

Research article

A Note on JNB-algebras

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Abstract: As a generalization of the self-distributive BE-algebra, the concept of JNB-algebras is introduced in 2025 by Y. B. Jun et al., and its basic properties are investigated. In this short note, we show that in every JNB algebra $\mathfrak{A} = (A, *, 1)$, the natural congruence relation $\equiv$ can be determined such that the factor-algebra $\mathfrak{A}/\equiv$ is an aBE-algebra, that is, it is a JNB-algebra that additionally satisfies the conditions (Ex) and (An).

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1 Introduction

Algebraic structures arising from non-classical logic have been widely studied due to their applications in many mathematical domains such as algebra, logic, and theoretical computer science. Various structures such as BCH/BCI/BCK-algebras, BE/GE-algebras, BI/QI-algebras, implicative algebras, (pre-)Hilbert algebras (as well as many others) and their generalizations have been introduced to model logical operations and algebraic properties of implication-type systems.

The theoretical basis for BE-algebras (determined by the axioms (Re), (M), (L) and (Ex)) was laid in 2007 by H. S. Kim and Y. H. Kim [6], which provided an algebraic framework for studying implication-like operations. Later, several extensions of BE-algebras were investigated. We denote this algebra by aBE (see, for example, [4]) if, in addition, it satisfies the condition (An). BE-algebra $\mathfrak{A}$ is said to be self-distributive if it satisfies the condition (pimpl). Note that a self-distributive BE algebra (in the sense of [6]) is in fact pimpl-BE algebra in the terminology of the article [4]. These developments motivated further exploration of related algebraic systems. Recently (more precisely, 2025), Jun et al. [5] formulated the concept of JNB-algebras, forming an extension of the self-distributive BE-algebraic framework constitute another class of algebraic structures defined by a binary operation and a distinguished element satisfying certain axioms (determined by the axioms (M), (JNB) and (BB)). One generalization of this concept is the subject of [7].



In this short note we show that for every JBN-algebra $\mathfrak{A} = (A, *, 1)$ there exists the congruence $\equiv$ on $\mathfrak{A}$ such that the factor-algebra $\mathfrak{A}/\equiv = (A/\equiv, *, [1])$ is an aBE-algebra.

2 Preliminaries

Abbreviations for many formulas that appear in this article are mostly taken from the paper [4].

Definition 1 ([3]): A Hilbert algebra is defined to be an algebra $\mathfrak{A} = (A, *, 1)$ of type $(2, 0)$ satisfying the following conditions:

- (K) $(\forall a, b \in A)(a * (b * a) = 1)$,
- (p-1) $(\forall a, b, c \in A)((a * (b * c)) * ((a * b) * (a * c)) = 1)$,
- (An) $(\forall a, b \in A)((a * b = 1 \wedge b * a = 1) \implies a = b)$.

Definition 2 ([6]): A BE-algebra is an algebra $\mathfrak{A} = (A, *, 1)$ of type $(2, 0)$ that satisfies:

- (Re) $(\forall a \in A)(a * a = 1)$,
- (M) $(\forall a \in A)(1 * a = a)$,
- (L) $(\forall a \in A)(a * 1 = 1)$,
- (Ex) $(\forall a, b, c \in A)(a * (b * c) = b * (a * c))$.

A BE-algebra $\mathfrak{A}$ is said to be self-distributive if it satisfies the condition

- (pimpl) $(\forall a, b, c \in A)(a * (b * c) = (a * b) * (a * c))$.

Definition 3 ([5], Definition 3.1): A JNB-algebra is defined to be an algebra $\mathfrak{A} = (A, *, 1)$, where A is a set with a special element 1 and $*$ is a binary operation on A that satisfies:

- (M) $(\forall x \in A)(1 * x = x)$,
- (JNB) $(\forall x, y, z \in A)(x * (y * z) = (y * x) * (y * z))$,
- (BB) $(\forall x, y, z \in A)((x * y) * ((y * z) * (x * z)) = 1)$.

Proposition 1 ([5], Theorem 3.4): Every self-distributive BE-algebra is a JNB-algebra.

The converse of the previous proposition does not have to be valid ([5], Example 3.3(ii)).

Proposition 2 ([5], Proposition 4.1): Every JNB-algebra $\mathfrak{A} = (A, *, 1)$ satisfies the following assertions:

- (D) $(\forall x, y \in A)(x * ((x * y) * y) = 1)$,
- (Re) $(\forall x \in A)(x * x = 1)$,
- (pi) $(\forall x, y \in A)(x * (x * y) = x * y)$,
- (K) $(\forall x, y \in A)(x * (y * x) = 1)$,
- (L) $(\forall x \in A)(x * 1 = 1)$,
- (*) $(\forall x, y, z \in A)(x * y = 1 \implies (z * x) * (z * y) = 1)$,

- (7) $(\forall x, y, z \in A)((x * y) * z) * (y * (x * z)) = 1$,
 (8) $(\forall x, y, z \in A)(x * (y * (z * x)) = 1)$,
 (9) $(\forall x, y, z \in A)((y * z) * (y * (x * z)) = 1)$,
 (B) $(\forall x, y, z \in A)((y * z) * ((x * y) * (x * z)) = 1)$,
 (11) $(\forall x, y, z \in A)((x * y) * (y * z) = y * z)$,
 (12) $(\forall x, y, z \in A)(x * (y * ((x * z) * z)) = 1)$,
 (13) $(\forall x, y, z \in A)(x * (((y * x) * z) * z) = 1)$,
 (14) $(\forall x, y, z \in A)(x * ((x * y) * (z * y)) = 1)$,
 (C) $(\forall x, y, z \in A)((x * (y * z)) * (y * (x * z)) = 1)$,
 (16) $(\forall x, y, z \in A)((x * (y * z)) * ((x * y) * (x * z)) = 1)$,
 (Tr) $(\forall x, y \in A)((x * y = 1 \wedge y * z = 1) \implies x * z = 1)$.

Definition 4 ([5], Definition 5.3): . Let $\mathfrak{A} =: (A, *, 1)$ be a JNB-algebra. A subset D of A is called a JNB-deductive system of $\mathfrak{A}$ if it satisfies:

- (D0) $1 \in D$.
 (D1) $(\forall x, y \in A)((x \in D \wedge x * y \in D) \implies y \in D)$.

Obviously, $D_0 = \{1\}$ and A are JNB-deductive systems of $\mathfrak{A}$. A JNB-deductive system D is said to be proper if $D \neq A$.

Example 1: ([5], Example 3.3(ii)) Consider the set $A = \{a, b, c, d, 1\}$ and the operation $*$ given by the following table:

$*$	a	b	c	d	1
1	a	b	c	d	1
a	1	b	c	c	1
b	a	1	d	d	1
c	a	1	1	1	1
d	a	1	1	1	1

Then, $\mathfrak{A} =: (A, *, 1)$ is a JNB-algebra. However, it is not self-distributive BE-algebra. $\square$

Example 2: ([5], Example 4.3) Consider the set $A = \{a, b, c, d, 1\}$ and the operation $*$ given by the following table:

$*$	a	b	c	d	1
1	a	b	c	d	1
a	1	1	c	1	1
b	1	1	c	1	1
c	b	b	1	d	1
d	a	a	c	1	1

Then, $\mathfrak{A} = (A, *, 1)$ is a JNB-algebra. $\square$

3 The Main results

As is customary in logical algebras, we will examine the properties of the relation $\preccurlyeq$ in a JNB-algebra: We define a binary relation “ $\preccurlyeq$ ” on a JNB-algebra $\mathfrak{A} = (A, *, 1)$ as follows:

$$(\forall x, y \in A)(x \preccurlyeq y \iff x * y = 1).$$

It is clear that the relation $\preccurlyeq$ is reflexive and transitive by (Re) and (Tr), respectively, that is, $\preccurlyeq$ is a quasi order compatible with the internal operation $*$ in $\mathfrak{A}$ in the following sense:

$$(q1) (\forall x, y, z \in A)(x \preccurlyeq y \implies z * x \preccurlyeq z * y) \text{ with respect to } (*).$$

$$(q2) (\forall x, y, z \in A)(x \preccurlyeq y \implies y * z \preccurlyeq x * z) \text{ with respect to (B) and (M).}$$

Let $\mathfrak{A} = (A, *, 1)$ be a JNB-algebra. For an equivalence relation ρ on A we say that it is a left congruence on $\mathfrak{A}$ if holds

$$(\forall x, y, z \in A)((x, y) \in \rho \implies (z * x, z * y) \in \rho).$$

Right congruence can be defined analogously. For an equivalence of ρ on A , we say that a is a congruence on $\mathfrak{A}$ if it is both a left and a right congruence on $\mathfrak{A}$.

Theorem 1: Let us define the relation $\equiv$ on the JNB-algebra $\mathfrak{A} = (A, *, 1)$ as follows

$$(\forall x, y \in A)(x \equiv y \iff (x \preccurlyeq y \wedge y \preccurlyeq x)).$$

Then $\equiv$ is a congruence on $\mathfrak{A}$.

Proof: (i) Let $x \in A$ be an arbitrary element. It is clear that $x \equiv x$ holds due to the reflexivity of the relation $\preccurlyeq$.

Let $x, y, z \in A$ be such that $x \equiv y$ and $y \equiv z$. This means $x \preccurlyeq y$, $y \preccurlyeq x$, $y \preccurlyeq z$ and $z \preccurlyeq y$. Then $x \preccurlyeq z$ and $z \preccurlyeq x$ in accordance with (Tr). Hence, $x \equiv z$ which proves that $\equiv$ is a transitive relation on A .

Since the symmetry of the relation $\equiv$ is obvious, we conclude that $\equiv$ is an equivalence relation on A .

(ii) Let $x, y \in A$ be such that $x \equiv y$. This means $x \leq y$ and $y \leq x$. Then $z * x \leq z * y$ by (q1) and $y * z \leq x * z$ according to (q2). Also, we have that $y \leq x$ implies $z * y \leq z * x$ and $x * z \leq y * z$ according to (q1) and (q2). So, $x * z \equiv y * z$. Therefore, the relation $\equiv$ is right compatible with the operation in $\mathfrak{A}$.

(iii) That the relation $\equiv$ is left compatible with the operation in $\mathfrak{A}$ can be proved analogously.

This shows that $\equiv$ is a congruence relation on the JNB-algebra $\mathfrak{A}$. $\square$

Let us show that:

Proposition 3: *The class $[1] =: \{y \in A : y \equiv 1\}$ of the congruence $\equiv$, generated by 1, is a JNB-deductive system in any JNB-algebra $\mathfrak{A}$.*

Proof: Let $x, y \in A$ be such that $x \in [1]$ and $x * y \in [1]$. This means $x \equiv 1$ and $x * y \equiv 1$. Furthermore, this means $x \leq 1 \wedge 1 \leq x$, and $x * y \leq 1 \wedge 1 \leq x * y$. Further on, this can be written in the form $x * 1 = 1$, $1 = 1 * x = x$ with respect to (L) and (M) and $1 = (x * y) * 1 = (1 * y) * 1 = y * 1$, $y = 1 * y = 1 * (1 * y) = 1$ also, with respect to (L) and (M). Finally, we have $1 = 1 * y = y$. Thus, $y \in [1]$. $\square$

As it was shown in the proof of the previous Proposition, for any JNB-algebra $\mathfrak{A} = (A, *, 1)$ holds

$$(18) (\forall x \in A)(1 \leq x \implies x = 1).$$

Denote $\mathfrak{A}/\equiv =: \{[x] : x \in A\}$, where $[x] =: \{y \in A : x \equiv y\}$, and define that

$$(\forall x, y \in A)([x] * [y] =: [x * y]).$$

So, for arbitrary $x, y \in A$ we have

$$[x] = [y] \iff x \equiv y \iff (x \leq y \wedge y \leq x).$$

Since $\equiv$ is a congruence relation on $\mathfrak{A}$, the operation $*$ is well-defined.

Theorem 2: *The structure $(A/\equiv, *, [1])$ is an aBE-algebra.*

Proof: In order to prove the validity of the theorem, it is necessary to prove the validity of formulas (An), (M), (L), (JNB), (BB) and (Ex) in the structure $\mathfrak{A}/\equiv$.

(An): It is obvious, according to the determination of the relation $\equiv$, that

$$[x] * [y] = [x * y] = [1] \wedge [y] * [x] = [y * x] = [1] \implies [x] = [y].$$

That means $[x * y] = [1]$ and $[y * x] = [1]$. That is this means that $x * y \equiv 1$ and $y * x \equiv 1$. So, we have $(x * y \leq 1 \wedge 1 \leq x * y)$ and $(y * x \leq 1 \wedge 1 \leq y * x)$. From here we get $x * y = 1$ and $y * x = 1$. Finally, according to the determination of the relation $\equiv$, we have $x \equiv y$ and $[x] = [y]$.

(BB): For arbitrary $x, y, z \in A$, we have

$$([x] \star [y]) \star ([y] \star [z]) \star ([x] \star [z]) = \dots [(x \star y) \star ((y \star x) \star (x \star z))] = [1].$$

(M) is valid because we have $[1] \star [x] = [1 \star x] = [x]$.

(L) is valid because we have $[x] \star [1] = [1 \star x] = [1]$.

(JBN): For arbitrary $x, y, z \in A$, we have $[x] \star ([y] \star [z]) = [x \star (y \star x)] = [(y \star x) \star (y \star z)] = ([y] \star [x]) \star ([y] \star [z])$.

(Ex): For arbitrary $x, y, z \in A$ we have $x \star (y \star z) \leq y \star (x \star z)$ according to (C). Substituting the variables in the previous inequality, we get $y \star (x \star z) \leq x \star (y \star z)$. Thus, we have $x \star (y \star z) \equiv y \star (x \star z)$. Finally, we have $[x] \star ([y] \star [z]) = [x \star (y \star z)] = [y \star (x \star z)] = [y] \star ([x] \star [z])$. $\square$

Example 3: Let $\mathfrak{A} := (A, \star, 1)$ be a $\mathcal{JNB}$ -algebra from Example 1. Then

$$\leq = \{1, 1), (a, a), (a, 1), (b, b), (b, 1), (c, b), (c, c), (c, d), (d, b), (d, c), (d, d), (d, 1)\}$$

and $[1] = \{1\}$, $[a] = \{a\}$, $[b] = \{b\}$ and $[c] = [d] = \{c, d\}$. So, $(A/\equiv, \star, [1]) = (\{\{1\}, \{a\}, \{b\}, \{c, d\}\}, \star, [1])$. $\square$

Example 4: Let $\mathfrak{A} =: (A, \star, 1)$ be a $\mathcal{JNB}$ -algebra from Example 2. Then $a \equiv b \leq d$ while the other elements ($\neq 1$) are not comparable to each other. So, $(A/\equiv, \star, [1]) = (\{\{a, b\}, \{c\}, \{d\}, \{1\}\}, \star, [1])$. $\square$

We conclude this section with the definition of dual BCK-algebras and implicative algebras.

Definition 5: ([2]) A dual BCK-algebra is a triple $(A, \star, 1)$ where A is a set with constant 1 and $\star$ as a binary operation on A that satisfies the axioms ($\mathcal{JNB}$), (D), (Re), (An) and (L).

Definition 6: ([1]) Implicative algebra is an algebra $(A, \star, 1)$ of type $(2, 0)$ which, in addition to the condition (Ex), also satisfies the conditions

$$(im) \quad (\forall x, y \in A)((x \star y) \star x = x)$$

$$(com) \quad (\forall x, y \in A)((x \star y) \star y = (y \star x) \star x).$$

In relation to the previous two definitions, we offer the following reflection:

Since, by Proposition 2, we conclude that the algebra $(A/\equiv, \star, [1])$ is a dual BCK-algebra as well.

Suppose that $\mathfrak{A}/\equiv$ is a positive implicative algebra, i.e. assume that it satisfies the condition (pimpl). Then, similarly as above, we would have: Since by [8], Proposition 3.2(i), we have $(Re) + (M) + (pimpl) \implies (pi)$, we conclude that the factor-algebra $\mathfrak{A}/\equiv$ is an implicative dual BCK-algebra, that is, by [2], it is an implicative algebra. So, it remains open: Can it be shown that the algebra $\mathfrak{A}/\equiv$ satisfies the condition (pimpl)?

4 Final comments

The concept of JNB-algebras was introduced in 2025 by Y. B. Jun et al. as a generalization of the self-distributive BE-algebra. In this paper we show that for every JNB-algebra $\mathfrak{A} = (A, *, 1)$ there exists the congruence $\equiv$ on $\mathfrak{A}$ such that the factor-algebra $\mathfrak{A}/\equiv = (A/\equiv, *, [1])$ is an aBE-algebra.

Our further investigation of the properties of JNB-algebras will be directed towards the following questions:

(1): It seems that it could be proved that in every JNB-algebra $\mathfrak{A} = (A, *, 1)$ there is at least one proper sub-algebra in $\mathfrak{A}$ isomorphic to the aBE-algebra $\mathfrak{A}/\equiv$.

(2): Can such a statement about the JNB-algebra $\mathfrak{A}$ be made if instead of the congruence $\equiv$ one considers any congruence on $\mathfrak{A}$?

(3): Also, it is quite justified to investigate various types of deductive systems in this class of logical algebras.

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