

Research article

The Exponentiated Cosine Gompertz Distribution: Properties, Estimation, and Application to Lifetime Data

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Abstract: The cosine Gompertz distribution has been proposed as an extension of the classical Gompertz model; however, its hazard rate is restricted to an increasing shape. To overcome this limitation, we introduce the Exponentiated Cosine Gompertz Distribution (ECGD) by applying the exponentiated family approach to the cosine Gompertz distribution. The proposed model offers greater flexibility in describing different hazard-rate patterns, including decreasing, increasing, and bathtub-shaped behaviors. Some mathematical properties of the ECGD are derived, and its parameters are estimated using the methods of maximum likelihood, maximum product of spacings, Anderson–Darling, and Cramér–von Mises. A simulation study is conducted to assess the performance of these estimation methods, with the results indicating satisfactory estimation performance across the considered scenarios. The practical applicability of the ECGD is demonstrated using a real-world dataset, where it provides a better fit than several competing models. These results indicate that the proposed distribution is a useful and flexible model for reliability analysis.

Mathematics Subject Classification: 62G20, 65C60

Keywords: Cosine Gompertz distribution; exponentiated family; hazard function; lifetime data; parameter estimation

1 Introduction

The Gompertz distribution by [1] is a well-known lifetime model that has attracted widespread attention in survival analysis, actuarial studies, reliability engineering, and medical research. Originally introduced to describe human mortality, it has since been applied in many areas due to its ability to capture monotonically increasing hazard rates. However, this feature also restricts its applicability, since some real-world data often exhibit more complex hazard patterns such as decreasing, bathtub-shaped, or non-monotonic behaviors. This limitation has motivated the development of several extensions and modifications of the Gompertz distribution to improve its flexibility in data modeling.



Over the years, numerous generalizations of the Gompertz distribution have been proposed. These include the Lomax-Gompertz distribution proposed by [2], the alpha power Gompertz distribution presented by [3], the Marshall-Olkin Gompertz distribution by [4], the inverse Gompertz distribution introduced by [5], the transmuted Gompertz distribution by [6], the transmuted generalized Gompertz distribution defined by [7], the Marshall-Olkin extended Gompertz distribution by [8], the Nadarajah-Haghighi Gompertz distribution by [9], the power Gompertz distribution introduced by [10], the unit Gompertz distribution by [11], the Topp-Leone inverse Gompertz distribution by [12], the new extended Gompertz distribution by [14] and the sine type II Topp-Leone Gompertz distribution by [13], among others.

Each extension aims to address some limitations of the Gompertz distribution by introducing new parameters or combining it with other distributional families. However, some existing models are still limited in the hazard-rate shapes they can describe. In particular, the cosine Gompertz distribution proposed by [15] has an increasing hazard rate and cannot describe decreasing or bathtub-shaped hazard rates. This limitation highlights the need for a more flexible model capable of describing a wider range of hazard-rate shapes.

Motivated by this limitation, we propose a new lifetime model called the Exponentiated Cosine Gompertz Distribution (ECGD). This distribution is developed by applying the exponentiated family approach of [16] to the cosine Gompertz distribution, resulting in an extended model with greater flexibility. The exponentiated family provides an additional shape parameter that can improve the flexibility of a baseline distribution. By applying this approach to the cosine Gompertz distribution, the ECGD retains its desirable properties while overcoming its restriction to an increasing hazard rate.

Some properties of the proposed distribution are derived, including its density, distribution function, and moments. A notable feature of the ECGD is its hazard rate function, which is capable of capturing bathtub, decreasing, and increasing shapes. This provides a significant advantage over its predecessor, the cosine Gompertz distribution, whose hazard function is limited to the increasing form. Such flexibility makes the ECGD a more versatile tool for reliability analysis and survival data modeling, where hazard rates often vary in complex ways.

For parameter estimation, we consider a range of classical techniques including maximum likelihood estimation, maximum product of spacings, the Anderson–Darling method, and the Cramér–von Mises method. A simulation study is carried out to assess the performance of these estimators, and the results indicate that all methods provide consistent and reliable estimates. To demonstrate the practical utility of the ECGD, it is applied to an engineering dataset. The proposed model outperforms several well-known competitors, confirming its potential as a useful addition to the family of lifetime distributions.

The structure of this study is as follows: Section 2 outlines the construction of the ECGD model and presents its reliability measures. Section 3 details the linear expansion of the model's probability density function (PDF) and cumulative distribution function (CDF). Statistical properties are derived in Section 4, while Section 5 describes the parameter estimation method. The results of the simulation study are presented in Section 6. Application results are given in Section ???. Finally, concluding remarks are presented in Section ???.

2 Proposed Model Formulation

The CDF and PDF of the exponentiated family of distributions are given in equations (1) and (2) below:

$$F(x) = [G(x)]^\lambda \quad (1)$$

$$f(x) = \lambda g(x)[G(x)]^{\lambda-1}; x > 0, \quad (2)$$

where $\lambda > 0$ is a shape parameter, $g(x)$ and $G(x)$ denote the PDF and CDF of a baseline distribution, respectively.

The CDF and PDF of the Cosine Gompertz distribution are given in (3) and (4) below:

$$G(x) = 1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \right) \right\} \quad (3)$$

$$g(x) = \frac{\pi}{2} d e^{cx} e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \sin \left\{ \frac{\pi}{2} \left(1 - e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \right) \right\} \quad (4)$$

By substituting (3) and (4) into (1) and (2), we obtain the CDF and PDF of the new exponentiated Cosine Gompertz Distribution (ECGD) as follows:

$$F(x) = \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \right) \right\} \right]^\lambda \quad (5)$$

$$f(x) = \lambda \frac{\pi}{2} d e^{cx} e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \sin \left\{ \frac{\pi}{2} \left(1 - e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \right) \right\} \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{\left(-\frac{d}{c}(e^{cx}-1)\right)} \right) \right\} \right]^{\lambda-1} \quad (6)$$

The PDF plot of the ECGD is given in Figure 1, from which it can be seen that the proposed ECGD is right-skewed and unimodal.

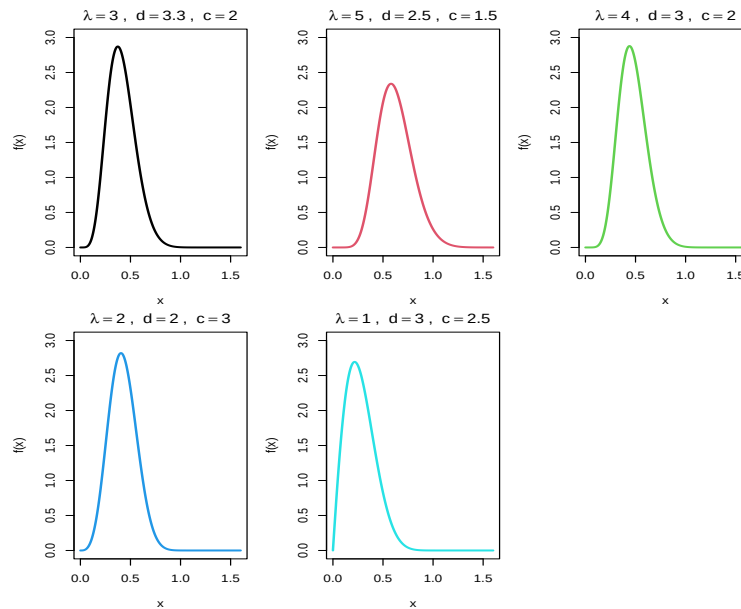


Figure 1: PDF plots of the ECGD.

2.1 Reliability Measures

The survival function of the ECGD is given as:

$$S(x) = 1 - \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^\lambda$$

The hazard function of the ECGD is given by

$$h(x) = \frac{\lambda \frac{\pi}{2} d e^{cx} e^{-\frac{d}{c}(e^{cx}-1)} \sin \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^{\lambda-1}}{1 - \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^\lambda}$$

The hazard function plots of the proposed model are given in Figure 2. From the plots, it can be seen that the proposed model exhibits increasing, decreasing, and bathtub-shaped hazard rates.

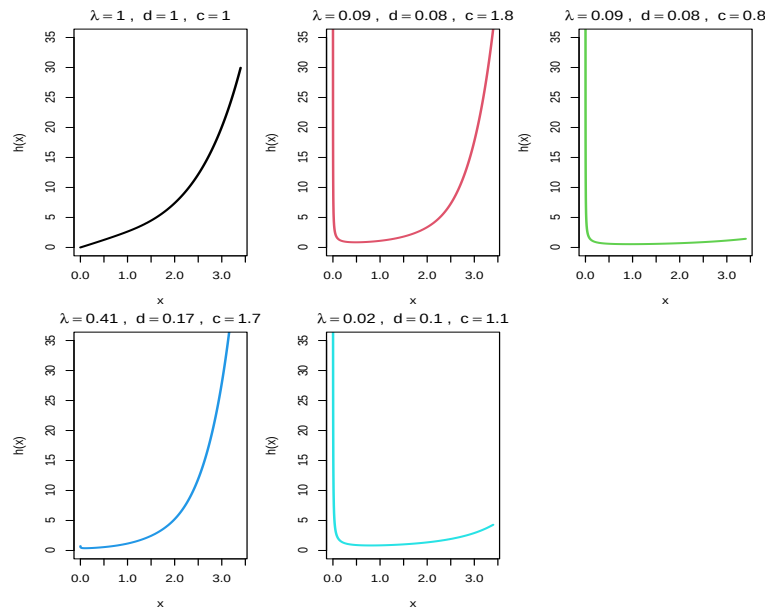


Figure 2: Hazard plots of the ECGD.

The cumulative hazard function plot of the proposed model is given as:

$$H(x) = -\ln \left\{ 1 - \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^\lambda \right\}.$$

3 Linear Expansion

The PDF and the CDF of the Exponentiated Cosine Gompertz distribution can be expanded using power series expansions as follows:

$$f(x) = \lambda \frac{\pi}{2} d e^{cx} e^{-\frac{d}{c}(e^{cx}-1)} \sin \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^{\lambda-1}.$$

Using the binomial expansion, we have

$$\left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^{\lambda-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\lambda-1}{i} \left[\cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^i.$$

For integer i , the power of the cosine function can be written as

$$(\cos z)^i = \frac{1}{2^i} \sum_{r=0}^i \binom{i}{r} \cos((i-2r)z).$$

Therefore,

$$\left[\cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^i = \frac{1}{2^i} \sum_{r=0}^i \binom{i}{r} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left[(i-2r) \frac{\pi}{2} \right]^{2j} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right)^{2j}.$$

Also,

$$\sin \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{2} \right)^{2k+1} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right)^{2k+1}.$$

Hence,

$$\left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right)^{2j+2k+1} = \sum_{l=0}^{2j+2k+1} (-1)^l \binom{2j+2k+1}{l} e^{-\frac{dl}{c}(e^{cx}-1)}.$$

Combining the above expressions gives

$$f(x) = \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{2j+2k+1} \frac{\lambda d \pi}{2^{i+1}} \frac{(-1)^{i+j+k+l}}{(2j)!(2k+1)!} \binom{\lambda-1}{i} \binom{i}{r} \binom{2j+2k+1}{l} \times \left[(i-2r) \frac{\pi}{2} \right]^{2j} \left(\frac{\pi}{2} \right)^{2k+1} e^{-\frac{d(l+1)}{c}(e^{cx}-1)} e^{cx}.$$

Using the exponential series,

$$e^{-\frac{d(l+1)}{c}(e^{cx}-1)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{d(l+1)}{c} \right)^m (e^{cx}-1)^m.$$

Furthermore,

$$(e^{cx}-1)^m = \sum_{n=0}^m (-1)^{m-n} \binom{m}{n} e^{cnx}.$$

Therefore,

$$f(x) = \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{2j+2k+1} \sum_{m=0}^{\infty} \sum_{n=0}^m A_{i,r,j,k,l,m,n} e^{c(n+1)x},$$

where

$$A_{i,r,j,k,l,m,n} = \frac{\lambda d \pi}{2^{i+1}} \frac{(-1)^{i+j+k+l+n}}{(2j)!(2k+1)!m!} \binom{\lambda-1}{i} \binom{i}{r} \binom{2j+2k+1}{l} \binom{m}{n} \times \left[(i-2r) \frac{\pi}{2} \right]^{2j} \left(\frac{\pi}{2} \right)^{2k+1} \left(\frac{d(l+1)}{c} \right)^m.$$

Hence, the reduced form of the PDF can be written as

$$f(x) = \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{2j+2k+1} \sum_{m=0}^{\infty} \sum_{n=0}^m A_{i,r,j,k,l,m,n} e^{c(n+1)x}. \quad (7)$$

We can expand the CDF as follows:

$$F(x) = \left[1 - \cos \left(\frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right) \right]^{\lambda}.$$

Using the binomial expansion,

$$F(x) = \sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i} \left[\cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right) \right\} \right]^i.$$

Using

$$(\cos z)^i = \frac{1}{2^i} \sum_{r=0}^i \binom{i}{r} \cos((i-2r)z),$$

we obtain

$$F(x) = \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{2^i (2j)!} \binom{\lambda}{i} \binom{i}{r} \left[(i-2r) \frac{\pi}{2} \right]^{2j} \times \left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right)^{2j}.$$

Also,

$$\left(1 - e^{-\frac{d}{c}(e^{cx}-1)} \right)^{2j} = \sum_{q=0}^{2j} (-1)^q \binom{2j}{q} e^{-\frac{dq}{c}(e^{cx}-1)}.$$

Using the exponential expansion,

$$e^{-\frac{dq}{c}(e^{cx}-1)} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{dq}{c} \right)^s (e^{cx} - 1)^s.$$

Furthermore,

$$(e^{cx} - 1)^s = \sum_{t=0}^s (-1)^{s-t} \binom{s}{t} e^{ctx}.$$

Therefore,

$$F(x) = \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \sum_{s=0}^{\infty} \sum_{t=0}^s B_{i,r,j,q,s,t} e^{ctx},$$

where

$$B_{i,r,j,q,s,t} = \frac{(-1)^{i+j+q+t}}{2^i(2j)!s!} \binom{\lambda}{i} \binom{i}{r} \binom{2j}{q} \binom{s}{t} \times \left[(i-2r) \frac{\pi}{2} \right]^{2j} \left(\frac{dq}{c} \right)^s.$$

Hence, the reduced form of the CDF can be written as

$$F(x) = \sum_{i=0}^{\infty} \sum_{r=0}^i \sum_{j=0}^{\infty} \sum_{q=0}^{2j} \sum_{s=0}^{\infty} \sum_{t=0}^s B_{i,r,j,q,s,t} e^{ctx}. \quad (8)$$

The expressions in equations (7) and (8) are the reduced forms of the new Exponentiated Cosine Gompertz Distribution.

4 Statistical Properties

4.1 Quantile Function

The quantile function of the proposed model is given as:

$$\phi(u) = \frac{1}{c} \left\{ \log \left[\frac{-c}{d} \left(\log \left(1 - \frac{\cos^{-1}(1 - u^{\frac{1}{\lambda}})}{\frac{\pi}{2}} \right) \right) + 1 \right] \right\}$$

4.2 Moments

Moments are fundamental statistical measures that provide insights into the shape, central tendency, and variability of a probability distribution. The r th moment of the proposed model can be obtained using the following expression:

$$\mu'_r = E(x^r) = \int_0^\infty x^r f(x) dx$$

5 Methods of Estimation

This section presents the methods used to estimate the parameters of the ECGD.

5.1 Maximum likelihood estimation

Let $x_1, x_2, \dots, x_n$ be a random sample from the proposed distribution with parameter vector $\Theta = (\lambda, c, d)^\top$. The log-likelihood function is given by

$$\begin{aligned} \ell(\Theta) = & n \log \lambda + n \log \left(\frac{\pi}{2} \right) + n \log d + c \sum_{i=1}^n x_i - \frac{d}{c} \sum_{i=1}^n (e^{cx_i} - 1) \\ & + \sum_{i=1}^n \log \left[\sin \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx_i} - 1)} \right) \right\} \right] \\ & + (\lambda - 1) \sum_{i=1}^n \log \left[1 - \cos \left\{ \frac{\pi}{2} \left(1 - e^{-\frac{d}{c}(e^{cx_i} - 1)} \right) \right\} \right]. \end{aligned} \quad (9)$$

The maximum likelihood estimates of λ , c , and d are obtained by solving the nonlinear likelihood equations

$$\frac{\partial \ell}{\partial \lambda} = 0, \quad \frac{\partial \ell}{\partial c} = 0, \quad \frac{\partial \ell}{\partial d} = 0.$$

Since these equations do not admit closed-form solutions, the maximum likelihood estimates are obtained numerically using an iterative optimization algorithm.

5.2 Method of maximum product of spacings

An effective alternative to the maximum likelihood estimation (MLE) method is the maximum product of spacings (MPS) technique. This approach, referred to as maximum product of spacings estimation (MPSE),

involves maximizing the product of the spacings between the ordered data points. For the proposed model, the MPS estimates of the parameters are obtained by maximizing the following function:

$$MPS = \frac{1}{n+1} \sum_{i=1}^{n+1} \log I_i(x_i)$$

where, $I_i(x_i) = F(x_{i:n}) - F(x_{i-1:n})$ and $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

5.3 Method of Anderson-Darling

The Anderson-Darling (AD) method is a widely used estimation technique based on goodness-of-fit principles. It places more emphasis on discrepancies in the tails of the distribution by comparing the empirical and theoretical cumulative distribution functions (CDFs). This characteristic makes it particularly useful in applications where tail behavior is of interest. For the proposed distribution, the AD estimates of the parameters are obtained by minimizing the following function:

$$A(x_i) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{n-i+1:n})].$$

5.4 Method of Cramér-von Mises

The Cramér-von Mises (CVM) estimation method determines the parameters of a theoretical distribution by minimizing the CVM statistic, which quantifies the discrepancy between the empirical and theoretical CDFs. For the ECGD model, the CVM estimates of the parameters are obtained by minimizing the following function:

$$C(x_i) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2.$$

6 Simulation

A simulation study was conducted to evaluate the performance of the parameter estimators for the Exponentiated Cosine Gompertz Distribution (ECGD). Two sets of true parameter values were considered: $(d = 0.5, c = 0.5, \lambda = 0.5)$ and $(d = 1.5, c = 1.7, \lambda = 2.2)$. Random samples of sizes $n = 30, 60, 90, 120, 150, 180, 230, 300, 350$, and 450 were generated. The model parameters were estimated using four estimation methods: Maximum Likelihood Estimation (MLE), Maximum Product of Spacings (MPS), Anderson-Darling (AD), and Cramér-von Mises (CVM).

Tables 1 and 2 present the bias and mean squared error (MSE) of the parameter estimates for the two parameter settings. The results show that both the bias and MSE generally decrease as the sample size increases, indicating that all four estimators are consistent. Overall, the MLE and MPS methods produce smaller bias and MSE than the AD and CVM methods, particularly for moderate and large sample sizes.

Figures 3 and 4 provide graphical comparisons of the bias and MSE for the three parameters (d, c, λ) under the four estimation methods. The plots confirm the numerical results reported in the tables, showing that both bias and MSE decrease steadily as the sample size increases. In general, the bias and MSE of the MLE and MPS estimators converge more rapidly toward zero than those of the AD and CVM estimators.

Table 1: Bias and mean squared error (MSE) of the ECGD parameter estimates for $(d, c, \lambda) = (0.5, 0.5, 0.5)$.

Method	n	Bias($\hat{d}$)	Bias($\hat{c}$)	Bias($\hat{\lambda}$)	MSE($\hat{d}$)	MSE($\hat{c}$)	MSE($\hat{\lambda}$)
MLE	30	0.2486	0.3063	0.1665	0.1071	0.1735	0.0625
	60	0.1650	0.1901	0.1034	0.0462	0.0587	0.0207
	90	0.1336	0.1514	0.0833	0.0290	0.0372	0.0124
	120	0.1168	0.1391	0.0694	0.0221	0.0322	0.0082
	150	0.1062	0.1229	0.0630	0.0176	0.0244	0.0065
	180	0.0941	0.1081	0.0572	0.0142	0.0187	0.0054
	230	0.0832	0.0931	0.0513	0.0113	0.0142	0.0044
	300	0.0705	0.0792	0.0428	0.0079	0.0103	0.0029
	350	0.0667	0.0774	0.0405	0.0074	0.0098	0.0028
	450	0.0604	0.0678	0.0373	0.0058	0.0073	0.0021
MPS	30	0.2407	0.3072	0.1579	0.0914	0.1557	0.0492
	60	0.1687	0.2014	0.1003	0.0465	0.0666	0.0168
	90	0.1380	0.1590	0.0788	0.0310	0.0412	0.0106
	120	0.1202	0.1403	0.0717	0.0237	0.0310	0.0085
	150	0.1003	0.1157	0.0615	0.0164	0.0214	0.0062
	180	0.0944	0.1095	0.0558	0.0142	0.0185	0.0050
	230	0.0831	0.0952	0.0503	0.0109	0.0138	0.0041
	300	0.0770	0.0867	0.0459	0.0095	0.0117	0.0033
	350	0.0678	0.0766	0.0402	0.0071	0.0091	0.0026
	450	0.0606	0.0697	0.0368	0.0059	0.0077	0.0022
AD	30	0.2704	0.3338	0.1747	0.1222	0.2011	0.0676
	60	0.1820	0.2153	0.1138	0.0552	0.0784	0.0243
	90	0.1537	0.1731	0.0946	0.0395	0.0474	0.0155
	120	0.1320	0.1524	0.0790	0.0291	0.0371	0.0109
	150	0.1143	0.1318	0.0694	0.0209	0.0271	0.0083
	180	0.1051	0.1230	0.0632	0.0178	0.0244	0.0065
	230	0.0934	0.1096	0.0551	0.0144	0.0190	0.0050
	300	0.0831	0.0947	0.0500	0.0105	0.0141	0.0040
	350	0.0756	0.0914	0.0450	0.0095	0.0134	0.0033
	450	0.0665	0.0793	0.0394	0.0070	0.0099	0.0024
CVM	30	0.3275	0.4307	0.2310	0.1807	0.3532	0.1267
	60	0.2235	0.3041	0.1383	0.0799	0.1593	0.0357
	90	0.1811	0.2426	0.1097	0.0534	0.0966	0.0216
	120	0.1553	0.1931	0.0921	0.0394	0.0586	0.0153
	150	0.1461	0.1840	0.0848	0.0350	0.0526	0.0130
	180	0.1262	0.1655	0.0762	0.0260	0.0452	0.0098
	230	0.1179	0.1477	0.0686	0.0226	0.0356	0.0081
	300	0.1019	0.1295	0.0602	0.0163	0.0264	0.0059
	350	0.0964	0.1205	0.0555	0.0152	0.0230	0.0051
	450	0.0833	0.1013	0.0482	0.0114	0.0165	0.0039

Table 2: Bias and mean squared error (MSE) of the ECGD parameter estimates for $(d, c, \lambda) = (1.5, 1.7, 2.2)$.

Method	n	Bias($\hat{d}$)	Bias($\hat{c}$)	Bias($\hat{\lambda}$)	MSE($\hat{d}$)	MSE($\hat{c}$)	MSE($\hat{\lambda}$)
MLE	30	0.5746	0.7384	0.7462	0.4741	1.0012	0.6784
	60	0.4747	0.5082	0.6305	0.3165	0.4622	0.4905
	90	0.4171	0.4234	0.5692	0.2468	0.3059	0.4174
	120	0.3987	0.3985	0.5304	0.2259	0.2754	0.3743
	150	0.3750	0.3585	0.5097	0.1967	0.2139	0.3463
	180	0.3361	0.3141	0.4598	0.1609	0.1605	0.2890
	230	0.3097	0.2825	0.4304	0.1366	0.1279	0.2582
	300	0.2746	0.2489	0.3814	0.1112	0.0985	0.2109
	350	0.2693	0.2459	0.3657	0.1052	0.0943	0.1965
	450	0.2422	0.2145	0.3377	0.0869	0.0704	0.1706
MPS	30	0.6430	0.7634	0.8161	0.5668	0.9772	0.8219
	60	0.5106	0.5476	0.6549	0.3765	0.5252	0.5536
	90	0.4530	0.4539	0.5753	0.3016	0.3436	0.4456
	120	0.4108	0.4077	0.5364	0.2478	0.2778	0.3955
	150	0.3508	0.3400	0.4721	0.1914	0.1892	0.3190
	180	0.3386	0.3234	0.4518	0.1787	0.1661	0.2951
	230	0.2912	0.2814	0.3990	0.1405	0.1288	0.2485
	300	0.2671	0.2535	0.3665	0.1224	0.1051	0.2181
	350	0.2385	0.2246	0.3277	0.1022	0.0855	0.1845
	450	0.2095	0.2023	0.2948	0.0854	0.0716	0.1596
AD	30	0.6262	0.7935	0.7789	0.5589	1.1152	0.7529
	60	0.5175	0.5828	0.6852	0.3801	0.6114	0.5899
	90	0.4477	0.4619	0.6210	0.2837	0.3819	0.4863
	120	0.4244	0.4261	0.5771	0.2553	0.3119	0.4318
	150	0.3909	0.3788	0.5401	0.2135	0.2419	0.3790
	180	0.3827	0.3652	0.5192	0.2061	0.2249	0.3552
	230	0.3549	0.3322	0.4807	0.1779	0.1719	0.3110
	300	0.3183	0.2863	0.4467	0.1432	0.1298	0.2737
	350	0.3060	0.2854	0.4151	0.1337	0.1276	0.2448
	450	0.2798	0.2520	0.3842	0.1129	0.0974	0.2112
CVM	30	0.6761	0.9286	0.8592	0.6345	1.4460	0.8721
	60	0.6106	0.7583	0.7915	0.5136	1.0438	0.7535
	90	0.5590	0.6662	0.7397	0.4320	0.8231	0.6757
	120	0.4905	0.5238	0.6571	0.3311	0.4947	0.5378
	150	0.4756	0.4947	0.6375	0.3081	0.4390	0.5075
	180	0.4545	0.4827	0.6055	0.2885	0.4150	0.4780
	230	0.4239	0.4257	0.5783	0.2523	0.3232	0.4338
	300	0.3958	0.3871	0.5364	0.2172	0.2486	0.3789
	350	0.3777	0.3544	0.5150	0.1969	0.2004	0.3510
	450	0.3326	0.3044	0.4633	0.1559	0.1494	0.2943

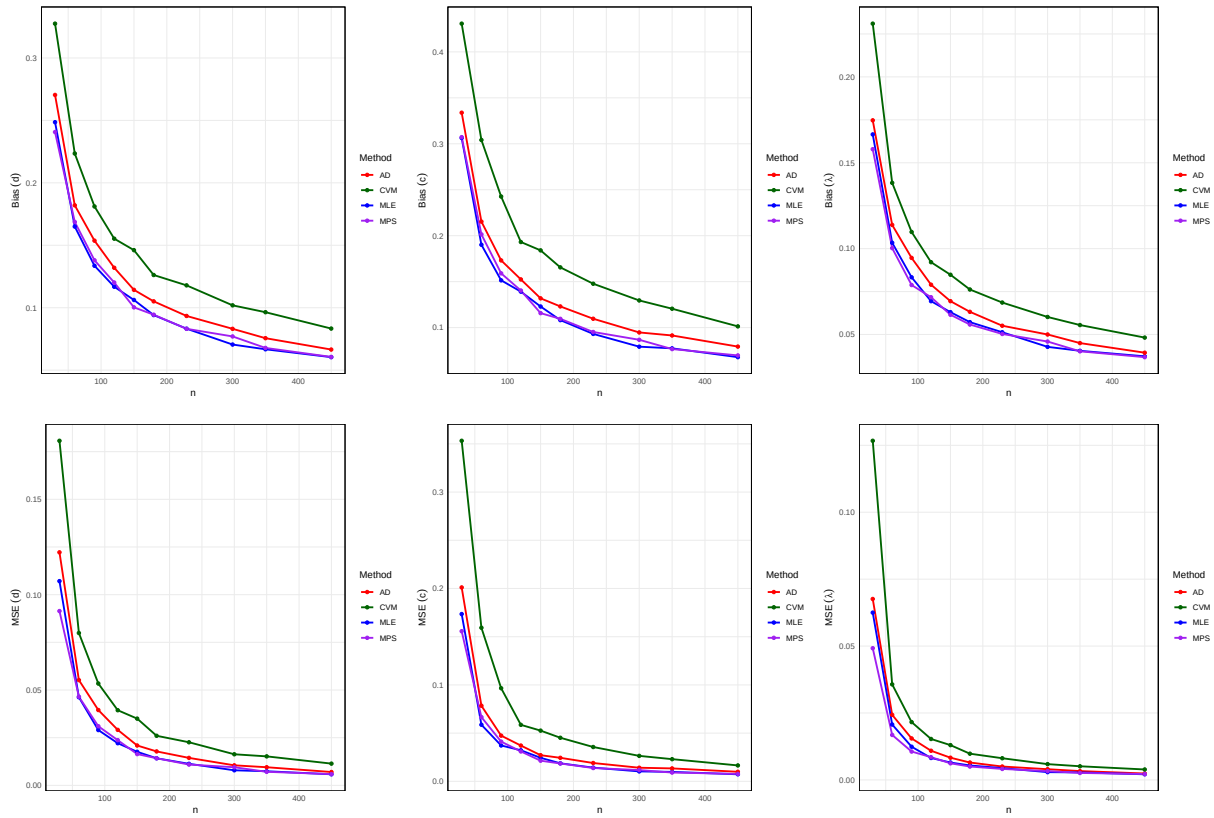


Figure 3: Bias and MSE of the ECGD parameter estimates for $(d, c, \lambda) = (0.5, 0.5, 0.5)$ under the four estimation methods. The first row shows the bias of $\hat{d}$, $\hat{c}$, and $\hat{\lambda}$, while the second row shows the corresponding MSE values.

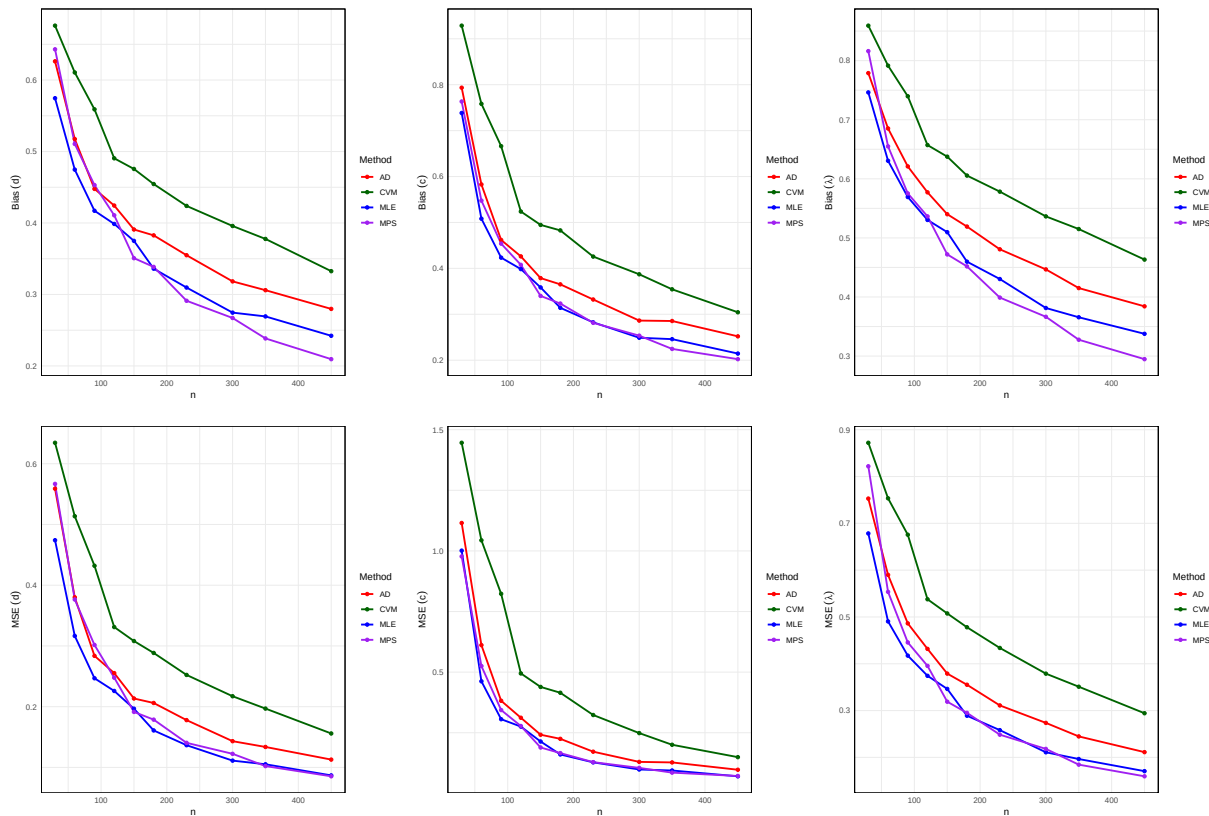


Figure 4: Bias and MSE of the ECGD parameter estimates for $(d, c, \lambda) = (1.5, 1.7, 2.2)$ under the four estimation methods. The first row shows the bias of $\hat{d}$, $\hat{c}$, and $\hat{\lambda}$, while the second row shows the corresponding MSE values.

7 Real-Life Application

In this section, the Exponentiated Cosine Gompertz distribution (ECGD) was applied to a real-life dataset to test its performance. The model was compared with three other distributions: the Cosine Gompertz distribution (CGD), the Gompertz distribution (GD), and the Inverse Gompertz distribution (IGD). To check how well each model fits the data, several statistical measures were used. These include the negative log-likelihood ($-\ell$), Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICc), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), root mean square error (RMSE), Kolmogorov–Smirnov (KS) test, and its p -value. Lower values of $-\ell$, AIC, AICc, BIC, HQIC, and RMSE show a better fit, while higher KS p -values indicate that the model fits the data well.

The dataset, recently analyzed by [17] and [18], consists of the failure times (in hours) of 24 mechanical components. The observations are as follows:

30.94, 18.51, 16.62, 51.56, 22.85, 22.38, 19.08, 49.56, 17.12, 10.67, 25.43, 10.24, 27.47, 14.70, 14.10, 29.93, 27.98, 36.02, 19.40, 14.97, 22.57, 12.26, 18.14, 18.84.

Table 3: Goodness-of-fit results for the dataset.

Model	$-\ell$	AIC	AICc	BIC	HQIC	RMSE	KS	p -value
ECGD	85.7507	177.5014	178.7014	181.0355	178.4390	0.0295	0.0932	0.9725
CGD	89.3765	182.7531	183.3245	185.1092	183.3781	0.0637	0.1372	0.7068
GD	92.5368	189.0737	189.6452	191.4298	189.6988	0.0878	0.1968	0.2728
IGD	88.2103	180.4207	180.9921	182.7768	181.0457	0.0527	0.1252	0.8021

Table 3 presents the goodness-of-fit results for the competing models. The ECGD model yields the smallest values of the negative log-likelihood ($-\ell$), AIC, AICc, BIC, HQIC, RMSE, and KS statistic. In addition, it has the largest Kolmogorov–Smirnov p -value (0.9725), which is greater than the 5% significance level, indicating excellent agreement between the fitted model and the observed data.

Overall, these results demonstrate that the Exponentiated Cosine Gompertz distribution provides a better fit to the failure time data than the competing CGD, GD, and IGD models. This indicates that the ECGD distribution is a flexible and effective model for analyzing lifetime data.

8 Conclusion

This paper proposed the Exponentiated Cosine Gompertz Distribution as a new extension of the cosine Gompertz model. By introducing the exponentiated family approach, the ECGD overcomes the limitation of the cosine Gompertz distribution and allows for bathtub, decreasing, and increasing hazard functions. Important statistical properties of the new model were established, and four estimation methods were studied through simulation, each showing reliable performance. The application to an engineering dataset confirmed that the ECGD outperforms existing models, making it a valuable addition to lifetime data analysis.

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