

Research article

On topological characterizations of rough automata and their primary decomposition

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Abstract: This paper presents a topological approach to the study of rough automata. We introduce the core of a rough automaton and show that it naturally induces a topology on the state set. We show that this topology is preserved under the product of rough automata. We further establish topological characterizations of definite and possible primary subautomata in terms of regular closed sets. Based on these characterizations, we prove primary decomposition theorems for compact rough automata. Overall, this work clarifies the theoretical interplay among rough sets, topology, and automata.

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1 Introduction

The topological study of automata was initiated by Srivastava and Shukla [7,8], who introduced natural topologies on the state set of an automaton through the source, successor, and core operators. They proved that the core operator is an interior operator of the induced topology. They also obtained topological characterizations of genetically closed subsets and primary subautomata by identifying them with regular closed sets and minimal regular closed sets, respectively. In addition, they introduced the notion of compact automata and established analogues of the classical primary decomposition and homomorphism decomposition theorems for finite automata.

These ideas were subsequently extended to fuzzy automata by Srivastava and Tiwari [9,11] by introducing suitable notions of source, successor, and core operators for fuzzy automata. They induced topologies on the state set and demonstrated that genetically closed subsets and primary subautomata admit analogous topological interpretations in terms of regular closed sets. They further proved a Primary Decomposition



Theorem for compact fuzzy automata. Furthermore, Srivastava and Tiwari [10] introduced the notion of f -primary of a fuzzy automaton and present another decomposition of a fuzzy automaton having infinite state-set. Later, Tiwari *et al.* [16] introduced the concept of the bifuzzy core of a fuzzy automaton, which induces a bifuzzy topology on its state set. Using this topology, they obtained bifuzzy characterizations of submachine families, separable families, and retrievable families of fuzzy automata. These developments reveal that topology provides an effective framework for investigating the structural and decomposition properties of fuzzy automata. Later, Jin *et al.* [4], further enriched the theory of fuzzy automata with fuzzy version of source and successor operators and studied under different algebraic settings, leading to several important structural results, including characterizations of primary submachines and unique decomposition theorems for fuzzy finite automata.

Pawlak's rough set theory [6] provides another successful mathematical framework for representing vagueness and uncertainty arising from incomplete or imperfect information. Motivated by Pawlak's theory, Basu [2] introduced the concept of a rough finite-state automaton by allowing the transition function to map each state-input pair to a rough subset of the state set, and further proposed a rough automata-based recognizer capable of processing imprecise statements. Subsequent research has considerably advanced the algebraic and structural theory of rough automata. Homomorphisms, coverings, and transformation semigroups of rough automata were investigated in [13], while several product constructions, including the direct product, restricted direct product, cascade product, and wreath product of rough finite-state machines, were developed in [15]. The structural properties of rough automata, such as separatedness, connectedness, and retrievability, were further studied by Tiwari, Srivastava, and Sharan [12]. Recently, Sharan *et al.* [14] established that every rough automaton, whether finite or infinite, naturally induces topologies on its state set, thereby providing a topological framework for investigating its structural properties.

The primary objective of the present paper is to further develop the topological theory of rough automata. Motivated by the work of Srivastava and Shukla [8] on classical automata and by the topological investigations of Tiwari *et al.* [9], we introduce the notions of *core* and *primary* for rough automata. We establish that the definite and possible primaries admit natural topological characterizations in terms of regular closed sets. Furthermore, we investigate decomposition properties of compact rough automata by establishing definite and possible primary decomposition theorems, thereby extending several classical results to the setting of rough automata.

2 Preliminaries

In this section, we recall the basic concepts and notation concerning rough automata that will be used throughout the paper. We also collect several auxiliary results required in the subsequent sections. In addition, we introduce some new notions and notation that will be used in the subsequent development of the paper.

We begin with the following concept of approximation space.

Definition 2.1: [18] An **approximation space** is an ordered pair $\mathbb{A} = (\Omega, \sim)$, where Ω is a nonempty universe of discourse and $\sim$ is an equivalence relation on Ω .

For each $\omega \in \Omega$, let $[\omega]_{\sim} = \{\eta \in \Omega : \omega \sim \eta\}$ denote the equivalence class of ω with respect to $\sim$. The collection of all equivalence classes is denoted by $\Omega/\sim$.

Definition 2.2: [18] Let $\mathbb{A} = (\Omega, \sim)$ be an approximation space and let $E \subseteq \Omega$. We define two approximation operators $I_{\sim}, C_{\sim} : 2^{\Omega} \rightarrow 2^{\Omega}$ by

$$I_{\sim}(E) = \bigcup \{[\omega]_{\sim} \in \Omega/\sim : [\omega]_{\sim} \subseteq E\}, \text{ and}$$

$$C_{\sim}(E) = \bigcup \{[\omega]_{\sim} \in \Omega/\sim : [\omega]_{\sim} \cap E \neq \emptyset\}.$$

The ordered pair $\mathcal{R}_{\sim}(E) = (I_{\sim}(E), C_{\sim}(E))$ is called a **rough set**.

The boundary region of E is defined by $B_{\sim}(E) = C_{\sim}(E) \setminus I_{\sim}(E)$. A subset $E \subseteq \Omega$ is said to be *definable* (or *exact*) whenever $I_{\sim}(E) = C_{\sim}(E)$, or equivalently, $B_{\sim}(E) = \emptyset$.

We now recall the union operation of rough sets from [1].

Definition 2.3: Let $\mathcal{R}_{\sim}(E) = (I_{\sim}(E), C_{\sim}(E))$ and $\mathcal{R}_{\sim}(F) = (I_{\sim}(F), C_{\sim}(F))$ be two rough approximations in the approximation space $\mathbb{A} = (\Omega, \sim)$. The **union** of $\mathcal{R}_{\sim}(E)$ and $\mathcal{R}_{\sim}(F)$ is defined by

$$\mathcal{R}_{\sim}(E) \sqcup \mathcal{R}_{\sim}(F) = (I_{\sim}(E) \cup I_{\sim}(F), C_{\sim}(E) \cup C_{\sim}(F)).$$

Let $\mathfrak{D}_{\sim} = \{E \subseteq \Omega : I_{\sim}(E) = C_{\sim}(E)\}$ denote the family of all definable subsets of Ω with respect to the equivalence relation $\sim$. Furthermore, for every $\omega \in \Omega$, the rough approximation $([\omega]_{\sim}, [\omega]_{\sim})$ is trivially exact. Indeed, if $E = [\omega]_{\sim}$, then $I_{\sim}(E) = [\omega]_{\sim} = C_{\sim}(E)$.

Definition 2.4: [2] A **rough automaton** is a quadruple $M = (Q, \sim, X, \delta)$, where

- (i) Q is a nonempty finite set, called the **state set**;
- (ii) $\sim$ is an equivalence relation on Q ;
- (iii) X is a nonempty finite set, called the **input alphabet**; and
- (iv) $\delta : Q \times X \rightarrow \mathfrak{R}(Q, \sim)$ is a mapping, called the **rough transition map**, where $\mathfrak{R}(Q, \sim) = \{\mathcal{R}_{\sim}(E) = (I_{\sim}(E), C_{\sim}(E)) : E \subseteq Q\}$ denotes the family of all rough approximations over the approximation space $(Q, \sim)$.

For convenience, if $\delta(q, a) = \mathcal{R}_{\sim}(E)$ for some $E \subseteq Q$, then we write $\delta(q, a) = (I_{\sim}(\delta(q, a)), C_{\sim}(\delta(q, a)))$, where $I_{\sim}(\delta(q, a))$ and $C_{\sim}(\delta(q, a))$ denote the lower and upper approximation components of the rough transition $\delta(q, a)$, respectively.

Remark 2.1: Every finite automaton (Q, X, δ) may be regarded as a rough automaton by considering the identity equivalence relation $\sim$ on Q , i.e., $[p]_{\sim} = \{p\}, \forall p \in Q$. Indeed, if $\delta(q, a) = p$, then the associated rough transition map $\hat{\delta} : Q \times X \rightarrow \mathfrak{R}(Q, \sim)$ is given by $\hat{\delta}(q, a) = ([p]_{\sim}, [p]_{\sim})$. Thus, the notion of a rough automaton naturally generalizes that of a finite automaton.

Definition 2.5: [15] Let $M = (Q, \sim, X, \delta)$ be a rough automaton and let $D \in \mathfrak{D}_{\sim}$ be a definable subset of Q . The **block transition map** $\delta^D : \mathfrak{D}_{\sim} \times X \rightarrow \mathfrak{R}(Q, \sim)$ is defined by

$$\delta^D(D, a) = \bigsqcup \{\delta(q, a) : q \in B, B \in Q/\sim, B \subseteq D\}, \text{ i.e.,}$$

$$\delta^D(D, a) = \left(\bigcup_{\substack{q \in B \\ B \subseteq D}} I_{\sim}(\delta(q, a)), \bigcup_{\substack{q \in B \\ B \subseteq D}} C_{\sim}(\delta(q, a)) \right).$$

In particular, if $B \in Q/\sim$ is an equivalence class (block), then

$$\delta^D(B, a) = \bigsqcup_{q \in B} \delta(q, a) = \left(\bigcup_{q \in B} I_{\sim}(\delta(q, a)), \bigcup_{q \in B} C_{\sim}(\delta(q, a)) \right).$$

Throughout this section, X^* is the set of all words on X (i.e., finite strings of elements of X , which form a monoid under concatenation of strings) including the empty word (which we shall denote by e).

Definition 2.6: [15] Let $M = (Q, \sim, X, \delta)$ be a rough automaton. The **extended rough transition map** $\delta^* : Q \times X^* \rightarrow \mathfrak{R}(Q, \sim)$ is defined recursively as follows:

- (i) For every $q \in Q$, $\delta^*(q, e) = ([q]_{\sim}, [q]_{\sim})$, where e denotes the identity element of the free monoid X^* .
- (ii) For every $q \in Q$, $x \in X^*$ and $a \in X$, $\delta^*(q, xa) = \delta^D(\delta^*(q, x), a)$, where D is the definable subset corresponding to $\delta^*(q, x)$. Equivalently,

$$\delta^*(q, xa) = \bigsqcup \{\delta(p, a) : p \in B, B \in Q/\sim, B \subseteq \delta^*(q, x)\}, \text{ i.e.,}$$

$$\delta^*(q, xa) = \left(\bigcup_{\substack{p \in B \\ B \subseteq \delta^*(q, x)}} I_{\sim}(\delta(p, a)), \bigcup_{\substack{p \in B \\ B \subseteq \delta^*(q, x)}} C_{\sim}(\delta(p, a)) \right).$$

Definition 2.7: [15] Let $M = (Q, \sim, X, \delta)$ be a rough automaton and let $D \in \mathfrak{D}_{\sim}$. The **extended block transition map** $\delta^{*D} : \mathfrak{D}_{\sim} \times X^* \rightarrow \mathfrak{R}(Q, \sim)$ is defined by

$$\delta^{*D}(D, x) = \bigsqcup \{\delta^*(q, x) : q \in B, B \in Q/\sim, B \subseteq D\}, \text{ i.e.,}$$

$$\delta^{*D}(D, x) = \left(\bigcup_{\substack{q \in B \\ B \subseteq D}} I_{\sim}(\delta^*(q, x)), \bigcup_{\substack{q \in B \\ B \subseteq D}} C_{\sim}(\delta^*(q, x)) \right).$$

In particular, if $B \in Q/\sim$ is an equivalence class, then

$$\delta^{*D}(B, x) = \bigsqcup_{q \in B} \delta^*(q, x) = \left(\bigcup_{q \in B} I_{\sim}(\delta^*(q, x)), \bigcup_{q \in B} C_{\sim}(\delta^*(q, x)) \right).$$

The following notion of a rough automaton extends the concept of rough finite automata introduced in [2] in two directions. Specifically, the state set is allowed to be infinite, and the input alphabet is replaced by a monoid.

Definition 2.8: [15] A **rough automaton** is a quadruple $M = (Q, \sim, X, \delta)$, where

- (i) Q is a nonempty set, called the **state set**;
- (ii) $\sim$ is an equivalence relation on Q ;
- (iii) X is a monoid with identity element e , whose elements are called **input symbols**;
- (iv) $\delta : Q \times X \rightarrow \mathfrak{R}(Q, \sim)$ is a mapping, called the **rough transition map**, satisfying the following conditions:
 - (a) For every $q \in Q$, $\delta(q, e) = ([q]_{\sim}, [q]_{\sim})$,
 - (b) For every $q \in Q$ and $x, y \in X$, $\delta(q, xy) = \delta^D(\delta(q, x), y)$, where D is the definable subset corresponding to the rough approximation $\delta(q, x)$. Equivalently,

$$\delta(q, xy) = \bigsqcup \{ \delta(p, y) : p \in B, B \in Q/\sim, B \subseteq D \}, \text{ i.e.,}$$

$$\delta(q, xy) = \left(\bigcup_{\substack{p \in B \\ B \subseteq D}} I_{\sim}(\delta(p, y)), \bigcup_{\substack{p \in B \\ B \subseteq D}} C_{\sim}(\delta(p, y)) \right).$$

Definition 2.9: [13] Let $M_1 = (Q_1, \sim_1, X_1, \delta_1)$ and $M_2 = (Q_2, \sim_2, X_2, \delta_2)$ be rough automata. A **homomorphism** from M_1 to M_2 is a pair (f, g) , where $f : Q_1 \rightarrow Q_2$, $g : X_1 \rightarrow X_2$, such that

- (i) For all $p, q \in Q_1$, $p \sim_1 q \implies f(p) \sim_2 f(q)$.
- (ii) For every $q \in Q_1$ and $x \in X_1$, $f(\delta_1(q, x)) \subseteq \delta_2(f(q), g(x))$, i.e.,

$$(f(I_{\sim_1}(\delta_1(q, x))), f(C_{\sim_1}(\delta_1(q, x)))) \subseteq (I_{\sim_2}(\delta_2(f(q), g(x))), C_{\sim_2}(\delta_2(f(q), g(x)))).$$

The class of all rough automata and their homomorphisms obviously forms a category (under obvious composition of maps). We shall denote it by **RAut**.

Definition 2.10: [2] Let $M = (Q, \sim, X, \delta) \in \mathbf{RAut}$ and let $A \subseteq Q$. The **definite successor** and **possible successor** of A are defined, respectively, by

$$SU_D(A) = \{p \in Q : p \in I_-(\delta(q, x)) \text{ for some } (q, x) \in A \times X\}, \text{ and}$$

$$SU_P(A) = \{p \in Q : p \in C_-(\delta(q, x)) \text{ for some } (q, x) \in A \times X\}.$$

Definition 2.11: [12] Let $M = (Q, \sim, X, \delta) \in \mathbf{RAut}$ and let $A \subseteq Q$. The **definite source** and **possible source** of A are defined, respectively, by

$$SO_D(A) = \{p \in Q : q \in I_-(\delta(p, x)) \text{ for some } (q, x) \in A \times X\}, \text{ and}$$

$$SO_P(A) = \{p \in Q : q \in C_-(\delta(p, x)) \text{ for some } (q, x) \in A \times X\}.$$

Definition 2.12: [12] Let $M = (Q, \sim, X, \delta) \in \mathbf{RAut}$. A rough automaton $M' = (Q', \sim', X, \delta') \in \mathbf{RAut}$ is called a **definite rough subautomaton** of M if $Q' \subseteq Q$, $SU_D(Q') = Q'$, $\sim' = \sim|_{Q' \times Q'}$, and $\delta' = \delta|_{Q' \times X}$. Similarly, M' is called a **possible rough subautomaton** of M if $Q' \subseteq Q$, $SU_P(Q') = Q'$, $\sim' = \sim|_{Q' \times Q'}$, and $\delta' = \delta|_{Q' \times X}$.

Proposition 2.1: [14] Let $M = (Q, \sim, X, \delta) \in \mathbf{RAut}$. Then the possible source and possible successor operators, $SO_P, SU_P : 2^Q \rightarrow 2^Q$, are Kuratowski closure operators on Q . Consequently, they induce two saturated topologies, denoted by $T(Q)$ and $T^*(Q)$, respectively. Moreover, these topologies are dual in the sense that a subset $A \subseteq Q$ is $T(Q)$ -open if and only if it is $T^*(Q)$ -closed.

Remark 2.2: It can be easily seen that the concept of definite source and definite successor are Kuratowski closure operators on Q . Thus these operators also induces two topologies, which we shall denoted as τ and τ^* respectively.

We close this section by showing that there is an interesting relationships between the topologies induced by definite (possible) source and definite (possible) successor.

Proposition 2.2: $\tau(Q) \subseteq T(Q)$ and $\tau^*(Q) \subseteq T^*(Q)$.

Proof: Let $A \in \tau(Q)$. Then $SO_D(Q - A) = Q - A \Rightarrow A = Q - SO_D(Q - A)$. We show that $SO_P(Q - A) = Q - A$, for which we only need to show that $SO_P(Q - A) \subseteq Q - A$. Let $q \in SO_P(Q - A)$. Then $q \in SO_P(p)$, for some $p \in Q - A \Rightarrow p \in C_-(\delta(q, x))$, for some $x \in X$ and $p \in Q - A$. Since, $p \in Q - A \Rightarrow p \notin A = Q - SO_D(Q - A) \Rightarrow p \in SO_D(Q - A) \subseteq SO_P(Q - A) \Rightarrow q \in SO_D(Q - A) \Rightarrow q \in Q - A$.

Conversely, let $A \in \tau^*(Q)$. Then $SU_D(A) = A$. We show that $SU_P(A) = A$, for which we only need to show that $SU_P(A) \subseteq A$. Let $q \in SU_P(A)$. Then $q \in SU_P(p)$, for some $p \in A \Rightarrow q \in C_-(\delta(p, x))$, for $x \in X$ and $p \in A$. Since, $p \in A = SU_D(A) \subseteq SU_P(A) \Rightarrow q \in SU_D(A) \Rightarrow q \in A$. Thus $SU_P(A) \subseteq A$. $\square$

3 Topological characterizations of rough automaton through core and primaries

Motivated by the work of Bavel [3], we introduce a new operator on Q , called the core operator, which is shown to act as an interior operator for the topology $T^*(Q)$ introduced in the preceding section. The introduction of this operator provides a convenient framework for characterizing several structural properties of rough automata and enables a more concise formulation of subsequent results.

Definition 3.1: The *definite core* (resp. *possible core*) of any subset A of the state-set Q of a rough automaton is the set

$$CO_D(A) = \{q \in A : SO_D(q) \subseteq A\}$$

$$(resp. CO_P(A) = \{q \in A : SO_P(q) \subseteq A\})$$

We shall frequently write $CO_D(\{q\})$ as just $CO_D(q)$ (resp. $CO_P(\{q\})$ as just $CO_P(q)$).

The proof of following proposition is similar as the corresponding result for finite automata in [7].

Proposition 3.1: The map $CO_D : 2^Q \rightarrow 2^Q$, (resp. $CO_P : 2^Q \rightarrow 2^Q$) which sends each $A \in 2^Q$ to $CO_D(A)$ (resp. $CO_P(A)$), is an interior operator on Q .

Proposition 3.2: The topologies on Q given by the interior operators CO_D and CO_P are precisely $\tau^*(Q)$ and $T^*(Q)$ resp.

Proof: To prove this proposition, we need to show that

- (i) $CO_D(A) = A \Leftrightarrow SO_D(A) = A$ and
- (ii) $CO_P(A) = A \Leftrightarrow SO_P(A) = A$.

We prove only (ii): First, let $SO_P(A) = A$. It is obvious that $CO_P(A) \subseteq A$. So, we only need to show that $A \subseteq CO_P(A)$. Let $q \in A \Rightarrow SO_P(q) \subseteq SO_P(A) = A \Rightarrow q \in CO_P(A)$; whence $CO_P(A) = A$.

Conversely, let $CO_P(A) = A$. It is obvious that $A \subseteq SO_P(A)$. We only need to show that $SO_P(A) \subseteq A$. Let $q \in SO_P(A)$. Then $p \in C_-(\delta(q, x))$, for some $p \in A$ and $x \in X \Rightarrow q \in SO_P(p)$, for some $p \in A$. But $p \in A = CO_P(A) \Rightarrow p \in CO_P(A) \Rightarrow SO_P(p) \subseteq A \Rightarrow q \in A$. Thus $SO_P(A) \subseteq A$ and so $SO_P(A) = A$.

The proof of (i) is similar to (ii). $\square$

Remark 3.1: We note that $CO_D(A)$ and $CO_P(A)$ are the largest $T(Q)$ -closed set contained in A for any $A \subseteq Q$. Also, $SO_D(A)$ and $SO_P(A)$ are the smallest $T^*(Q)$ -open set contained in A .

The notion of a *primary* of a finite automaton $M = (Q, X, \delta)$ was introduced in [8]. In this section, we provide a topological characterization of primaries by establishing their connection with the notion of *regular closed sets*. Throughout this section, the state set Q is assumed to be endowed with the topology $T^*(Q)$ introduced in the previous section, unless stated otherwise. Accordingly, the terms *open* and *closed* subsets of Q will always be understood with respect to the topology $T^*(Q)$.

Definition 3.2: A subset $A \subseteq Q$ is called

- (i) **definite genetic** (resp. **possible genetic**) if $SO_D(A) \subseteq SU_D(A)$ (resp. if $SO_P(A) \subseteq SU_P(A)$);
- (ii) **definite genetically closed** (resp. **possible genetically closed**) if $\exists B \subseteq A$ such that $SO_D(B) \subseteq SU_D(B)$ and $SU_D(B) = A$ (resp. $SO_P(B) \subseteq SU_P(B)$ and $SU_P(B) = A$);
- (iii) a **definite primary** (resp. **possible primary**) subset of Q if A is a nonempty minimal definite genetically closed (resp. possible genetically closed) subset of Q .

Definition 3.3: A **definite primary** (resp. **possible primary**) of a rough automaton $M = (Q, \sim, X, \delta)$ is definite rough subautomaton (resp. possible rough subautomaton) of M whose state-set is a definite primary (resp. possible primary) subset of Q .

Before stating the next proposition, we first recall the following definition from Willard [17].

Definition 3.4: A closed subset of topological space is called **regular closed** if it is equal to the closure of its interior

Proposition 3.3: A closed subset A of Q is definite genetically closed iff A is a regular closed subset of Q .

Proof: Let A be definite genetically closed. Then it is easy to show that $SU_D(CO_D(A)) = A$, whereby A is regular closed subset of Q .

Conversely, let A be a regular closed subset of Q , so that $SU_D(CO_D(A)) = A$. Put $B = CO_D(A)$. Then $B \subseteq A$. Also, let $q \in SO_D(B) = SO_D(CO_D(A)) \Rightarrow p \in \mathcal{I}_\sim(\delta(q, x))$, for some $(x, p) \in X \times CO_D(A) \Rightarrow q \in SO_D(p)$, for some $p \in CO_D(A)$. But $p \in CO_D(A) \Rightarrow SO_D(p) \subseteq A$. Thus, $q \in A = SU_D(CO_D(A)) = SU_D(B) \Rightarrow q \in SU_D(B)$. This shows that $SO_D(B) \subseteq SU_D(B) = A$, and so A is definite genetically closed. $\square$

Proposition 3.4: A closed subset A of Q is possible genetically closed iff A is a regular closed subset of Q .

Proof: The proof is similar to that of Proposition 3.3. $\square$

Proposition 3.5: A definite (possible) primary of a rough automaton is precisely a minimal regular closed subset of Q .

Proposition 3.6: For every $A \subseteq Q$, $SU_D(SO_D(A))$ (resp. $SU_P(SO_P(A))$) is a definite (resp. possible) genetically closed subset of Q .

Proposition 3.7: Let (Q', R', X, δ') be a definite primary rough subautomaton of $(Q, R, X, \delta) \in \mathbf{RAut}$. Then $p \in CO_D(A) \Rightarrow SU_D(SO_D(p)) = A$.

Proof: Let $p \in CO_D(A)$. Then $SO_D(p) \subseteq A$ and so $SU_D(SO_D(p)) \subseteq A$ (as $SU_D(A) = A$). But as $SU_D(SO_D(p))$ is a regular closed subset (cf. Proposition 3.6) and A is a minimal regular closed subset of Q , $SU_D(SO_D(p)) = A$. $\square$

Proposition 3.8: *Let (Q', R', X, δ') be a possible primary rough subautomaton of $(Q, R, X, \delta) \in \mathbf{RAut}$. Then $SU_P(SO_P(p)) = A, \forall p \in CO_P(A)$.*

Proof: The proof is similar to that of Proposition 3.7. $\square$

Proposition 3.9: *For each $q \in Q$, $SU_D(CO_D(q))$ is a definite primary of Q , if $CO_D(q) \neq \phi$.*

Proof: Let $CO_D(q) \neq \phi$. Then $CO_D(q) = \{q\}$ and so $SU_D(CO_D(q)) = SU_D(q)$. As, closures of open sets are regular closed, $SU_D(q)$ is regular closed. Let A be any regular closed subset of Q such that $A \subseteq SU_D(q)$. As A is definite genetically closed, $\exists B \subseteq A$ such that $SO_D(B) \subseteq SU_D(B) = A$. But then $SO_D(B) \subseteq SU_D(q)$. Now $t \in SO_D(B) \Rightarrow t \in SU_D(q) \Rightarrow t \in \mathcal{I}_-(\delta(q, x))$, for some $x \in X \Rightarrow q \in SO_D(t) \Rightarrow q \in SO_D(SO_D(B)) = SO_D(B)$. Thus, as $SO_D(B) \subseteq A, q \in A$, showing that $SU_D(q) \subseteq A$. Hence $SU_D(CO_D(q))$ is a minimal regular closed subset of Q and so is a definite primary. $\square$

Proposition 3.10: *For each $q \in Q$, $SU_P(CO_P(q))$ is a possible primary of Q , if $CO_P(q) \neq \phi$.*

Proof: The proof is similar to that of Proposition 3.9. $\square$

Remark 3.2: *From the proof of above proposition it is clear that whenever $SU_D(p)$ ($SU_P(p)$) is regular closed, it must be minimal regular closed.*

We recall that a subset of a topological space is called *nowhere dense* if its closure has empty interior (cf. [17]).

Proposition 3.11: *Let (Q, X, δ) be a rough automaton and $p \in Q$. Then the following statements are equivalent:*

- (i) $SU_D(p)$ ($SU_P(p)$) is a definite (possible) primary of Q ;
- (ii) $SU_D(p)$ ($SU_P(p)$) is a regular closed subset of Q ;
- (iii) $\{p\}$ is not a nowhere dense subset of Q ;
- (iv) $SU_D(p)$ ($SU_P(p)$) is a minimal regular closed subset of Q ;
- (v) $p \in CO_D(SU_D(p))$.

4 Compact rough automata and their primary decomposition

The *decomposability* of a rough automaton refers to the possibility of expressing it as a union of certain proper rough subautomata satisfying suitable decomposition properties. One of the cornerstone results in the theory of finite automata is the celebrated *Primary Decomposition Theorem*, which states that every finite automaton can be decomposed into its primaries. In this section, we establish an analogous decomposition theorem for rough automata by showing that every rough automaton admits a decomposition in terms of its primaries. This result extends the classical decomposition theory to the framework of rough automata. Throughout this section, unless stated otherwise, the state set Q is assumed to be endowed with the topology

$T^*(Q)$.

We begin by introducing the following definition.

Definition 4.1: A rough automaton $(Q, \sim, X, \delta)$ is called **compact** if the topology $T(Q)$ is compact.

We describe below a construction due to Lorrain [5] and a related observation, which will help us to give a nice description of compact rough automata.

Given a saturated topological space (X, T) , sending each $R \subseteq X$ to $\hat{R}$, where $\hat{R} = \cap\{U \in T : R \subseteq U\}$, defines a closure operator on X , giving rise to another topology on X , say T^* (cf. [5]).

Theorem 4.1: ([5]) A subset R of saturated topological space (X, T) is compact if and only if there exists a finite subset of X , which is dense in X with respect to the topology T^* .

The following is now immediate.

Proposition 4.1: A rough automaton $(Q, \sim, X, \delta)$ is compact if and only if there exists a finite subset Q' of Q such that $SU_D(Q') = Q$, or equivalently, if and only if $SO_D(Q)$ is finite.

Remark 4.1: Let $M = (Q, \sim, X, \delta)$ be a rough automaton and let $q, q' \in Q$. We say that q' is reachable from q if there exists $x \in X$ such that $q' \in C_-(\delta(q, x))$. As observed earlier, M is compact if there exists a finite subset $Q' \subseteq Q$ such that $SU_{\mathcal{D}_-}(Q') = Q$, or equivalently, $M = \langle Q' \rangle$. Hence every state of the (possibly infinite) state set Q is reachable from some state in the finite generating set Q' . Therefore, a compact rough automaton may be viewed as finitely generated (or equivalently, finitely reachable). Consequently, compact rough automata, despite possibly having infinitely many states, retain several structural properties analogous to those of finite rough automata.

Lemma 4.1: Let (R, X, λ) be a definite primary rough subautomaton of $(Q, X, \delta) \in \mathbf{RAut}$. Then for every finite subset T of R , $T \subseteq SU_D(r)$, for some $r \in R$.

Proof: This can be proved by induction on the number of elements in T . $\square$

Proposition 4.2: A definite (possible) primary of a compact rough automaton is a maximal singly generated definite (possible) rough subautomaton.

Proposition 4.3: A compact rough automaton $M = (Q, X, \delta)$ has only a finite number of distinct primaries.

Remark 4.2: We note that not only are the primaries of a compact rough automaton $M = (Q, X, \delta)$ finite in number but each primary is moreover of the form $\langle q \rangle$, for some $q \in Q'$, where Q' is finite subset of Q such that $Q = SU_D(Q')$.

Proposition 4.4 (Definite Primary Decomposition Theorem): *Let $P_1, P_2, \dots, P_n$ be the distinct definite primary rough subautomata of a compact rough automaton $M = (Q, \sim, X, \delta)$. Then the following assertions hold:*

- (i) $M = \bigcup_{i=1}^n P_i$.
- (ii) For each $j, 1 \leq j \leq n, M \neq \bigcup_{i \neq j}^n P_i$.

Proof: (i) As M is compact, $Q = SU_D(Q')$, for a finite subset Q' of Q , and each definite primary P_i is a maximal singly generated definite rough subautomaton of the form $\langle q_i \rangle, 1 \leq i \leq n$, for some $q_i \in Q'$ (Remark 4.2). Thus $Q = \bigcup_{i=1}^n SU_D(q_i)$. Hence $M = \bigcup_{i=1}^n P_i$.

(ii) Let $P_i = \langle q_i \rangle$ and $R = \bigcup_{i=1, i \neq j}^n SU_D(q_i)$. Then $q_j \in R \Rightarrow q_j \in SU_D(q_i)$ for some $i \neq j$, implying that $SU_D(q_j) \subseteq SU_D(q_i) \Rightarrow SU_D(q_j) = SU_D(q_i)$ (as $SU_D(q_j)$ is a maximal definite genetically closed subset). But then $P_i = P_j$, which is impossible as $P_i \neq P_j$. Thus $q_j \notin R$, whereby $M \neq \bigcup_{i=1, i \neq j}^n P_i$. $\square$

Proposition 4.5 (Possible Primary Decomposition Theorem): *Let $P_1, P_2, \dots, P_n$ be the distinct possible primary rough subautomata of a compact rough automaton $M = (Q, \sim, X, \delta)$. Then the following statements hold:*

- (i) $M = \bigcup_{i=1}^n P_i$.
- (ii) For each $j, 1 \leq j \leq n, M \neq \bigcup_{i \neq j}^n P_i$.

Definition 4.2: [12] Let $M_1 = (Q_1, \sim_1, X_1, \delta_1)$ and $M_2 = (Q_2, \sim_2, X_2, \delta_2)$ be two rough automata. Their **direct product** is the rough automaton $M_1 \times M_2 = (Q_1 \times Q_2, \sim_1 \times \sim_2, X_1 \times X_2, \delta_1 \times \delta_2)$, where

$\delta_1 \times \delta_2 : (Q_1 \times Q_2) \times (X_1 \times X_2) \longrightarrow \mathfrak{R}(Q_1 \times Q_2, \sim_1 \times \sim_2)$ is defined by

$$(\delta_1 \times \delta_2)((q_1, q_2), (x_1, x_2)) = \left(I_{\sim_1}(\delta_1(q_1, x_1)) \times I_{\sim_2}(\delta_2(q_2, x_2)), C_{\sim_1}(\delta_1(q_1, x_1)) \times C_{\sim_2}(\delta_2(q_2, x_2)) \right),$$

for all $(q_1, q_2) \in Q_1 \times Q_2$ and $(x_1, x_2) \in X_1 \times X_2$.

Proposition 4.6: Let $M_1 = (Q_1, \sim_1, X_1, \delta_1)$ and $M_2 = (Q_2, \sim_2, X_2, \delta_2)$ be rough automata. Then their direct product $M_1 \times M_2 = (Q_1 \times Q_2, \sim_1 \times \sim_2, X_1 \times X_2, \delta_1 \times \delta_2)$ is the categorical product of M_1 and M_2 in the category **RAut**.

Proof: Let $M_i = (Q_i, \sim_i, X_i, \delta_i), i = 1, 2$, and let $M_1 \times M_2 = (Q_1 \times Q_2, \sim_1 \times \sim_2, X_1 \times X_2, \delta_1 \times \delta_2)$ be their direct product. Define the canonical projection maps $h_i : Q_1 \times Q_2 \longrightarrow Q_i, k_i : X_1 \times X_2 \longrightarrow X_i, i = 1, 2$, by $h_1(q_1, q_2) = q_1, h_2(q_1, q_2) = q_2$, and $k_1(x_1, x_2) = x_1, k_2(x_1, x_2) = x_2$. We first show that $(h_i, k_i) : M_1 \times M_2 \longrightarrow M_i, i = 1, 2$, are morphisms in **RAut**. Indeed, if $((p_1, p_2), (q_1, q_2)) \in \sim_1 \times \sim_2$, then $p_i \sim_i q_i$, for $i = 1, 2$, and hence condition (i) of Definition 2.9 is satisfied. For condition (ii), let $(q_1, q_2) \in Q_1 \times Q_2, (x_1, x_2) \in X_1 \times X_2$. By the definition of the product transition map,

$$(\delta_1 \times \delta_2)((q_1, q_2), (x_1, x_2)) = \left(I_{\sim_1}(\delta_1(q_1, x_1)) \times I_{\sim_2}(\delta_2(q_2, x_2)), C_{\sim_1}(\delta_1(q_1, x_1)) \times C_{\sim_2}(\delta_2(q_2, x_2)) \right).$$

Applying the projection h_1 componentwise, then

$$h_1((\delta_1 \times \delta_2)((q_1, q_2), (x_1, x_2))) \subseteq \delta_1(q_1, x_1) = \delta_1(h_1(q_1, q_2), k_1(x_1, x_2)).$$

Hence (h_1, k_1) is a morphism. The proof for (h_2, k_2) is identical. Now let $M' = (Q', \sim', X', \delta')$ be another rough automaton, and suppose $(f_1, g_1) : M' \rightarrow M_1, (f_2, g_2) : M' \rightarrow M_2$ are morphisms.

Define $f : Q' \rightarrow Q_1 \times Q_2, f(q) = (f_1(q), f_2(q))$, and $g : X' \rightarrow X_1 \times X_2, g(x) = (g_1(x), g_2(x))$. Since (f_1, g_1) and (f_2, g_2) preserve equivalence relations, so does (f, g) . Moreover, for every $q \in Q'$ and $x \in X'$,

$$f(\delta'(q, x)) \subseteq \delta_1(f_1(q), g_1(x)) \times \delta_2(f_2(q), g_2(x)) = (\delta_1 \times \delta_2)(f(q), g(x)),$$

so that condition (ii) of Definition 2.9 is satisfied. Hence $(f, g) : M' \rightarrow M_1 \times M_2$ is a morphism. Furthermore,

$$(h_1, k_1) \circ (f, g) = (f_1, g_1), \quad (h_2, k_2) \circ (f, g) = (f_2, g_2), \text{ so the following diagram commutes.}$$

Finally, suppose $(f', g') : M' \rightarrow M_1 \times M_2$ is another morphism satisfying $(h_i, k_i) \circ (f', g') = (f_i, g_i), i = 1, 2$.

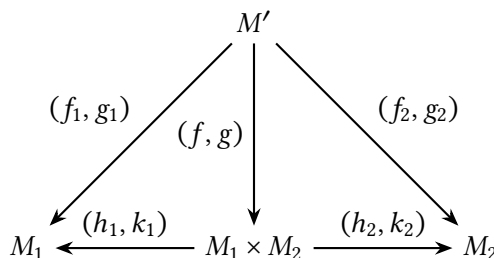


Figure 1: Commutative diagram

Then $h_i \circ f' = h_i \circ f, k_i \circ g' = k_i \circ g, i = 1, 2$. Since the Cartesian product is characterized by its projection maps, $f' = f, g' = g$. Thus (f, g) is unique. Therefore, $M_1 \times M_2$ is the categorical product of M_1 and M_2 in the category **RAut**. $\square$

Proposition 4.7: Let $M_1 = (Q_1, \sim_1, X_1, \delta_1)$ and $M_2 = (Q_2, \sim_2, X_2, \delta_2)$ be two rough automata. Then $T^*(Q_1 \times Q_2) = T^*(Q_1) \times T^*(Q_2)$, where $T^*(Q_1) \times T^*(Q_2)$ denotes the product topology of the topological spaces $(Q_1, T^*(Q_1))$ and $(Q_2, T^*(Q_2))$.

Proof: Let W be open in the product topology $T^*(Q_1) \times T^*(Q_2)$, and let $(q, r) \in W$. Then there exist open sets $U \in T^*(Q_1)$ and $V \in T^*(Q_2)$ such that $(q, r) \in U \times V \subseteq W$. Since U and V are open, $\mathcal{P}_{Q_1}(U) = U$ and $\mathcal{P}_{Q_2}(V) = V$. Now let $(q', r') \in \mathcal{P}_{Q_1 \times Q_2}(q, r)$. Then there exists $(x_1, x_2) \in X_1 \times X_2$ such that $(q', r') \in (\delta_1 \times \delta_2)((q, r), (x_1, x_2))$. Hence $q' \in C_{\sim_1}(\delta_1(q, x_1))$ and $r' \in C_{\sim_2}(\delta_2(r, x_2))$. Therefore, $q' \in \mathcal{P}_{Q_1}(q) \subseteq U$, and $r' \in \mathcal{P}_{Q_2}(r) \subseteq V$, which implies $(q', r') \in U \times V \subseteq W$. Thus $\mathcal{P}_{Q_1 \times Q_2}(q, r) \subseteq W$, and consequently, $\mathcal{P}_{Q_1 \times Q_2}(W) \subseteq W$. Hence $W \in T^*(Q_1 \times Q_2)$. Conversely, let $W \in T^*(Q_1 \times Q_2)$, so that $\mathcal{P}_{Q_1 \times Q_2}(W) = W$, and let $(q, r) \in W$. Since $\mathcal{P}_{Q_1}(q) \times \mathcal{P}_{Q_2}(r)$ is a basic open neighbourhood of (q, r) in the product topology, it suffices to show that $\mathcal{P}_{Q_1}(q) \times \mathcal{P}_{Q_2}(r) \subseteq W$. Let $(q', r') \in \mathcal{P}_{Q_1}(q) \times \mathcal{P}_{Q_2}(r)$. Then there exist $x_1 \in X_1$ and

$x_2 \in X_2$ such that $q' \in C_{\sim_1}(\delta_1(q, x_1))$ and $r' \in C_{\sim_2}(\delta_2(r, x_2))$. Hence $(q', r') \in (\delta_1 \times \delta_2)((q, r), (x_1, x_2))$, which implies $(q', r') \in \mathcal{P}_{Q_1 \times Q_2}(q, r) \subseteq \mathcal{P}_{Q_1 \times Q_2}(W) = W$. Therefore, $\mathcal{P}_{Q_1}(q) \times \mathcal{P}_{Q_2}(r) \subseteq W$, showing that $W \in T^*(Q_1) \times T^*(Q_2)$. Hence, $T^*(Q_1 \times Q_2) = T^*(Q_1) \times T^*(Q_2)$. $\square$

5 Conclusion

In this paper, we introduced the notion of the core of a rough automaton and showed that it induces a natural topology on its state set. The resulting topology was employed to investigate several structural properties of rough automata, including product constructions, and to obtain topological characterizations of definite and possible primaries in terms of regular closed sets, leading to primary decomposition theorems for compact rough automata. These results strengthening the interplay between topology, rough set theory, and automata theory.

The present work also opens several promising directions for future research. One natural extension is the investigation of topological properties of other rough computational models, such as rough tree automata, rough weighted automata, and rough fuzzy automata. Another interesting direction is the study of topological invariants, quotient constructions, and categorical aspects of rough automata, as well as extensions to generalized rough set models induced by coverings, fuzzy relations, and interval type-2 fuzzy relations.

From an application viewpoint, the proposed framework may provide useful mathematical foundations for modeling computational systems under uncertainty and incomplete information. Potential applications include knowledge representation, granular computing, machine learning, pattern recognition, data mining, and automata-based models for uncertain computational systems.

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