Post algebras [10]. Their implication $x \Rightarrow y$ is the *greatest element of the Birkhoff center* $B(A)$ whose meet with x lies below y , a construction tied to the complemented elements of the lattice in the sense of [1]. Semi-B-algebras, presented in [24], retain only the three equational conditions (B1)–(B3) of B-algebras, dropping the two full distributivity laws.

This weakening fits a broader pattern in the algebraization of non-classical logics. Hilbert algebras [9] and their supremum-extensions [2] relax the implicational fragment of intuitionistic logic. The variety of weak Heyting algebras [3] relaxes the Heyting implication axioms and remains an active area of study, including work on its implicative-infimum subreducts [4] and its tense and difference expansions [5]. (The term “weak Heyting algebra” carries an unrelated meaning in [19], which studies a distinct, completeness-based weakening of the Heyting axioms rather than a subvariety of [3]; we flag this only to avoid a terminological clash.) Brouwerian semilattices [18] drop the lattice meet in favour of relative pseudocomplementation on a semilattice. Residuated lattices [13,22] supply the general substructural setting for residuation, and subresiduated lattices [12] weaken the Epstein–Horn adjunction of Section 2 to a distinguished sublattice rather than the whole Birkhoff center, generalising several of the structures considered below. A representation-theoretic account of finite distributive nearlattices appears in [14], and [20] surveys the algebra-logic correspondence. General lattice-theoretic background is in [17]; nearlattice congruence theory specifically is treated in [7]; and recent work on distributive nearlattice duality appears in [21].

Distributive nearlattices arise naturally in subalgebra lattices, where each principal upset is a bounded distributive lattice carrying its own Birkhoff center and hence a natural domain for the sectional Epstein–Horn implication. Semi-B-algebras are strictly weaker than B-algebras (Example 3), so conditions that hold automatically in B-algebras become non-trivial axioms in the semi-B setting, and this motivates a sectional theory of semi-B-algebras on distributive nearlattices. The open problems of [23] have precise counterparts here, and their resolution in the semi-B setting may shed light on the Hilbert case.

Throughout the paper we use the following class abbreviations: **sBalg** for the class of *semi-B-algebras*, **NsB** for the class of *near-semi-B-algebras*, **DNSB** for the class of *distributive nearlattice semi-B-algebras*, and **SSBN** for the class of *sectional semi-B-nearlattices*, as defined precisely in Sections 2 and 3.

Section 3 introduces SSBN-algebras and the intermediate class of DNSB-algebras. Together with near-semi-B-algebras and global semi-B-algebras, these are placed into the hierarchy $\text{NsB} \subseteq \text{SSBN} \subseteq \text{DNSB}$, alongside an incomparability result for **sBalg** and **NsB** (Theorem 4); the two classes are incomparable rather than nested, as witnessed respectively by Examples 3 and 4. Section 4 develops a filter theory for downward-directed SSBN-algebras. Implicative semi-B-filters form a complete lattice (Theorem 5), and an order-theoretic characterisation of generated filters through a sectional meet (Proposition 7) yields an unconditional prime extension theorem (Theorem 6) and a separation theorem (Theorem 7); the resulting prime spectrum embeds the algebra order into a powerset lattice (Theorem 8). Finally, every implicative semi-B-filter induces a congruence on the semi-B-algebra reduct, with a semi-B-algebra quotient (Theorem 9, Corollary 3).

Semi-B-algebras also admit a logical reading, as models of a fragment of B-logic in which the residuation property is restricted to Birkhoff-center elements: a formula y is derivable from x together with a Boolean hypothesis b exactly when b lies below $x \Rightarrow y$, so that implication encodes deduction relative to a distinguished classical (complemented) sublanguage rather than the whole algebra. The sectional version studied here corresponds to a logic whose models carry local complementation structure on each principal filter, without a global Boolean complement.

Condition (SSBN3) governs how the results below are stated. It constrains $\Rightarrow$ only for pairs of elements that already lie in a common section $[a]$; it says nothing about $x \Rightarrow y$ when x and y have no common lower bound, which can happen in a general distributive nearlattice (Example 4 is such a case). Any argument that compares or combines elements not already known to share a section must first produce one, and this requires a common *lower* bound of the elements involved, not their join. The same caveat applies throughout Sections 3 and 4: the order-theoretic results of Section 3 are stated sectionally to respect it, while the filter and congruence theory of Section 4 is developed under an explicit downward-directedness hypothesis that guarantees the needed common lower bounds exist.

Convention on Birkhoff centers. For a bounded distributive lattice L with least element 0_L and greatest element 1_L , $B(L)$ denotes the Boolean subalgebra of all complemented elements. Since 0_L and 1_L are always complemented, $\{0_L, 1_L\} \subseteq B(L)$. For a distributive nearlattice $(A, \vee, 1)$ and $a \in A$, the upset $[a]$ is a bounded distributive lattice with bounds a and 1 ; its Birkhoff center $B([a])$ satisfies $\{a, 1\} \subseteq B([a])$.

Organisation. Section 2 recalls the background structures and establishes the well-definedness of sectional meets used throughout the paper. Section 3 introduces SSBN-algebras, establishes the Birkhoff-center

membership observation, and develops the basic sectional order theory. Section 4 covers filters, prime extension, and congruences for downward-directed SSBN-algebras. Section 5 gives concrete examples. Section 6 summarises results and states open problems.

2 Preliminaries

Notation. For ease of reading, we collect here the symbols used throughout, all of which follow established conventions in the nearlattice and Epstein–Horn literature cited above: $\vee$ is the nearlattice join, and $[a]$ the principal upset of a ; $\wedge_a$ is the meet computed inside $[a]$ (Definition 1 below), written $\wedge$ once a lower bound is understood to exist (Remark 6); $B(L)$ is the Birkhoff center of a bounded distributive lattice L (Definition 3); $\Rightarrow$ is implication, sectional unless A has a global bottom; **sBalg**, **NsB**, **DNSB**, **SSBN** are the class abbreviations introduced in Section 1; and $\text{Fil}(A)$, $\text{Spec}(A)$ denote the filter lattice and prime spectrum of Section 4.

2.1 Distributive Nearlattices

Definition 1 ([6]): *A distributive nearlattice is a join-semilattice $(A, \vee, 1)$ with greatest element 1 such that, for every $a \in A$, the principal upset $[a] = \{x \in A : a \leq x\}$ is a bounded distributive lattice under the induced order. The meet within $[a]$ is denoted $\wedge_a$; its bounds are a (least) and 1 (greatest).*

The order on A is the join-semilattice order: $x \leq y$ if and only if $x \vee y = y$. The global meet $x \wedge y$ need not exist; when it does, it coincides with $x \wedge_a y$ for any $a \leq x, y$. Throughout the paper, $1 \wedge_a b = b$ for all $b \in [a]$, since 1 is the greatest element of $[a]$; no global meet with 1 is assumed.

Definition 1 attaches the symbol $\wedge_a$ to a specific base point a , and every later use of $\wedge_a$ implicitly treats this value as depending only on x, y and on the fact that a is *some* common lower bound of x, y , not on the particular choice of a . The next lemma records that this is legitimate, before the notation is put to use in Definition 2 below.

Lemma 1 (Well-definedness of sectional meets): *Let $(A, \vee, 1)$ be a distributive nearlattice and let $x, y \in A$. If $a, a' \in A$ both satisfy $a \leq x, y$ and $a' \leq x, y$, then $x \wedge_a y = x \wedge_{a'} y$.*

Proof: Put $d = a \vee a'$. Since $a \leq x$ and $a' \leq x$, x is an upper bound of $\{a, a'\}$, so $d \leq x$ by the defining property of the join; symmetrically $d \leq y$. Hence d is a common lower bound of x, y , and $x, y \in [d]$.

Since $d \geq a$, the upset $[d]$ is a subset of $[a]$, and $[d]$ is itself a principal upset of the lattice $[a]$. For elements $u, v \in [d]$, the meet $u \wedge_a v$ computed in $[a]$ is a lower bound of u, v lying in $[a]$; because $d \leq u, v$ and $u \wedge_a v$ is the greatest such lower bound in $[a]$, and $d \in [a]$ is itself a candidate, we get $d \leq u \wedge_a v$, so $u \wedge_a v \in [d]$. Thus $u \wedge_a v$ is a lower bound of u, v lying in $[d]$, hence $u \wedge_a v \leq u \wedge_d v$. Conversely $u \wedge_d v \in [d] \subseteq [a]$ is a lower bound of u, v lying in $[a]$, hence $u \wedge_d v \leq u \wedge_a v$. Combining the two inequalities, $u \wedge_a v = u \wedge_d v$ for all $u, v \in [d]$; taking $u = x, v = y$ gives $x \wedge_a y = x \wedge_d y$.

The symmetric argument, with the roles of a and a' exchanged, gives $x \wedge_{a'} y = x \wedge_d y$. Hence $x \wedge_a y = x \wedge_{a'} y$. $\square$

By Lemma 1, whenever $x, y \in A$ have at least one common lower bound, the notation $x \wedge_a y$ may be written without reference to a ; we use this freely from Definition 2 onward.

Definition 2: A subset $F \subseteq A$ is a filter of a distributive nearlattice if $1 \in F$, F is upward-closed, and $x \wedge_a y \in F$ whenever $x, y \in F$ and $a \leq x, y$.

2.2 The Birkhoff Center

Definition 3: Let L be a bounded distributive lattice with bounds 0_L and 1_L . The Birkhoff center $B(L)$ is the set of all complemented elements:

$$B(L) = \{b \in L : \exists b' \in L, b \wedge b' = 0_L \text{ and } b \vee b' = 1_L\}.$$

$B(L)$ is a Boolean subalgebra of L , and $\{0_L, 1_L\} \subseteq B(L)$. For a distributive nearlattice $(A, \vee, 1)$ and $a \in A$, the sectional Birkhoff center $B([a])$ satisfies $\{a, 1\} \subseteq B([a])$.

2.3 Epstein–Horn B-Algebras

Definition 4 ([11]): A distributive lattice $(A, \vee, \wedge, 0, 1)$ with Birkhoff center $B(A)$ is a B-algebra if, for every $x, y \in A$, there exists a greatest $b \in B(A)$ satisfying $x \wedge b \leq y$. This greatest element is denoted $x \Rightarrow y$.

The equational characterisation of B-algebras (Theorem 3.2 of [24]) shows that A is a B-algebra if and only if, for all $x, y, z \in A$ and $b \in B(A)$:

$$x \Rightarrow x = 1, \tag{BA1}$$

$$(x \Rightarrow b) \wedge b = b, \tag{BA2}$$

$$x \wedge (x \Rightarrow b) = x \wedge b, \tag{BA3}$$

$$z \Rightarrow (x \wedge y) = (z \Rightarrow x) \wedge (z \Rightarrow y), \tag{BA4}$$

$$(x \vee y) \Rightarrow z = (x \Rightarrow z) \wedge (y \Rightarrow z). \tag{BA5}$$

2.4 Semi-B-Algebras

Definition 5 ([24]): A distributive lattice $(A, \vee, \wedge, 0, 1)$ with Birkhoff center $B(A)$ is a semi-B-algebra if the following hold for all $x, y \in A$ and $b \in B(A)$:

$$x \Rightarrow x = 1, \tag{B1}$$

$$x \wedge (x \Rightarrow b) = x \wedge b, \tag{B2}$$

$$b \wedge (x \Rightarrow y) = b \wedge ((b \wedge x) \Rightarrow (b \wedge y)). \tag{B3}$$

The class of all semi-B-algebras is denoted **sBalg**.

Every B-algebra is a semi-B-algebra: (B1) is (BA1), (B2) is (BA3), and (B3) is derived from (BA4)–(BA5) in the proof of Theorem 3.2 of [24], to which we refer the reader for the equational computation rather than repeating it here. The converse does not hold; see Example 3. Axioms (B2) and (B3) are stated for $b \in B(A)$ only; the implication $x \Rightarrow y$ is defined for all $x, y \in A$. In a general semi-B-algebra, (B1)–(B3) alone do not guarantee that $x \Rightarrow y \in B(A)$ for arbitrary $x, y \in A$: this membership is additional content, supplied directly in the B-algebra definition and, in the sectional setting of Section 3, by axiom (SSBN3). Any argument that applies (B3) to an expression such as $(b \wedge x) \Rightarrow (b \wedge y)$ therefore requires this element to be known in advance to lie in $B(A)$, and we flag explicitly every place this is needed below.

The following properties hold in every semi-B-algebra (Theorems 3.7–3.10 of [24]):

- (a) If $b \in B(A)$ and $b \leq x$, then $b \wedge (x \Rightarrow c) = b \wedge c$ for any $c \in B(A)$.
- (b) If $b \in B(A)$, $b \leq x$, and $b \leq y$, then $b \leq (x \Rightarrow y)$.
- (c) For $b \in B(A)$ with $b \leq x$: $b \leq (b \Rightarrow x)$ and $b \leq (x \Rightarrow b)$.
- (d) $(x \Rightarrow y) = 1$ implies $x \leq y$.
- (e) $b \leq ((b \wedge x) \Rightarrow x)$ for all $b \in B(A)$.

Property (b) gives $y \leq x \Rightarrow y$ only when y itself is a Birkhoff element ($b \leq x$ and $b \leq y$ with $b = y$ requires $y \in B(A)$). For a general, non-Birkhoff element y , the inequality $y \leq x \Rightarrow y$ need not hold, even in a genuine Epstein–Horn B-algebra; Example 1 exhibits this explicitly, and Section 4 relies on the distinction repeatedly.

2.5 Near-Semi-B-Algebras

Definition 6: An algebra $(A, \vee, \Rightarrow, 1)$ is a near-semi-B-algebra if $(A, \vee, 1)$ is a distributive nearlattice and the following hold for all $x, y \in A$:

$$y \vee (x \Rightarrow y) = x \Rightarrow y, \quad (\text{NsB1})$$

$$x \Rightarrow x = 1, \quad (\text{NsB2})$$

$$1 \Rightarrow x = x, \quad (\text{NsB3})$$

$$(x \vee y) \Rightarrow y = x \Rightarrow y, \quad (\text{NsB4})$$

and, for every $a \in A$, the restriction of $\Rightarrow$ to $[a]$ satisfies (B1)–(B3) with respect to $B([a])$ and the sectional adjunction

$$x \wedge_a b \leq z \iff b \leq x \Rightarrow z \quad (\text{NsB5})$$

holds for all $x, z \in [a]$ and $b \in B([a])$. The class of all near-semi-B-algebras is denoted **NsB**.

Axiom (NsB1) forces $y \leq x \Rightarrow y$ for all x, y , which is strictly stronger than property (b) above guarantees in a general semi-B-algebra, where this inequality is secured only when y is itself a Birkhoff element. Axiom (NsB4) is the global-linking identity; SSBN-algebras (Definition 7) do not require it. Axiom (NsB3) is postulated independently rather than derived from (NsB1): substituting $x = 1$ into (NsB1)

gives $y \vee (1 \Rightarrow y) = 1 \Rightarrow y$, which only yields $y \leq 1 \Rightarrow y$ and does not by itself force $1 \Rightarrow y = y$. Condition (NsB5) is precisely the sectional adjunction (SSBN3) of Definition 7 below, included here as a hypothesis on near-semi-B-algebras rather than as a consequence of (NsB1)–(NsB4); Proposition 2 records the resulting relationship.

3 Sectional Semi-B-Nearlattices

3.1 Definition

Definition 7: A sectional semi-B-nearlattice (SSBN-algebra) is an algebra $(A, \vee, \Rightarrow, 1)$ satisfying:

- (SSBN1) $(A, \vee, 1)$ is a distributive nearlattice.
- (SSBN2) For every $a \in A$, the restriction of $\Rightarrow$ to $[a]$ satisfies (B1)–(B3) with respect to $B([a])$.
- (SSBN3) (Sectional adjunction) For every $a \in A$ and $x, z \in [a]$, the element $x \Rightarrow z$ lies in $B([a])$ and is the greatest element of $B([a])$ satisfying $x \wedge_a (x \Rightarrow z) \leq z$; equivalently, for every $b \in B([a])$,

$$x \wedge_a b \leq z \iff b \leq x \Rightarrow z.$$

The class of all SSBN-algebras is denoted **SSBN**. Condition (SSBN3) is a genuine additional postulate on the operation $\Rightarrow$: it is not derivable from (SSBN1)–(SSBN2), since (B1)–(B3) alone place no constraint on how $\Rightarrow$ interacts with $\wedge_a$. The same postulational status attaches to $x \Rightarrow y \in B(A)$ in ordinary B-algebras (Definition 4), where membership in the Birkhoff center is part of what the definition demands of $\Rightarrow$ rather than something proved about it. The membership $x \Rightarrow z \in B([a])$ stated in (SSBN3) is used repeatedly below and is collected as Remark 2 for ease of citation.

Because of this, (SSBN3) constrains $x \Rightarrow z$ only when x and z already lie in a common section $[a]$, i.e. only when some $a \in A$ satisfies $a \leq x$ and $a \leq z$. If x and z have no common lower bound, nothing in (SSBN1)–(SSBN3) pins down the value of $x \Rightarrow z$.

Remark 1 (Consistency across sections): Condition (SSBN3) is quantified over every $a \in A$ simultaneously, since $\Rightarrow$ is a single global operation. So whenever $x, z \in [a] \cap [a']$ for $a, a' \in A$, the value $x \Rightarrow z$ must agree with the greatest element of $B([a])$ and of $B([a'])$ satisfying the adjunction; this agreement is part of the axiom's content, not a derived consequence. Every example in Section 5 overlaps sections only trivially, so each pair (x, z) has a canonical minimal section once a lower bound is fixed; for a richer nearlattice, requiring a single $\Rightarrow$ to satisfy (SSBN3) on all sections at once remains a non-trivial constraint.

Definition 8: A distributive nearlattice semi-B-algebra (DNSB-algebra) is an algebra $(A, \vee, \Rightarrow, 1)$ satisfying (SSBN1) and (SSBN2) only, i.e. a distributive nearlattice whose restriction of $\Rightarrow$ to every principal upset $[a]$ is a semi-B-algebra with respect to $B([a])$, without requiring the sectional adjunction (SSBN3). The class of all DNSB-algebras is denoted **DNSB**.

The sectional adjunction (SSBN3) is independent of (SSBN1) and (SSBN2): the axioms (B1)–(B3) impose no adjunction between $\wedge_a$ and $\Rightarrow$ within $[a]$, which is why DNSB-algebras need not be SSBN-algebras (Corollary 4). Condition (SSBN3) says that $x \Rightarrow z$ is the greatest element of $B([a])$ satisfying $x \wedge_a (x \Rightarrow z) \leq z$,

i.e., it is the sectional Epstein–Horn implication. When A is a lattice with global bottom 0 , so that $[0] = A$, condition (SSBN3) reduces to the global Epstein–Horn adjunction, recovering Definition 4.

Remark 2 (Birkhoff membership): *In any SSBN-algebra A , for every $a \in A$ and $x, z \in [a]$, $x \Rightarrow z \in B([a])$. This membership is part of the content postulated by (SSBN3) and is not proved independently; it is invoked by name throughout the paper to licence the substitution $b := x \Rightarrow z$ into (SSBN3) without restating the definition each time.*

3.2 Basic Properties

Proposition 1: *Let A be an SSBN-algebra.*

- (1) *If $x \leq y$ in the nearlattice order, then $x \Rightarrow y = 1$.*
- (2) *1 is the greatest element of A , and $x \Rightarrow 1 = 1$ for every $x \in A$.*
- (3) *If $x \leq y$, then $x \vee z \leq y \vee z$ for every $z \in A$, and hence $(x \vee z) \Rightarrow (y \vee z) = 1$ by part (1).*

Proof: (1) Take $a = x$; since $x \leq x$ and $x \leq y$, both $x, y \in [a]$, and $1 \in B([a])$. From $x \wedge_a 1 = x \leq y$, the $(\Rightarrow)$ direction of (SSBN3) gives $1 \leq x \Rightarrow y$, i.e. $x \Rightarrow y = 1$.

(2) That 1 is the greatest element of A is part of Definition 1. For $x \Rightarrow 1 = 1$: take $a = x$; then $x \wedge_a 1 = x \leq 1$, so the $(\Rightarrow)$ direction of (SSBN3) gives $x \Rightarrow 1 = 1$.

(3) Monotonicity of $\vee$ with respect to $\leq$ holds in every join-semilattice; the displayed implicational consequence is part (1) applied to the pair $x \vee z \leq y \vee z$. $\square$

The order-theoretic content of (SSBN3) is stated separately from the elementary facts above, since it is the result invoked most heavily in the remainder of the paper.

Theorem 1 (Sectional order agreement): *Let A be an SSBN-algebra and $a \in A$. The relation $x \leq_{\Rightarrow} y : \Leftrightarrow x \Rightarrow y = 1$, restricted to $[a]$, is a partial order on $[a]$, and it coincides there with the nearlattice order $\leq$ restricted to $[a]$.*

This statement is confined to a single section rather than to all of A , since (SSBN3) constrains $x \Rightarrow y$ only for pairs with a common lower bound. Every later use of this theorem applies it inside a section already fixed by hypothesis, as in Lemmas 2 and 3 below.

Proof: Fix $a \in A$; by (SSBN2) (Definition 7), the restriction of $\Rightarrow$ to $[a]$ satisfies (B1)–(B3) with respect to $B([a])$, so $[a]$ is in particular a semi-B-algebra. Reflexivity of $\leq_{\Rightarrow}$ on $[a]$ is (B1). For $x, y \in [a]$, Proposition 1(1) (applied within $[a]$, since $a \leq x, y$ already places them in a common section) gives that $x \leq y$ implies $x \Rightarrow y = 1$, and general semi-B property (d) gives the converse: $x \Rightarrow y = 1$ implies $x \leq y$. Hence, for $x, y \in [a]$,

$$x \Rightarrow y = 1 \quad \Longleftrightarrow \quad x \leq y.$$

Antisymmetry and transitivity of $\leq$ on $[a]$ are therefore inherited directly from antisymmetry and transitivity of the lattice order $\leq$ on $[a]$, via this identification. $\square$

Theorem 2: *Let A be an SSBN-algebra and $a \in A$. The algebra $([a], \wedge_a, \vee, \Rightarrow_{[a]}, a, 1)$ is a semi-B-algebra with respect to $B([a])$. Moreover, for every $x, z \in [a]$, the element $x \Rightarrow z$ is the greatest element of $B([a])$ satisfying $x \wedge_a (x \Rightarrow z) \leq z$.*

Proof: *Closure.* For $x, z \in [a]$, Remark 2 gives $x \Rightarrow z \in B([a]) \subseteq [a]$.

Semi-B axioms. Axioms (B1)–(B3) hold in $[a]$ by (SSBN2).

Adjunction. Condition (SSBN3) provides $x \wedge_a b \leq z \Leftrightarrow b \leq x \Rightarrow z$ for all $b \in B([a])$. Setting $b = x \Rightarrow z$ (which lies in $B([a])$ by Remark 2) in the $(\Leftarrow)$ direction gives $x \wedge_a (x \Rightarrow z) \leq z$, confirming $x \Rightarrow z$ is the greatest such element by (SSBN3). $\square$

Corollary 1: *Every principal upset $[a]$ of an SSBN-algebra carries the Epstein–Horn sectional adjunction and is a semi-B-algebra with respect to $B([a])$.*

Corollary 2 (Sectional modus ponens): *For every $a \in A$ and $x, z \in [a]$, one has $x \wedge_a (x \Rightarrow z) \leq z$.*

Proof: Set $b = x \Rightarrow z \in B([a])$ (Remark 2) and apply the $(\Leftarrow)$ direction of (SSBN3). $\square$

Theorem 3 (Uniqueness): *For every $a \in A$ and $x, z \in [a]$, the element $x \Rightarrow z$ is the unique greatest $b \in B([a])$ satisfying $x \wedge_a b \leq z$.*

Proof: By (SSBN3) every $b \in B([a])$ with $x \wedge_a b \leq z$ satisfies $b \leq x \Rightarrow z$. Uniqueness of the greatest element is then immediate. $\square$

Lemma 2 (Sectional monotonicity of implication): *Let A be an SSBN-algebra, $a \in A$, and $x, u, y, v \in [a]$ with $x \leq u$ and $y \leq v$. For every $b \in B([a])$ with $b \leq u \Rightarrow y$, one has $b \leq x \Rightarrow v$.*

Proof: **Step 1.** Let $b \in B([a])$ with $b \leq u \Rightarrow y$. Applying (B3) within $[a]$:

$$b \wedge_a (u \Rightarrow y) = b \wedge_a ((b \wedge_a u) \Rightarrow (b \wedge_a y)).$$

Since $b \leq u \Rightarrow y$, the left side equals b , giving

$$b \leq (b \wedge_a u) \Rightarrow (b \wedge_a y). \quad (\star)$$

Step 2(a) (Birkhoff membership and the Boolean order). By Remark 2, both $(b \wedge_a u) \Rightarrow (b \wedge_a y)$ and $(b \wedge_a x) \Rightarrow (b \wedge_a v)$ lie in $B([a])$, so the argument below uses (SSBN3) rather than (B1)–(B3) alone: without (SSBN3) these implications need not lie in $B([a])$, and the Boolean-order argument that follows would not apply. Since $B([a])$ is a Boolean algebra, its implication $p \Rightarrow_{B([a])} q = \neg p \vee_{B([a])} q$ is order-reversing in p and order-preserving in q . By (SSBN3) this Boolean implication coincides with the restriction of the

global $\Rightarrow$ to elements of $B([a])$, since for $p, q \in B([a])$ the greatest $b \in B([a])$ with $p \wedge_a b \leq q$ is exactly the Boolean relative complement $\neg p \vee_{B([a])} q$.

Step 2(b) (Applying monotonicity). From $x \leq u$ and $y \leq v$ we get $b \wedge_a x \leq b \wedge_a u$ and $b \wedge_a y \leq b \wedge_a v$. Order-reversal in the first argument applied to $b \wedge_a x \leq b \wedge_a u$ gives $(b \wedge_a u) \Rightarrow (b \wedge_a y) \leq (b \wedge_a x) \Rightarrow (b \wedge_a y)$ in $B([a])$; order-preservation in the second argument applied to $b \wedge_a y \leq b \wedge_a v$ then gives $(b \wedge_a x) \Rightarrow (b \wedge_a y) \leq (b \wedge_a x) \Rightarrow (b \wedge_a v)$. Combining the two displayed inequalities,

$$(b \wedge_a u) \Rightarrow (b \wedge_a y) \leq (b \wedge_a x) \Rightarrow (b \wedge_a v) \quad \text{in } B([a]).$$

Step 3. Combining Step 2(b) with $(*)$: $b \leq (b \wedge_a x) \Rightarrow (b \wedge_a v)$. A second application of (B3) to the pair (x, v) with coefficient b gives

$$b \wedge_a (x \Rightarrow v) = b \wedge_a ((b \wedge_a x) \Rightarrow (b \wedge_a v)) = b,$$

hence $b \leq x \Rightarrow v$. $\square$

Remark 3: Lemma 2 records a basic structural fact about the sectional order of an SSBN-algebra, in the spirit of the standard monotonicity property of Heyting implication; it is not needed for the filter-generation argument of Section 4 (Proposition 7), which instead argues directly through sectional meets.

Lemma 3 (Sectional syllogism): *Let A be an SSBN-algebra, $a \in A$, and $x, y, z \in [a]$. Then*

$$(y \Rightarrow z) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z)) = 1.$$

Proof: By Remark 2, $x \Rightarrow y$, $y \Rightarrow z$, and $x \Rightarrow z$ lie in $B([a])$, and so does their meet $(x \Rightarrow y) \wedge_a (y \Rightarrow z)$. By Corollary 2, $x \wedge_a (x \Rightarrow y) \leq y$, so associativity and monotonicity of $\wedge_a$ give

$$x \wedge_a ((x \Rightarrow y) \wedge_a (y \Rightarrow z)) \leq y \wedge_a (y \Rightarrow z) \leq z,$$

the last step by Corollary 2. Since $(x \Rightarrow y) \wedge_a (y \Rightarrow z) \in B([a])$, the $(\Rightarrow)$ direction of (SSBN3) applied to (x, z) gives

$$(x \Rightarrow y) \wedge_a (y \Rightarrow z) \leq x \Rightarrow z.$$

Applying (SSBN3) again to $(x \Rightarrow y, x \Rightarrow z)$ with $b = y \Rightarrow z \in B([a])$ gives $y \Rightarrow z \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$. Theorem 1 identifies this with the stated equality, since all elements involved lie in the single section $[a]$ fixed above. $\square$

Remark 4: Lemmas 2 and 3 take x, y, z (or x, u, y, v) as already given in a common section $[a]$; this is essential, since setting $a := x \vee y \vee z$ and treating x, y, z as members of $[a]$ would be invalid ($a \leq x$ would then force $y \vee z \leq x$, which the hypotheses do not give). The remainder of the paper accordingly assumes or establishes a common lower bound rather than manufacturing one from a join.

3.3 Relationship with Near-Semi-B-Algebras

The next two propositions state that the defining clauses of a larger class are a subset of those of each smaller class; both follow immediately from comparing definitions.

Proposition 2: *Every near-semi-B-algebra is an SSBN-algebra.*

Proof: Condition (SSBN1) holds as part of Definition 6; (SSBN2) holds because Definition 6 requires (B1)–(B3) on each $[a]$; and (SSBN3) is precisely axiom (NsB5). The global-linking identity (NsB4) plays no role here: it is what might distinguish near-semi-B-algebras from SSBN-algebras (Theorem 4 and Remark 5). $\square$

Proposition 3: *Every SSBN-algebra is a DNSB-algebra.*

Proof: Conditions (SSBN1) and (SSBN2) are exactly the defining conditions of a DNSB-algebra (Definition 8). $\square$

Theorem 4 (Hierarchy and incomparability): *The inclusions $\mathbf{NsB} \subseteq \mathbf{SSBN} \subseteq \mathbf{DNSB}$ hold, with the last inclusion strict. The classes $\mathbf{sBalg}$ and $\mathbf{NsB}$ are incomparable: $\mathbf{NsB} \not\subseteq \mathbf{sBalg}$ and $\mathbf{sBalg} \not\subseteq \mathbf{NsB}$.*

Proof: $\mathbf{NsB} \subseteq \mathbf{SSBN}$: Proposition 2.

$\mathbf{SSBN} \subseteq \mathbf{DNSB}$: Proposition 3.

$\mathbf{SSBN} \subsetneq \mathbf{DNSB}$: Corollary 4 exhibits a DNSB-algebra that fails (SSBN3).

$\mathbf{NsB} \not\subseteq \mathbf{sBalg}$: Example 4 is a near-semi-B-algebra whose underlying structure is not a lattice, hence is not a global semi-B-algebra (Definition 5 requires a distributive lattice).

$\mathbf{sBalg} \not\subseteq \mathbf{NsB}$: Example 3 is a semi-B-algebra (on $A = \{0, 1\}$ with $0 \Rightarrow 1 = 0$) in which axiom (NsB1) fails: taking $x = 0$ and $y = 1$,

$$y \vee (x \Rightarrow y) = 1 \vee (0 \Rightarrow 1) = 1 \vee 0 = 1 \neq 0 = x \Rightarrow y.$$

Hence $A \in \mathbf{sBalg} \setminus \mathbf{NsB}$, so $\mathbf{sBalg} \not\subseteq \mathbf{NsB}$. $\square$

Remark 5: *Whether $\mathbf{NsB} \subsetneq \mathbf{SSBN}$ is strict remains open: a separating example requires an SSBN-algebra failing the global-linking identity (NsB4); see Section 6 for discussion.*

Proposition 4 (Characterisation): *An algebra $(A, \vee, \Rightarrow, 1)$ satisfying (SSBN1) is an SSBN-algebra if and only if every $[a]$ is a semi-B-algebra carrying the Epstein–Horn adjunction with respect to $B([a])$.*

Proof: $(\Rightarrow)$: Corollary 1.

$(\Leftarrow)$: The adjunction holding in each $[a]$ is (SSBN3); each $[a]$ being a semi-B-algebra is (SSBN2). $\square$

3.4 Structural Comparison

Table 1 places the new classes alongside earlier related structures. The columns for Sect. (B1)–(B3) and Sect. adj. (SSBN3) are separated to emphasise that (SSBN3) is independent of (B1)–(B3). In particular, the mere fact that a semi-B-algebra is a distributive lattice (hence its own single section) does *not* guarantee that it carries the Epstein–Horn adjunction: Example 3, verified in Corollary 4 to fail (SSBN3), is a semi-B-algebra for which the global and sectional adjunction columns are both “no.”

Structure	Global lattice	Global adj.	Sect. (B1)–(B3)	Sect. adj. (SSBN3)	Linking id.
Semi-B-algebra	yes	not guaranteed	yes	not guaranteed	no
Near-semi-B-algebra	no	no	yes	yes	yes
SSBN-algebra	no	no	yes	yes	no
DNSB-algebra	no	no	yes	not req.	no
Heyting algebra	yes	yes	n/a	n/a	n/a
SHN-algebra [23]	no	no	n/a	n/a	no

Table 1: Structural comparison of semi-B and Heyting-type classes. “n/a” means the concept does not apply within this paper’s framework. The semi-B-algebra row reads “not guaranteed” in the adjunction columns because (B1)–(B3) alone do not force $x \Rightarrow y \in B(A)$ (Section 2); Example 3 exhibits a semi-B-algebra failing (SSBN3) (Corollary 4), so the adjunction cannot be asserted in general for this row. The row has “no” in the Linking id. column because **sBalg** and **NsB** are incomparable; the global-linking identity (NsB4) is an axiom of **NsB**, not of **sBalg** (Theorem 4).

4 Filters and Congruences

4.1 Implicative Semi-B-Filters

Definition 9: A nonempty subset $F \subseteq A$ is an implicative semi-B-filter (sB-filter) of an SSBN-algebra A if:

- (1) $1 \in F$;
- (2) $x \in F$ and $x \leq y$ imply $y \in F$ (upward closure);
- (3) $x \in F$ and $x \Rightarrow y \in F$ imply $y \in F$ (implicative closure);
- (4) $x, y \in F$ and $a \leq x, y$ imply $x \wedge_a y \in F$ (meet closure).

We use the shorter name sB-filter throughout the remainder of the paper. Clause (4) mirrors the meet-closure clause of nearlattice filters (Definition 2), of which an sB-filter is a refinement, and is well posed by Lemma 1.

The trivial sB-filters $\{1\}$ and A exist in every SSBN-algebra. Arbitrary intersections of sB-filters are sB-filters.

Proposition 5: Let F be an sB-filter of A and $x \in F$ with $x \leq y$. Then $x \Rightarrow y = 1 \in F$.

Proof: By Proposition 1(1), $x \leq y$ gives $x \Rightarrow y = 1$, and $1 \in F$ by clause (1). $\square$

4.2 A Standing Hypothesis: Downward Directedness

The results from Lemma 4 onward compare or combine implications $x \Rightarrow y$, $y \Rightarrow z$, ... built from several elements of A at once. Such comparisons are only licensed by (SSBN3) when the elements concerned already share a common lower bound. We therefore adopt an explicit hypothesis for the remainder of this section.

Definition 10: An SSBN-algebra A is downward directed if every nonempty finite subset $S \subseteq A$ has a lower bound in A , i.e. there exists $d \in A$ with $d \leq s$ for every $s \in S$.

Remark 6: Every result from Lemma 4 to Corollary 3 below assumes A is downward directed. Each argument in this range must place several elements of A into a single section $[d]$ before invoking (SSBN3) or Lemma 3, and $[d]$ requires d to lie below each of them, not above. Every worked example of Section 5 is downward directed except Example 4, whose two minimal elements a, b have no common lower bound; that example is used solely to witness Theorem 4, which does not rely on any result of this section. For readers who wish to construct further directed examples: a finite join-semilattice that is downward directed is automatically a lattice, since the join of all common lower bounds of any pair is itself a lower bound of both and hence they meet. Non-lattice directed examples, if wanted, must therefore be sought among infinite nearlattices.

Remark 7 (Consistency of directed meets): When $x, y \in A$ have a common lower bound, Lemma 1 already guarantees that $x \wedge_d y$ takes the same value for every common lower bound $d \leq x, y$; downward directedness is used here only to guarantee that such a common lower bound exists for every finite subset of A occurring in an argument of this section. We write this common value as $x \wedge y$ whenever a lower bound of x, y exists.

4.3 Hypothetical Syllogism in Filters

Lemma 4: Let A be a downward-directed SSBN-algebra and F an sB-filter of A . If $a, b, c \in A$ with $a \Rightarrow b \in F$ and $b \Rightarrow c \in F$, then $a \Rightarrow c \in F$.

Proof: By Definition 10, choose $d \in A$ with $d \leq a, b, c$, so $a, b, c \in [d]$. Lemma 3 applied within $[d]$ gives

$$(b \Rightarrow c) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c)) = 1 \in F.$$

Two applications of implicative closure (using $b \Rightarrow c \in F$ then $a \Rightarrow b \in F$) yield $a \Rightarrow c \in F$. $\square$

4.4 Generated sB-Filters

Proposition 6: For every $a \in A$, the sB-filter generated by $\{a\}$ is $\langle a \rangle = [a]$.

Proof: $[a]$ is an sB-filter. (1) $1 \in [a]$ since $a \leq 1$. (2) $[a]$ is upward-closed by definition. (3) Let $x, x \Rightarrow y \in [a]$. Then $a \leq x \Rightarrow y$. Since $x \Rightarrow y \in B([a])$ (Remark 2), the $(\Leftarrow)$ direction of (SSBN3) with $b = a \in B([a])$ gives $x \wedge_a a \leq y$, i.e. $a \leq y$, so $y \in [a]$. (4) $[a]$ is a bounded distributive lattice with meet $\wedge_a$, so it is closed under $\wedge_a$.

Minimality. Any sB-filter containing a contains $\lceil a \rceil$ by upward closure. $\square$

We now turn to filters generated by adjoining a *single further element* s to an existing sB-filter F . A natural first guess is to mimic the classical Hilbert-algebra construction and describe the generated filter through witnesses of the form $p \leq s \Rightarrow c$ with $p \in F$. The next example shows that this guess is false in general, which is why Proposition 7 below is instead phrased through a sectional meet.

Example 1 (Failure of an implication-based witness): *The inequality $f \leq s \Rightarrow f$ need not hold for a general (not necessarily Birkhoff) element f , even in an honest Epstein–Horn B-algebra. Let $L = \{0, 1\} \times \{0, m, 1\}$ be the product of the two-element chain and the three-element chain, ordered, met, and joined componentwise. This is a bounded distributive lattice with bottom $(0, 0)$ and top $(1, 1)$. Its Birkhoff center is $B(L) = \{0, 1\} \times \{0, 1\}$, four elements in all, since m has no complement in the three-element chain: any candidate complement c of (x_1, m) would need $m \wedge c_2 = 0$ and $m \vee c_2 = 1$ in the chain $\{0, m, 1\}$, and no $c_2 \in \{0, m, 1\}$ satisfies both. Regarded as a nearlattice with global bottom, L has the single section $\lceil (0, 0) \rceil = L$, and $B(L)$ is a finite Boolean algebra closed under the joins required in Definition 4, so the Epstein–Horn implication $x \Rightarrow y = (\text{greatest } b \in B(L) \text{ with } x \wedge b \leq y)$ is well defined and $L \in \mathbf{sBalg}$.*

Take $s = (0, 1)$ and $f = (1, m)$. For $b = (b_1, b_2) \in B(L)$, $s \wedge b = (0, b_2)$, and $(0, b_2) \leq (1, m)$ holds automatically in the first coordinate and requires $b_2 \leq m$ in the second; since $b_2 \in \{0, 1\}$ and $1 \not\leq m$, this forces $b_2 = 0$, while b_1 is unconstrained. The greatest such b is therefore $(1, 0)$, so $s \Rightarrow f = (1, 0)$. But $f = (1, m) \not\leq (1, 0)$, since $m \not\leq 0$ in the three-element chain. Thus $f \not\leq s \Rightarrow f$. Consistently with property (b) of Section 2, note that $f = (1, m) \notin B(L)$: the failure occurs precisely because f is not a Birkhoff element. This is the gap that an implication-based description of the generated filter would inherit, and Proposition 7 below is proved using the sectional meet $p \wedge s$ rather than the implication $s \Rightarrow c$ for this reason.

Lemma 5 (Sectional distributivity bound): *Let A be a downward-directed SSBN-algebra and $p, x, y, c \in A$. If $p \wedge x \leq c$ and $p \wedge y \leq c$ (meets in the sense of Remark 7), then $p \wedge (x \vee y) \leq c$.*

Proof: Choose $d \in A$ with $d \leq p, x, y$ (downward-directedness). Since $x, y \geq d$, also $x \vee y \geq d$, so $p, x, y, x \vee y \in \lceil d \rceil$, a distributive lattice by Definition 1. Distributivity gives $p \wedge_d (x \vee y) = (p \wedge_d x) \vee (p \wedge_d y)$. By hypothesis both terms on the right are $\leq c$, and the join of two elements each $\leq c$ is itself $\leq c$; hence $p \wedge (x \vee y) = p \wedge_d (x \vee y) \leq c$. $\square$

Proposition 7 (Generated sB-filter): *Let A be downward directed, F an sB-filter, and $s \in A$. The sB-filter generated by $F \cup \{s\}$ is*

$$G := \{c \in A : \exists p \in F, p \wedge s \leq c\},$$

where $p \wedge s$ is the sectional meet of Remark 7.

Proof: G contains $F \cup \{s\}$. Taking $p = 1 \in F$ and $c = 1$: $1 \wedge s = s \leq 1$, so $1 \in G$. For $f \in F$: taking $p = c = f$, $f \wedge s \leq f$ always, since a meet is a lower bound of its arguments; so $f \in G$, giving $F \subseteq G$. For s : taking $p = 1 \in F$ and $c = s$, $1 \wedge s = s \leq s$, so $s \in G$.

Upward closure. If $c \in G$ via $p \wedge s \leq c$, and $c \leq e$, then $p \wedge s \leq c \leq e$, so $e \in G$ via the same p .

Meet closure. Let $c_1, c_2 \in G$ via $p_1 \wedge s \leq c_1$ and $p_2 \wedge s \leq c_2$. Put $p = p_1 \wedge p_2 \in F$ (meet-closure of F , well defined by downward-directedness). Since $p \leq p_1, p_2$, monotonicity of the meet in any common section gives $p \wedge s \leq p_1 \wedge s \leq c_1$ and $p \wedge s \leq p_2 \wedge s \leq c_2$, so $p \wedge s$ is a lower bound of both c_1 and c_2 , and therefore $p \wedge s \leq c_1 \wedge c_2$. Hence $c_1 \wedge c_2 \in G$ via p .

Implicative closure. Let $c, c \Rightarrow e \in G$, with witnesses $p_1 \wedge s \leq c$ and $p_2 \wedge s \leq (c \Rightarrow e)$. Put $p = p_1 \wedge p_2 \in F$; as above, $p \wedge s \leq c$ and $p \wedge s \leq c \Rightarrow e$. Choose $a \in A$ with $a \leq p \wedge s, c, e$ (downward-directedness); then $c, e \in [a]$, so $c \Rightarrow e \in B([a])$ by Remark 2, and Corollary 2 gives $c \wedge_a (c \Rightarrow e) \leq e$. Since $p \wedge s$ is a lower bound of both c and $c \Rightarrow e$,

$$p \wedge s \leq c \wedge_a (c \Rightarrow e) \leq e,$$

so $e \in G$ via p .

Minimality. Let H be any sB-filter with $F \cup \{s\} \subseteq H$. If $c \in G$ via $p \wedge s \leq c$ with $p \in F \subseteq H$, then $p, s \in H$; meet-closure of H gives $p \wedge s \in H$, and upward closure gives $c \in H$. Hence $G \subseteq H$. $\square$

Remark 8: *The meet-based description of Proposition 7 is what makes the prime extension theorem below unconditional: because G is described through $p \wedge s$ rather than through $s \Rightarrow c$, no hypothesis is needed on how $\vee$ and $\Rightarrow$ interact at the point where a prime filter is enlarged by a single element. Set-theoretically, G is the upward closure, inside the section fixed by downward-directedness, of $\{p \wedge s : p \in F\}$; this is the sB-filter analogue of the standard construction of a generated filter in a distributive lattice by adjoining one element and closing under the meet with the existing filter.*

4.5 The sB-Filter Lattice

Theorem 5: *The family $\text{Fil}(A)$ of sB-filters of A , ordered by inclusion, is a complete lattice: for any family $\{F_i\}_{i \in I} \subseteq \text{Fil}(A)$, the meet is $\bigcap_i F_i$ and the join is the sB-filter generated by $\bigcup_i F_i$.*

Proof: Closure under arbitrary intersections is immediate; A is the top element. The join of a family $\{F_i\}_{i \in I}$ is the sB-filter generated by $\bigcup_i F_i$. $\square$

4.6 Prime sB-Filters and Separation

Definition 11: *A proper sB-filter $P \subsetneq A$ is prime if $x \vee y \in P$ implies $x \in P$ or $y \in P$. The prime spectrum $\text{Spec}(A)$ is the set of all prime sB-filters of A .*

Establishing that every sB-filter extends to a prime sB-filter requires combining information coming from two different sections (those generated by x and by y) with the section containing the join $x \vee y$. Because Proposition 7 characterises generated filters through a sectional *meet* rather than an implication,

this combination can be carried out directly through the distributivity of a single common section (Lemma 5), with no further hypothesis relating $\vee$ and $\Rightarrow$.

Theorem 6 (Prime extension): *Let A be downward directed, F an sB-filter of A , and $b \notin F$. Then there exists a prime sB-filter P with $F \subseteq P$ and $b \notin P$.*

Proof: Let $S = \{G \in \text{Fil}(A) : F \subseteq G, b \notin G\}$. Then $F \in S$, so $S \neq \emptyset$.

Zorn's lemma applies. Let C be a nonempty chain in S and $U = \bigcup C$. Each clause of Definition 9 involves only finitely many elements at a time: clause (1) involves none, clauses (2) and (3) involve two elements of U , and clause (4) involves two elements of U together with a lower bound. Given any such finite set of elements of U , each lies in some member of the chain C , and since C is totally ordered by inclusion, all of them lie in a single member $G_0 \in C$; the relevant clause holds in G_0 , and its consequence is again an element of $G_0 \subseteq U$. Hence U satisfies all four clauses and is an sB-filter containing F . Since $b \notin G$ for every $G \in C$, also $b \notin U$, so $U \in S$ is an upper bound for C in S . By Zorn's lemma, S has a maximal element P .

P is prime. Suppose $x \vee y \in P$ but $x \notin P$ and $y \notin P$. Since $x \notin P$, the sB-filter $\langle P \cup \{x\} \rangle$ generated by $P \cup \{x\}$ strictly contains P ; by maximality of P in S , this larger filter cannot itself lie in S , so it must contain b . By Proposition 7, $\langle P \cup \{x\} \rangle = \{c \in A : \exists p \in P, p \wedge x \leq c\}$, so there is $p_x \in P$ with $p_x \wedge x \leq b$. Symmetrically there is $p_y \in P$ with $p_y \wedge y \leq b$.

Put $p = p_x \wedge p_y \in P$ (meet-closure of P). Since $p \leq p_x, p_y$, monotonicity of the meet gives $p \wedge x \leq p_x \wedge x \leq b$ and $p \wedge y \leq p_y \wedge y \leq b$. Lemma 5 then gives $p \wedge (x \vee y) \leq b$. Since $p \in P$ and $x \vee y \in P$, meet-closure gives $p \wedge (x \vee y) \in P$, and upward closure (as $p \wedge (x \vee y) \leq b$) gives $b \in P$, contradicting $P \in S$. Hence $x \in P$ or $y \in P$, and P is prime. $\square$

Theorem 7 (Separation): *Let A be downward directed. If $a \not\leq b$ then there exists a prime sB-filter containing a but not b .*

Proof: Since $a \not\leq b$, $b \notin [a] = \langle a \rangle$ (Proposition 6). Apply Theorem 6 with $F = [a]$. $\square$

4.7 Prime Spectrum Embedding

Theorem 8: *Let A be downward directed. The map $a \mapsto \hat{a} = \{P \in \text{Spec}(A) : a \in P\}$ is an order-embedding of $(A, \leq)$ into $(2^{\text{Spec}(A)}, \subseteq)$.*

Proof: If $a \leq b$, upward closure gives $\hat{a} \subseteq \hat{b}$. If $a \not\leq b$, Theorem 7 yields $P \in \hat{a} \setminus \hat{b}$, so $\hat{a} \not\subseteq \hat{b}$. $\square$

4.8 Congruences Induced by sB-Filters

Definition 12: *For an sB-filter F , define $\Theta_F = \{(x, y) \in A^2 : x \Rightarrow y \in F \text{ and } y \Rightarrow x \in F\}$.*

Theorem 9: *Let A be downward directed and F an sB-filter. Then Θ_F is a congruence on the semi-B-algebra reduct $(A, \Rightarrow, 1)$.*

Proof: *Reflexivity.* $x \Rightarrow x = 1 \in F$.

Symmetry. The definition is symmetric.

Transitivity. $x \Rightarrow y, y \Rightarrow z \in F$ implies $x \Rightarrow z \in F$ by Lemma 4; symmetrically $z \Rightarrow x \in F$.

Compatibility with $\Rightarrow$. Let $(x, u), (y, v) \in \Theta_F$. By downward directedness choose $d \in A$ with $d \leq x, u, y, v$. Lemma 3 applied to $u, x, y \in [d]$ gives $(u \Rightarrow x) \Rightarrow ((x \Rightarrow y) \Rightarrow (u \Rightarrow y)) = 1$. Since $u \Rightarrow x \in F$ (from $(x, u) \in \Theta_F$), two implicative closures yield $(x \Rightarrow y) \Rightarrow (u \Rightarrow y) \in F$. Lemma 3 applied to $u, y, v \in [d]$ then gives $(y \Rightarrow v) \Rightarrow ((u \Rightarrow y) \Rightarrow (u \Rightarrow v)) = 1$. Since $y \Rightarrow v \in F$ (from $(y, v) \in \Theta_F$), two implicative closures give $(u \Rightarrow y) \Rightarrow (u \Rightarrow v) \in F$. Lemma 4 applied to these two elements of F yields $(x \Rightarrow y) \Rightarrow (u \Rightarrow v) \in F$; by symmetry in (x, u) and (y, v) , $(u \Rightarrow v) \Rightarrow (x \Rightarrow y) \in F$. Hence $(x \Rightarrow y, u \Rightarrow v) \in \Theta_F$. $\square$

Remark 9: Extending Θ_F to a congruence on the full algebra $(A, \vee, \Rightarrow, 1)$ requires some further condition linking $\vee$ and $\Rightarrow$; this remains open in general.

Corollary 3: Let A be downward directed and F an sB-filter. The quotient $(A/\Theta_F, \Rightarrow, [1])$ with $[x] \Rightarrow [y] := [x \Rightarrow y]$ is a semi-B-algebra.

Proof: Well-definedness follows from Theorem 9, and each of (B1)–(B3) passes to the quotient. $\square$

5 Examples

Example 2 (Four-element Boolean algebra as SSBN-algebra): Let $A = \{0, a, b, 1\}$ be the four-element Boolean algebra with $a \wedge b = 0$, $a \vee b = 1$, and $x \Rightarrow y := \neg x \vee y$. Since $B(A) = A$, A is an Epstein–Horn B-algebra and an SSBN-algebra (Proposition 4). A has global bottom 0, so it is downward directed (Definition 10), and Theorem 6 applies to it directly.

Example 3 (A semi-B-algebra that is not a B-algebra, and not in NsB): Let $A = \{0, 1\}$ with $0 < 1$ and the implication:

$\Rightarrow$	0	1
0	1	0
1	0	1

Here $B(A) = \{0, 1\}$. Axiom (B1) holds ($0 \Rightarrow 0 = 1$, $1 \Rightarrow 1 = 1$). Table 2 verifies (B2) and (B3) for all $b \in \{0, 1\}$, confirming $A \in \mathbf{sBalg}$. A has global bottom 0, so it is downward directed.

x	b	$x \Rightarrow b$	$x \wedge (x \Rightarrow b)$	$x \wedge b$
0	0	1	0	0 ✓
0	1	0	0	0 ✓
1	0	0	0	0 ✓
1	1	1	1	1 ✓

Verification of (B2).

b	x	y	$b \wedge (x \Rightarrow y)$	$b \wedge x$	$(b \wedge x) \Rightarrow (b \wedge y)$
0	.	.	0	0	1 ✓
1	0	0	1	0	1 ✓
1	0	1	0	0	0 ✓
1	1	0	0	1	0 ✓
1	1	1	1	1	1 ✓

Verification of (B3).

Table 2: Verification of (B2)–(B3) in Example 3.

A is not a B-algebra: $0 \leq 1$ but $0 \Rightarrow 1 = 0 \neq 1$, contradicting Theorem 3.17(ii) of [24]. As shown in Corollary 4 below, A also fails the sectional adjunction (SSBN3) and so is not an SSBN-algebra; the discussion of NsB-membership below concerns only the semi-B-algebra structure. Moreover, $A \notin \text{NsB}$: axiom (NsB1) fails for $x = 0, y = 1$:

$$y \vee (x \Rightarrow y) = 1 \vee 0 = 1 \neq 0 = x \Rightarrow y.$$

This witnesses $\mathbf{sBalg} \not\subseteq \text{NsB}$ in Theorem 4.

Corollary 4 (SSBN $\subsetneq$ DNSB): *Regard $A = \{0, 1\}$ of Example 3 as a distributive nearlattice with global bottom 0, so that A has the single section $[0] = A$. Then A is a DNSB-algebra that fails (SSBN3), hence is not an SSBN-algebra; consequently SSBN $\subsetneq$ DNSB. In particular, (SSBN1) and (SSBN2) together do not imply (SSBN3).*

Proof: (SSBN1) holds since A is a two-element lattice. (SSBN2) holds in the single section $[0] = A$ by Table 2. Hence A is a DNSB-algebra. To see (SSBN3) fails, take $x = 0, z = 1, b = 1 \in B(A)$. Then $x \wedge_0 b = 0 \leq 1 = z$, so the $(\Rightarrow)$ direction of (SSBN3) requires $1 \leq 0 \Rightarrow 1 = 0$, which is false. $\square$

Example 4 (A near-semi-B-algebra not in $\mathbf{sBalg}$): *Let $A = \{a, b, 1\}$ with $a < 1, b < 1, a$ and b incomparable, $a \vee b = 1$. No global meet $a \wedge b$ exists, so A is not a lattice and not in $\mathbf{sBalg}$. The upsets $[a] = \{a, 1\}$ and $[b] = \{b, 1\}$ are two-element Boolean algebras. Define:*

$\Rightarrow$	a	b	1
a	1	b	1
b	a	1	1
1	a	b	1

Table 3 verifies axioms (NsB1)–(NsB4).

Axiom	Check
(NsB1): $y \leq x \Rightarrow y$	$a \leq b \Rightarrow a = a; b \leq a \Rightarrow b = b$; all cases with $y = 1$ hold since $\Rightarrow y = 1$.
(NsB2): $x \Rightarrow x = 1$	$a \Rightarrow a = 1; b \Rightarrow b = 1; 1 \Rightarrow 1 = 1$.
(NsB3): $1 \Rightarrow x = x$	$1 \Rightarrow a = a; 1 \Rightarrow b = b$.
(NsB4): $(x \vee y) \Rightarrow y = x \Rightarrow y$	$(a \vee b) \Rightarrow b = 1 \Rightarrow b = b = a \Rightarrow b; (a \vee b) \Rightarrow a = 1 \Rightarrow a = a = b \Rightarrow a$.

Table 3: Verification of (NsB1)–(NsB4) in Example 4.

The restrictions to $[a]$ and $[b]$ are Boolean implications and satisfy the sectional adjunction (NsB5). Hence $A \in \text{NsB}$, and Proposition 2 gives $A \in \text{SSBN}$. Since a and b have no common lower bound, A is not downward directed (Definition 10), so the filter and congruence theory of Section 4 does not apply to it; this example witnesses only $\text{NsB} \not\subseteq \mathbf{sBalg}$ in Theorem 4, since A, not being a lattice, lies outside $\mathbf{sBalg}$.

Example 5 (Three-element chain): Let $A = \{0, m, 1\}$ with $0 < m < 1$ and the Heyting chain implication: $x \Rightarrow y = 1$ if $x \leq y$, and $x \Rightarrow y = y$ otherwise. Then A is a B-algebra and an SSBN-algebra. Being a chain, A is downward directed: for any finite $S \subseteq A$, $\min(S)$ is a lower bound, so Theorem 6 applies to it unconditionally.

The congruence $\Theta_{\{m,1\}}$ does not relate 0 and m ($m \Rightarrow 0 = 0 \notin \{m, 1\}$), illustrating that Θ_F can be non-trivial. The quotient $A/\Theta_{\{1\}} = A$ shows Corollary 3 can act trivially.

6 Conclusion and Open Problems

Sectional semi-B-nearlattices (SSBN-algebras) carry an Epstein–Horn semi-B-algebra structure sectionwise on every principal upset, with the Birkhoff-center-based adjunction holding in each section. Remark 2, that $x \Rightarrow z \in B([a])$ for all $x, z \in [a]$, underlies the sectional order theory of Section 3 and the filter and congruence theory of Section 4; this membership is part of what (SSBN3) postulates, not a consequence of (B1)–(B3) alone, and Corollary 4 shows it fails when (SSBN3) is dropped. Theorem 1 and Remark 6 make explicit that (SSBN3) says nothing about $x \Rightarrow y$ once x, y lack a common lower bound; the order-theoretic results of Section 3 are accordingly stated sectionally, and the filter/congruence theory of Section 4 is developed under an explicit downward-directedness hypothesis rather than claimed for every SSBN-algebra.

A further outcome of the filter theory is that the naive implication-based description of a generated sB-filter fails whenever the witnessing element is not a Birkhoff element (Example 1). Replacing it with the order-theoretic, meet-based description of Proposition 7 both repairs the argument and removes a join-compatibility hypothesis that would otherwise be needed for the prime extension theorem: primality follows directly from the distributivity of a single common section (Lemma 5).

The main results are as follows.

- (1) $\mathbf{NsB} \subseteq \mathbf{SSBN} \subsetneq \mathbf{DNSB}$ holds with the last inclusion strict; $\mathbf{sBalg}$ and $\mathbf{NsB}$ are incomparable (Propositions 2, 3; Theorem 4; Corollary 4; Examples 3, 4). Strictness of $\mathbf{NsB} \subseteq \mathbf{SSBN}$ is open (Remark 5).
- (2) Every upset $[a]$ is a semi-B-algebra with a unique sectional Epstein–Horn implication, and the relation $x \Rightarrow y = 1$ agrees with the nearlattice order within any single section (Theorem 2; Theorem 3; Theorem 1).
- (3) For downward-directed A , implicative semi-B-filters form a complete lattice, and generated filters admit an order-theoretic, meet-based description that repairs an implication-based formula shown to fail in general (Theorem 5; Proposition 7; Example 1). Every proper sB-filter extends unconditionally to a prime sB-filter (Theorem 6).
- (4) For downward-directed A , the prime spectrum embeds $(A, \leq)$ into a powerset lattice, again without extra hypotheses (Theorem 8).
- (5) For downward-directed A , every sB-filter induces a semi-B-algebra congruence whose quotient is a semi-B-algebra (Theorem 9; Corollary 3).

The following questions remain open.

- (1) **Is $\mathbf{NsB} \subsetneq \mathbf{SSBN}$?** A separating example requires an SSBN-algebra failing (NsB4); a candidate construction would adapt the free algebra over the SSBN axioms on a small generating set and check (NsB4) directly.

- (2) **Does the downward-directedness hypothesis of Section 4 admit a genuine non-lattice example?** As noted in Remark 6, a finite downward-directed join-semilattice is automatically a lattice, so a directed, non-lattice SSBN-algebra (if one exists) would have to be infinite; constructing one, or proving none exists, would clarify how much of Section 4 extends to the non-lattice setting of Example 4.
- (3) **Does Θ_F extend to a congruence on $(A, \vee, \Rightarrow, 1)$?** See Remark 9.
- (4) **Is there a Stone-type duality for SSBN-algebras?**
- (5) **What are the subdirectly irreducible SSBN-algebras?**
- (6) **Does the semi-B setting admit a sequent calculus whose Lindenbaum–Tarski algebras are SSBN-algebras?**
- (7) **What is the connection with sectionally pseudocomplemented posets [8]?** Both weaken global complementation to a sectional condition; a formal comparison is left for future work.
- (8) **Is there a genuine DNSB-algebra with $|B([a])| > 2$ on some section that fails (SSBN3)?** Corollary 4 treats only the Birkhoff-trivial case; a larger example would sharpen the SSBN–DNSB boundary.
- (9) **Do the open problems of [23] simplify in the Epstein–Horn semi-B setting?**
- (10) **Applications to algebraic semantics.** Section 1 sketches a logical reading in which implication encodes deduction relative to a distinguished Boolean sublanguage on each section; beyond the Lindenbaum–Tarski question above, natural non-classical logics whose algebraic models are (downward-directed) SSBN-algebras remain to be located, for instance arising from partial or paraconsistent reasoning where only a designated classical fragment supports full residuation.

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1. G. Birkhoff, *Lattice Theory*, 3rd ed., American Mathematical Society Colloquium Publications, Vol. 25, American Mathematical Society, Providence, RI, **(1967)**.
2. S. A. Celani and D. Montangie, Hilbert algebras with supremum, *Algebra Universalis* 67(3) **(2012)**, 237–255. <https://doi.org/10.1007/s00012-012-0178-z>
3. S. A. Celani and R. Jansana, Bounded distributive lattices with strict implication, *Mathematical Logic Quarterly* 51(3) **(2005)**, 219–246. <https://doi.org/10.1002/malq.200410022>
4. S. Celani and H. J. San Martín, On the implicative-infimum subreducts of weak Heyting algebras, *Mathematical Logic Quarterly* 70(2) **(2024)**, 178–196. <https://doi.org/10.1002/malq.202300021>
5. S. Celani, A. Nagy and W. Zuluaga Botero, Bounded distributive lattices with strict implication and weak difference, *Archive for Mathematical Logic* 64(3) **(2025)**, 387–422. <https://doi.org/10.1007/s00153-024-00945-2>
6. I. Chajda and M. Kolařík, Nearlattices, *Discrete Mathematics* 308(21) **(2008)**, 4906–4913. <https://doi.org/10.1016/j.disc.2007.09.009>
7. I. Chajda and M. Kolařík, Ideals, congruences and annihilators on nearlattices, *Acta Universitatis Palackianae Olomucensis, Facultas Rerum Naturalium, Mathematica* 46 **(2007)**, 25–33.
8. J. Cîrulis, Implication in sectionally pseudocomplemented posets, *Acta Scientiarum Mathematicarum (Szeged)* 74(3–4) **(2008)**, 477–491.
9. A. Diego, *Sur les algèbres de Hilbert*, Collection de Logique Mathématique, Série A, No. 21, Gauthier-Villars, Paris, **(1966)**.
10. G. Epstein, The lattice theory of Post algebras, *Transactions of the American Mathematical Society* 95 **(1960)**, 300–317.
11. G. Epstein and A. Horn, P-algebras, an abstraction from Post algebras, *Algebra Universalis* 4(1) **(1974)**, 195–206. <https://doi.org/10.1007/BF02485725>
12. G. Epstein and A. Horn, Logics which are characterized by subresiduated lattices, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 22(1) **(1976)**, 199–210. <https://doi.org/10.1002/malq.19760220128>
13. N. Galatos, P. Jipsen, T. Kowalski and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logic and the Foundations of Mathematics, Vol. 151, Elsevier, Amsterdam, **(2007)**.
14. L. J. González and I. Calomino, Finite distributive nearlattices, *Discrete Mathematics* 344(9) **(2021)**, Article 112511. <https://doi.org/10.1016/j.disc.2021.112511>
15. L. J. González and M. B. Lattanzi, Congruences on near-Heyting algebras, *Algebra Universalis* 79(4) **(2018)**, Article 78. <https://doi.org/10.1007/s00012-018-0560-6>
16. L. J. González, M. B. Lattanzi, I. Calomino and S. A. Celani, Characterizations of near-Heyting algebras, *European Journal of Mathematics* 9(3) **(2023)**, Article 68. <https://doi.org/10.1007/s40879-023-00663-8>
17. G. Grätzer, *General Lattice Theory*, Academic Press, New York, **(1978)**.
18. P. Köhler, Brouwerian semilattices: The lattice of total subalgebras, In: *Universal Algebra and Applications*, Banach Center Publications, Vol. 9, PWN, Warsaw, **(1982)**, 47–56.
19. F. Pérez-Gámez and C. Bejines, An exploration of weak Heyting algebras: Characterization and properties, *International Journal of Approximate Reasoning* 179 **(2025)**, Article 109365. <https://doi.org/10.1016/j.ijar.2025.109365>
20. H. Rasiowa, *An Algebraic Approach to Non-Classical Logics*, Studies in Logic and the Foundations of Mathematics, Vol. 78, North-Holland, Amsterdam, **(1974)**.
21. S. Celani and I. Calomino, Stone style duality for distributive nearlattices, *Algebra Universalis* 71(2) **(2014)**, 127–153. <https://doi.org/10.1007/s00012-014-0269-0>
22. M. Ward and R. P. Dilworth, Residuated lattices, *Transactions of the American Mathematical Society* 45 **(1939)**, 335–354. <https://doi.org/10.1090/S0002-9947-1939-1501995-3>

23. N. K. Kakumanu, B. Tharuni, S. S. Raju and B. Surendranath Reddy, Sectional Hilbert nearlattices: Filters, congruences and sectional residuation, *Preprint* (2026). <https://doi.org/10.13140/RG.2.2.29167.39847>
24. N. K. Kakumanu, B. Tharuni, D. A. Romano and G. C. Rao, Semi B-algebra, *Bulletin of the International Mathematical Virtual Institute* 14(1) (2024), 13–22.

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