Annals of Communications in Mathematics
Volume 9, Number 3 (2026), 11
DOI: 10.62072/acm.2026.09043
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

Research article

On Bipolar Complex Intuitionistic Fuzzy Multi-Polygroups

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Received: 06 July 2026; Accepted: 06 September 2026; Published: 14 September 2026

Citation: A.I. Isah, P. Ayuba, N. Mubarak, P.A. Ejegwa, On Bipolar Complex Intuitionistic Fuzzy Multi-Polygroups. Ann. Commun. Math. 9 **(2026)**, 11. https://doi.org/10.62072/acm.2026.09043

Abstract: This paper introduces the idea of complex numbers to intuitionistic fuzzy sets which enhances the representation of uncertain, hesitant and ambiguous data. By integrating the complex-valued intuitionistic fuzzy membership and non-membership functions with the algebraic structure of multi-polygroups known as Complex Intuitionistic Fuzzy Multi-Polygroups (CIFMPGs). This extension addresses the limitations of classical fuzzy and intuitionistic fuzzy multi-polygroups by introducing both amplitude and phase into the representation of membership values. The basic operations and structural properties were studied such as complement, intersection, containment and cartesian product. Their closure properties on those operations were also established. Furthermore, the concept of Bipolar Complex Intuitionistic Fuzzy Multi-Polygroups (BCIFMPGs) was introduced by integrating the degrees of positive and negative complex-valued membership and non-membership into the multi-polygroups structure. The bipolar extension leads to an algebraic representation of positive and negative information together with amplitude and phase. Basic operations on BCIFMPGs such as intersection and complement were defined, and their closure properties. The introduced structures generalize intuitionistic fuzzy multi-polygroups to complex-valued settings, opening the way for further theoretical study of fuzzy algebraic structures.

Mathematics Subject Classification: (2020): 16S34
Keywords: intuitionistic fuzzy sets, multi-polygroups, complex intuitionistic fuzzy multi-polygroups, bipolar complex intuitionistic fuzzy multi-polygroups.

1 Introduction

The concept of fuzzy sets was first proposed by Zadeh [1], introducing a membership function that assigns a value in the interval $[0, 1]$ to represent the degree of belongingness of an element to a set. Building on this, Atanassov [2] introduced the notion of intuitionistic fuzzy sets (IFS) by including a non-membership function alongside the membership function, thus providing a richer framework to model uncertainty and vagueness. Sets are traditionally well-defined collections of distinct objects, the concept of multisets relaxes this restriction by allowing repeated occurrences of elements. The concept of groups was integrated into multisets to obtain multigroups. Various properties of multisets and multigroups have been explored in [3–9].

As a generalization of fuzzy sets, Yager [10] introduced fuzzy multisets (FMS), allowing elements to occur multiple times with potentially varying membership values. Later, Shinoj and Sunil [11] extended this idea by combining IFSs and FMSs to introduce intuitionistic fuzzy multisets (IFMSs). This hybrid framework enabled a more nuanced approach to representing uncertainty and ambiguity. By using the idea of FMSs, the concept of fuzzy multigroups (FMGs) has been developed and explored in several studies [11–16]. Similarly, the notion of intuitionistic fuzzy multigroups (IFMGs) has been presented in [17] and extended to intuitionistic fuzzy multi-polygroups (IFMPGs) in [18].

The integration of IFMSs with algebraic structures such as polygroups has opened new frontiers in mathematical modelling. This study builds on these advancements by introducing and exploring bipolar complex intuitionistic fuzzy multi-polygroups (BCIFMPGs) based on complex intuitionistic fuzzy groups (CIFGs) [19] and bipolar complex intuitionistic fuzzy sets (BCIFSs) [19], which is an advanced framework that integrates bipolar and complex-valued functions under the IFMS setting. This structure not only addresses the challenges of dual-polarity and hesitancy but also provides a powerful tool for future applications in artificial intelligence, pattern recognition, and decision-making processes.

2 Preliminaries

Definition 1 ([1])**:** *Let X be the universal set. A fuzzy set A drawn from X is a set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0, 1]$ is called the membership function. For each $x \in X$, the value $\mu_A(x)$ is called the grade of membership of x in A.*

Definition 2 ([2])**:** *Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A in X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A, \nu_A : X \to [0, 1]$ define, respectively, the degree of membership and degree of non-membership of the element $x \in X$ to A, which is a subset of X, and for every element $x \in X$ we have $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, called the hesitation margin of x in A. In addition, $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to A and $\pi_A(x) \in [0, 1]$, i.e., $\pi_A : X \to [0, 1]$ and $0 \leq \pi_A(x) \leq 1$, for every $x \in X$.*

Definition 3 ([4])**:** *Let $X = \{x_1, x_2, \dots, x_n, \dots\}$ be a set. Then a multiset depicted by A over X is a cardinal-valued function, $C_A : X \to \mathbb{N}$ such that $x \in \mathrm{Dom}(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denotes the number of times an object x occurs in A. Whenever $C_A(x) = 0$, this implies $x \notin \mathrm{Dom}(A)$. The set X is called the ground or generic set of the class of all multisets containing objects from X, and $\mathbb{N}$ is the set of non-negative integers.*

Definition 4 ([10])**:** *Let X be a nonempty set. Then a FMS denoted by A drawn from X is characterized by a function, the count membership of A represented by CM_A, such that $CM_A : X \to Q$, where Q is the set of all crisp multisets drawn from the unit interval $[0, 1]$. Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from $[0, 1]$. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$, which is denoted by $(\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^n(x))$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq \dots \geq \mu_A^n(x)$.*

Definition 5 ([19])**:** *A complex fuzzy set (CFS) A defined in a universe of discourse X is characterized by a membership function $\mu_A(x)$ that assigns to any element $x \in X$ a complex-valued grade of membership in A. By definition, the values of $\mu_A(x)$ lie within the unit circle in the complex plane and are thus of the form $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$, where $i = \sqrt{-1}$ and each of $r_A(x)$ and $\omega_A(x)$ is real-valued, with $r_A(x) \in [0,1]$. The CFS A may be represented as the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\} = \{(x, r_A(x)e^{i\omega_A(x)}) : x \in X\}$.*

Definition 6 ([20])**:** *A CIFS depicted by S, defined on a universe of discourse X, is characterized by membership and non-membership functions $\mu_S(x)$ and $\gamma_S(x)$, respectively, that assign to any element $x \in U$ a complex-valued grade of both membership and non-membership in S. By definition, the values of $\mu_S(x)$, $\gamma_S(x)$, and their sum may all be lying within the unit circle in the complex plane, and are on the form $\mu_S(x) = r_S(x) \cdot e^{i\omega_{(\mu_S)}(x)}$ for membership function in S and $\gamma_S(x) = k_S(x) \cdot e^{i\omega_{(\gamma_S)}(x)}$ for non-membership function in S, where $i = \sqrt{-1}$, each of $r_S(x)$ and $k_S(x)$ are real-valued and both belong to the interval $[0,1]$ such that $0 \le r_S(x) + k_S(x) \le 1$, also $\omega_{(\mu_S)}(x)$ and $\omega_{(\gamma_S)}(x)$ are real-valued. We represent the CIFS S as $S = \{\langle x, \mu_S(x), \gamma_S(x)\rangle : x \in X\}$ where $\mu_S(x) : U \to \{a \mid a \in \mathbb{C}, |a| \le 1\}$, $\gamma_S(x) : U \to \{a' \mid a' \in \mathbb{C}, |a'| \le 1\}$ and $|\mu_S(x) + \gamma_S(x)| \le 1$.*

Definition 7 ([20])**:** *Let X be a non-empty set. A bipolar fuzzy set (BFS) A in X is an object having the form $A = \{(x, r_A^+(x), r_A^-(x)) : x \in X\}$, where $r^+ : X \to [0,1]$ and $r^- : X \to [-1,0]$ are mappings.*

In [21], it has been proved that every BFS is an IFS. Thus, the discussion of the BFS has become meaningless.

Definition 8 ([22])**:** *Let X be a non-empty set. A bipolar intuitionistic fuzzy set (BIFS) denoted by A in a non-empty set X is an object having the form $A = \{\langle x, \mu_A^P(x), \gamma_A^P(x), \mu_A^N(x), \gamma_A^N(x)\rangle : x \in X\}$, where $\mu_A^P : X \to [0,1]$, $\gamma_A^P : X \to [0,1]$, $\mu_A^N : X \to [-1,0]$, $\gamma_A^N : X \to [-1,0]$ are mappings such that $0 \le \mu_A^P(x) + \gamma_A^P(x) \le 1$ and $-1 \le \mu_A^N(x) + \gamma_A^N(x) \le 0$.*

Similar to Definition 7 and [21], it follows that every BIFS is representable as an interval-valued IFS. Thus, work on BIFS is meaningless since interval-valued IFS has been hitherto introduced in [24], [25].

Definition 9 ([20])**:** *Let X be a non-empty set. A bipolar complex fuzzy set (BCFS) represented by A in X is an object having the form: $A = \{(x,\ r_A^+ e^{i\theta_A^+},\ r_A^- e^{i\theta_A^-}) : x \in X\}$, where $r_A^+ : X \to [0,1]$ and $r_A^- : X \to [-1,0]$ are mappings, $r_A^+ e^{i\theta_A^+}$ is the positive complex membership degree and $r_A^- e^{i\theta_A^-}$ is the negative complex membership degree. Also, the phase term of the bipolar complex positive membership function and bipolar complex negative membership function belongs to $(0, 2\pi]$, and $r_A^+ \in [0,1]$, $r_A^- \in [-1,0]$.*

Definition 10 ([20])**:** *Let X be a non-empty set. A BCIFS A in X is an object having the form: $A = \{(x,\ r_A^+ e^{i\theta_A^+},\ r_A^- e^{i\theta_A^-},\ s_A^+ e^{i\theta_A^+},\ s_A^- e^{i\theta_A^-}) : x \in X\}$, where $r_A^+ : X \to [0,1]$, $r_A^- : X \to [-1,0]$, $s_A^+ : X \to [0,1]$ and $s_A^- : X \to [-1,0]$ are mappings such that $0 \le r_A^+ + s_A^+ \le 1 \quad \text{and} \quad -1 \le r_A^- + s_A^- \le 0$.*

Definition 11 ([18])**:** *Let $\langle M, \circ_1, e_1, {}^{-1}\rangle$ and $\langle N, \circ_1, e_1, {}^{-1}\rangle$ be two polygroups. Then, the productional polygroup $\langle M \times N, \circ, (e_1, e_2), {}^{-1}\rangle$ is defined as follows: $(x_1, y_1) \circ (x_2, y_2) = \{(x_3, y_3) : x_3 \in x_1 \circ_1 x_2,\ y_3 \in y_1 \circ_2 y_2\}$. In the above definition, $(M \times N, \circ)$ with identity (e_1, e_2) and unitary operation ${}^{-1}$ is a polygroup.*

Definition 12 ([23])**:** *Let X be a group. Then a FMS depicted by A over X is said to be a FMG of X if it satisfies the following two conditions:*

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \quad \forall x, y \in X,$

(ii) $CM_A(x^{-1}) \geq CM_A(x) \quad \forall x \in X,$

where $\wedge$ and $\vee$ denote the minimum and maximum operators. It follows immediately that $CM_A(x^{-1}) = CM_A(x) \quad \forall x \in X$, *since* $CM_A(x) = CM_A\big((x^{-1})^{-1}\big) \geq CM_A(x^{-1})$. *Also,* $CM_A(x^n) \geq CM_A(x) \quad \forall x \in X,\ n \in \mathbb{N}$.

Definition 13 ([18])**:** *Let $\langle P, \circ_1, e, ^{-1}\rangle$ be a polygroup. An IFMS I over P is an intuitionistic fuzzy multi-polygroup of P if for all $a, b \in P$ the following conditions are satisfied:*

(i) $CM_I(a) \wedge CM_I(b) \leq \inf\{CM_I(c) \mid c \in a \circ b\}$ *and* $CN_I(a) \vee CN_I(b) \geq \sup\{CN_I(c) \mid c \in a \circ b\}$,

(ii) $CM_I(a) \leq CM_I(a^{-1})$ *and* $CN_I(a) \leq CN_I(a^{-1})$.

3 Complex Intuitionistic Fuzzy Multi-Polygroups

In this section, the concept of complex intuitionistic fuzzy multi-polygroups (CIFMPGs) has been developed by integrating complex-valued membership and non-membership functions with the theory of multi-polygroups. CIFMPGs provide a powerful framework for modeling and analyzing complex data in various applications, including artificial intelligence and decision-making systems.

Definition 14: *Let $\langle P, \circ, e, ^{-1}\rangle$ be a polygroup. An IFMS I over P is a CIFMPG of P if for all $a, b \in P$, the following conditions are satisfied:*

(i) $CM_I(c)\, e^{i\alpha_I(c)} \geq \langle CM_I(a)e^{i\alpha_I(a)} \wedge CM_I(b)e^{i\alpha_I(b)}\ :\ c \in a \circ b\rangle,$

$CN_I(c)\, e^{i\beta_I(c)} \leq \langle CN_I(a)e^{i\beta_I(a)} \vee CN_I(b)e^{i\beta_I(b)}\ :\ c \in a \circ b\rangle,$

(ii) $CM_I(a^{-1})e^{i\alpha_I(a^{-1})} \geq CM_I(a)e^{i\alpha_I(a)}, \qquad CN_I(a^{-1})e^{i\beta_I(a^{-1})} \leq CN_I(a)e^{i\beta_I(a)}.$

It is an objective of the form: $I = \{(x,\ CM_I(c)e^{i\alpha_I(c)},\ CN_I(c)e^{i\beta_I(c)})\ :\ c \in a \circ b\}$ *for all* $a, b \in P$. By definition, the values of $CM_I(c)e^{i\alpha_I(c)}$ for the membership function in P and $CN_I(c)e^{i\beta_I(c)}$ for the non-membership function in P, and their sum may receive all lying within the unit circle in the complex plane, where $i = \sqrt{-1}$. Each of $CM_I(c)$ and $CN_I(c)$ are real-valued functions and both belong to the interval $[0, 1]$ such that $0 \leq CM_I(c) + CN_I(c) \leq 1$, also $e^{i\alpha_I(c)}$ and $e^{i\beta_I(c)}$ are periodic functions whose periodic laws and principal periods are 2π and $0 < e^{i\alpha_I(c)},\ e^{i\beta_I(c)} \leq 2\pi$.

Example 1: *Let $P_1 = \{e, x\}$ and $\langle P_1, \circ, e, ^{-1}\rangle$ be the polygroup. Then,*

$$I = \begin{Bmatrix} (e\ :\ (0.9e^{1.8\pi i}, 0.1e^{0.2\pi i}),\ (0.8e^{1.4\pi i}, 0.2e^{0.6\pi i}),\ (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i})) \\ (x\ :\ (0.7e^{1.6\pi i}, 0.3e^{0.4\pi i}),\ (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i}),\ (0.4e^{\pi i}, 0.6e^{\pi i})) \end{Bmatrix}$$

is a CIFMPG of P_1.

Also, $P_2 = \{e, x\}$ and $\langle P_2, \circ, e, ^{-1}\rangle$ be the polygroup. Then

$$J = \left\{ \begin{matrix} (e\ :\ (0.6e^{1.3\pi i}, 0.4e^{0.7\pi i}),\ (0.5e^{0.5\pi i}, 0.5e^{1.5\pi i}),\ (0.4e^{0.3\pi i}, 0.6e^{1.7\pi i})) \\ (x\ :\ (0.5e^{1.2\pi i}, 0.5e^{0.8\pi i}),\ (0.4e^{\pi i}, 0.6e^{\pi i}),\ (0.3e^{0.8\pi i}, 0.7e^{1.2\pi i})) \end{matrix} \right\}$$

is a CIFMPG of P_2. Now, we check whether I is a CIFMPG as follows:
i. *Thus, we see that*

$$\begin{aligned}
CM_I(ex)e^{i\alpha_I(ex)} &= CM_I(x)e^{i\alpha_I(x)} = 0.7e^{1.6\pi i},\ 0.6e^{1.2\pi i},\ 0.4e^{\pi i} \\
&\geq CM_I(e)e^{i\alpha_I(e)} \wedge CM_I(x)e^{i\alpha_I(x)} = 0.7e^{1.6\pi i},\ 0.6e^{1.2\pi i},\ 0.4e^{\pi i} \\
CN_I(ex)e^{i\beta_I(ex)} &= CN_I(x)e^{i\beta_I(x)} = 0.3e^{0.4\pi i},\ 0.4e^{0.8\pi i},\ 0.6e^{\pi i} \\
&\leq CN_I(e)e^{i\beta_I(e)} \vee CN_I(x)e^{i\beta_I(x)} = 0.3e^{0.4\pi i},\ 0.4e^{0.8\pi i},\ 0.6e^{\pi i} \\
CM_I(xx)e^{i\alpha_I(xx)} &= CM_I(e)e^{i\alpha_I(e)} = 0.9e^{1.8\pi i},\ 0.8e^{1.4\pi i},\ 0.6e^{1.2\pi i} \\
&\geq CM_I(x)e^{i\alpha_I(x)} \wedge CM_I(x)e^{i\alpha_I(x)} = 0.7e^{1.6\pi i},\ 0.6e^{1.2\pi i},\ 0.4e^{\pi i} \\
CN_I(xx)e^{i\beta_I(xx)} &= CN_I(e)e^{i\beta_I(e)} = 0.1e^{0.2\pi i},\ 0.2e^{0.6\pi i},\ 0.4e^{0.8\pi i} \\
&\leq CN_I(x)e^{i\beta_I(x)} \vee CN_I(x)e^{i\beta_I(x)} = 0.3e^{0.4\pi i},\ 0.4e^{0.8\pi i},\ 0.6e^{\pi i} \\
CM_I(ee)e^{i\alpha_I(ee)} &= CM_I(e)e^{i\alpha_I(e)} = 0.9e^{1.8\pi i},\ 0.8e^{1.4\pi i},\ 0.6e^{1.2\pi i} \\
&\geq CM_I(e)e^{i\alpha_I(e)} \wedge CM_I(e)e^{i\alpha_I(e)} = 0.9e^{1.8\pi i},\ 0.8e^{1.4\pi i},\ 0.6e^{1.2\pi i} \\
CN_I(ee)e^{i\beta_I(ee)} &= CN_I(e)e^{i\beta_I(e)} = 0.1e^{0.2\pi i},\ 0.2e^{0.6\pi i},\ 0.4e^{0.8\pi i} \\
&\leq CN_I(e)e^{i\beta_I(e)} \vee CN_I(e)e^{i\beta_I(e)} = 0.1e^{0.2\pi i},\ 0.2e^{0.6\pi i},\ 0.4e^{0.8\pi i}.
\end{aligned}$$

ii. In addition,

$$CM_I(x^{-1})e^{i\alpha_I(x^{-1})} = CM_I(x)e^{i\alpha_I(x)} = 0.7e^{1.6\pi i},\ 0.6e^{1.2\pi i},\ 0.4e^{\pi i},$$

$$CN_I(x^{-1})e^{i\beta_I(x^{-1})} = CN_I(x)e^{i\beta_I(x)} = 0.3e^{0.4\pi i},\ 0.4e^{0.8\pi i},\ 0.6e^{\pi i},$$

$$CM_I(e^{-1})e^{i\alpha_I(e^{-1})} = CM_I(e)e^{i\alpha_I(e)} = 0.9e^{1.8\pi i},\ 0.8e^{1.4\pi i},\ 0.6e^{1.2\pi i},$$

$$CN_I(e^{-1})e^{i\beta_I(e^{-1})} = CN_I(e)e^{i\beta_I(e)} = 0.1e^{0.2\pi i},\ 0.2e^{0.6\pi i},\ 0.4e^{0.8\pi i}.$$

Therefore, I is a CIFMPG.

Similarly, J is a CIFMPG because we have **i.**

$$\begin{aligned}
CM_I(ex)e^{i\alpha_I(ex)} &= CM_I(x)e^{i\alpha_I(x)} = 0.5e^{i1.2\pi},\ 0.4e^{i\pi},\ 0.3e^{i0.8\pi} \\
&\geq CM_I(e)e^{i\alpha_I(e)} \wedge CM_I(x)e^{i\alpha_I(x)} = 0.5e^{i1.2\pi},\ 0.4e^{i\pi},\ 0.3e^{i0.8\pi}
\end{aligned}$$

$$\begin{aligned}
CM_I(xx)e^{i\alpha_I(xx)} &= CM_I(e)e^{i\alpha_I(e)} = 0.6e^{i1.3\pi},\ 0.5e^{i0.5\pi},\ 0.4e^{i0.3\pi} \\
&\geq CM_I(x)e^{i\alpha_I(x)} \wedge CM_I(x)e^{i\alpha_I(x)} = 0.5e^{i1.2\pi},\ 0.4e^{i\pi},\ 0.3e^{i0.8\pi}
\end{aligned}$$

$$CM_I(ee)e^{i\alpha_I(ee)} = CM_I(e)e^{i\alpha_I(e)} = 0.6e^{i1.3\pi},\ 0.5e^{i0.5\pi},\ 0.4e^{i0.3\pi}$$

$$\begin{aligned}
CN_I(ex)e^{i\beta_I(ex)} &= CN_I(x)e^{i\beta_I(x)} = 0.5e^{i0.8\pi},\ 0.6e^{i\pi},\ 0.7e^{i1.2\pi} \\
&\leq CN_I(e)e^{i\beta_I(e)} \vee CN_I(x)e^{i\beta_I(x)} = 0.4e^{i0.7\pi},\ 0.5e^{i1.5\pi},\ 0.6e^{i1.7\pi}
\end{aligned}$$

$$\begin{aligned}
CN_I(xx)e^{i\beta_I(xx)} &= CN_I(e)e^{i\beta_I(e)} = 0.4e^{i0.7\pi},\ 0.5e^{i1.5\pi},\ 0.6e^{i1.7\pi} \\
&\leq CN_I(x)e^{i\beta_I(x)} \vee CN_I(x)e^{i\beta_I(x)} = 0.5e^{i0.8\pi},\ 0.6e^{i\pi},\ 0.7e^{i1.2\pi}
\end{aligned}$$

$$\begin{aligned}
CN_I(ee)e^{i\beta_I(ee)} &= CN_I(e)e^{i\beta_I(e)} = 0.4e^{i0.7\pi},\ 0.5e^{i1.5\pi},\ 0.6e^{i1.2\pi} \\
&\leq CN_I(e)e^{i\beta_I(e)} \vee CN_I(e)e^{i\beta_I(e)} = 0.4e^{i0.7\pi},\ 0.5e^{i1.5\pi},\ 0.6e^{i1.7\pi}.
\end{aligned}$$

ii.

$$CM_I(x^{-1})e^{i\alpha_I(x^{-1})} = CM_I(x)e^{i\alpha_I(x)} = 0.5e^{i1.2\pi},\ 0.4e^{i\pi},\ 0.3e^{i0.8\pi}$$

$$CM_I(e^{-1})e^{i\alpha_I(e^{-1})} = CM_I(e)e^{i\alpha_I(e)} = 0.6e^{i1.3\pi},\ 0.5e^{i0.5\pi},\ 0.4e^{i0.3\pi}$$

$$CN_I(x^{-1})e^{i\beta_I(x^{-1})} = CN_I(x)e^{i\beta_I(x)} = 0.5e^{i0.8\pi},\ 0.6e^{i\pi},\ 0.7e^{i1.2\pi}$$

$$CN_I(e^{-1})e^{i\beta_I(e^{-1})} = CN_I(e)e^{i\beta_I(e)} = 0.4e^{i0.7\pi},\ 0.5e^{i1.5\pi},\ 0.6e^{i1.7\pi}.$$

Therefore, J is a CIFMPG.

Example 2: *Let $\langle P, \circ, e, ^{-1}\rangle$ be a polygroup. Then, the complement of a CIFMPG $I = \{\, x,\ CM_I(a)e^{i\alpha_I(a)},\ CN_I(a)e^{i\beta_I(a)} : x \in X \,\}$ is denoted by I^c and defined by: $I^c = \{\, x,\ CN_I(a)e^{i\beta_I(a)},\ CM_I(a)e^{i\alpha_I(a)} : x \in X \,\}$ for all $a \in P$.*

Remark 1: *Let $\langle P, \circ, e, ^{-1}\rangle$ be a polygroup and I be an IFMS of P. If I is a CIFMPG of P, then I^c is a CIFMPG of P.*

Example 3: *Let I be a CIFMPG. Determine the complement, I^c. Now if*

$$I = \begin{Bmatrix} (e \ : \ (0.9e^{1.8\pi i}, 0.1e^{0.2\pi i}), (0.8e^{1.4\pi i}, 0.2e^{0.6\pi i}), (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i})) \\ (x \ : \ (0.7e^{1.6\pi i}, 0.3e^{0.4\pi i}), (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i}), (0.4e^{\pi i}, 0.6e^{\pi i})) \end{Bmatrix}.$$

Then,

$$I^c = \begin{Bmatrix} (e \ : \ (0.1e^{0.2\pi i}, 0.9e^{1.8\pi i}), (0.2e^{0.6\pi i}, 0.8e^{1.4\pi i}), (0.4e^{0.8\pi i}, 0.6e^{1.2\pi i})) \\ (x \ : \ (0.3e^{0.4\pi i}, 0.7e^{1.6\pi i}), (0.4e^{0.8\pi i}, 0.6e^{1.2\pi i}), (0.6e^{\pi i}, 0.4e^{\pi i})) \end{Bmatrix}.$$

is a CIFMPG of P_1.

Definition 15: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup and I, J be two IFMSs of P. Then $I \cap J$ is defined via the following complex intuitionistic fuzzy count function for $c \in P$, where*

$$CM_{I\cap J}(c)\, e^{i\alpha_{I\cap J}(c)} = CM_I(a)\, e^{i\alpha_I(a)} \wedge CM_J(b)\, e^{i\alpha_J(b)},$$
$$CN_{I\cap J}(c)\, e^{i\beta_{I\cap J}(c)} = CN_I(a)\, e^{i\beta_I(a)} \vee CN_J(b)\, e^{i\beta_J(b)},$$

for all $a, b \in P$ and $c \in a \circ b$.

Remark 2: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup and I, J be two IFMSs of P. If I and J are CIFMPGs of P, then $I \cap J$ is a CIFMPG of P.*

Example 4: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup with I and J be two CIFMPGs of P defined by*

$$I = \begin{Bmatrix} (e \ : \ (0.9e^{1.8\pi i}, 0.1e^{0.2\pi i}), (0.8e^{1.4\pi i}, 0.2e^{0.6\pi i}), (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i})) \\ (x \ : \ (0.7e^{1.6\pi i}, 0.3e^{0.4\pi i}), (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i}), (0.4e^{\pi i}, 0.6e^{\pi i})) \end{Bmatrix} \quad \textit{and}$$

$$J = \begin{Bmatrix} (e \ : \ (0.6e^{1.3\pi i}, 0.4e^{0.7\pi i}), (0.5e^{0.5\pi i}, 0.5e^{1.5\pi i}), (0.4e^{0.3\pi i}, 0.6e^{1.7\pi i})) \\ (x \ : \ (0.5e^{1.2\pi i}, 0.5e^{0.8\pi i}), (0.4e^{\pi i}, 0.6e^{\pi i}), (0.3e^{0.8\pi i}, 0.7e^{1.2\pi i})) \end{Bmatrix}.$$

Then,

$$I \cap J = \begin{Bmatrix} (e \ : \ (0.6e^{1.3\pi i}, 0.4e^{0.7\pi i}), (0.5e^{0.5\pi i}, 0.5e^{1.5\pi i}), (0.4e^{0.3\pi i}, 0.6e^{1.7\pi i})) \\ (x \ : \ (0.5e^{1.2\pi i}, 0.5e^{0.8\pi i}), (0.4e^{\pi i}, 0.6e^{\pi i}), (0.3e^{0.8\pi i}, 0.7e^{1.2\pi i})) \end{Bmatrix}.$$

Example 5: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup. If I and J are two CIFMPGs of P, then J is said to be equal to or similar to I, denoted $J = I$, if*

$$CM_J(c)e^{i\alpha_J(c)} \leq CM_I(c)e^{i\alpha_I(c)},$$
$$CN_J(c)e^{i\beta_J(c)} \geq CN_I(c)e^{i\beta_I(c)},$$
$$CM_J(c)e^{i\alpha_J(c)} \geq CM_I(c)e^{i\alpha_I(c)},$$
$$CN_J(c)e^{i\beta_J(c)} \leq CN_I(c)e^{i\beta_I(c)},$$

for all $a, b \in P$ and $c \in a \circ b$.

Remark 3: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup with I and J two CIFMPGs of P. Then I is equal to J if $I \subseteq J$ and $J \subseteq I$.*

Definition 16: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup, and I and J two CIFMPGs of P. Then J is contained in I, denoted $J \subseteq I$, if $CM_J(c)e^{i\alpha_J(c)} \leq CM_I(c)e^{i\alpha_I(c)}$ and $CN_J(c)e^{i\beta_J(c)} \geq CN_I(c)e^{i\beta_I(c)}$, for all $a, b \in P$ and $c \in a \circ b$. Then J is a subset of I and I is a superset of J.*

Example 6: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup, and I and J be two CIFMPGs of P defined by:*

$$I = \left\{ \begin{array}{l} (e \,:\, (0.9e^{1.8\pi i}, 0.1e^{0.2\pi i}), (0.8e^{1.4\pi i}, 0.2e^{0.6\pi i}), (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i})) \\ (x \,:\, (0.7e^{1.6\pi i}, 0.3e^{0.4\pi i}), (0.6e^{1.2\pi i}, 0.4e^{0.8\pi i}), (0.4e^{\pi i}, 0.6e^{\pi i})) \end{array} \right\} \quad \text{and}$$

$$J = \begin{Bmatrix} (e \,:\, (0.6e^{1.3\pi i}, 0.4e^{0.7\pi i}), (0.5e^{0.5\pi i}, 0.5e^{1.5\pi i}), (0.4e^{0.3\pi i}, 0.6e^{1.7\pi i})) \\ (x \,:\, (0.5e^{1.2\pi i}, 0.5e^{0.8\pi i}), (0.4e^{\pi i}, 0.6e^{\pi i}), (0.3e^{0.8\pi i}, 0.7e^{1.2\pi i})) \end{Bmatrix}.$$

Then, $J \subseteq I$.

Definition 17: *Let $\langle P_1, \circ_1, e_1, ^{-1}\rangle$ and $\langle P_2, \circ_2, e_2, ^{-1}\rangle$ be two polygroups with IFMSs I and J, respectively. If I and J are CIFMPGs of P_1 and P_2, respectively, then $I\times J$ is a CIFMPG of the productional polygroup $\langle P_1\times P_2, \circ, e^{-1}\rangle$, where $CM_{I\times J}(a, b)e^{i\alpha_{I\times J}(a,b)} = CM_I(a)e^{i\alpha_I(a)} \wedge CM_J(b)e^{i\alpha_J(b)}$, $CN_{I\times J}(a, b)e^{i\beta_{I\times J}(a,b)} = CN_I(a)e^{i\beta_I(a)} \vee CN_J(b)e^{i\beta_J(b)}$, for all $(a, b) \in P_1 \times P_2$.*

Example 7: *Let $\langle P_1, \circ_1, e_1, ^{-1}\rangle$ and $\langle P_2, \circ_2, e_2, ^{-1}\rangle$ be two polygroups, and let I and J be two CIFMPGs for these polygroups. With $P_1 = \{a_1, a_2\}$ and $P_2 = \{b_1, b_2\}$, we have*

$$I = \begin{Bmatrix} (a_1 \ : \ (0.8e^{1.2\pi i}, 0.2e^{0.8\pi i})), \\ (a_2 \ : \ (0.7e^{1.4\pi i}, 0.3e^{0.6\pi i})), \\ (b_1 \ : \ (0.6e^{1.8\pi i}, 0.4e^{0.4\pi i})), \\ (b_2 \ : \ (0.5e^{1.8\pi i}, 0.5e^{0.2\pi i})) \end{Bmatrix} \quad and$$

$$J = \begin{Bmatrix} (a_1 \ : \ (0.7e^{1.1\pi i}, 0.3e^{0.9\pi i})), \\ (a_2 \ : \ (0.6e^{1.3\pi i}, 0.4e^{0.7\pi i})), \\ (b_1 \ : \ (0.5e^{1.5\pi i}, 0.5e^{0.5\pi i})), \\ (b_2 \ : \ (0.4e^{1.7\pi i}, 0.6e^{0.3\pi i})) \end{Bmatrix}.$$

For I and J, we need to check if the IFMPG conditions are satisfied. Recall $I \times J$ implies $CM_{I\times J}(a,b)e^{i\alpha_{I\times J}(a,b)} = CM_I(a)e^{i\alpha_I(a)} \wedge CM_J(b)e^{i\alpha_J(b)}$ and $CN_{I\times J}(a,b)e^{i\beta_{I\times J}(a,b)} = CN_I(a)e^{i\beta_I(a)} \vee CN_J(b)e^{i\beta_J(b)}$. If I and J satisfy the conditions, then their Cartesian product should also satisfy the conditions. We check if $CM_I(a_1)e^{i\alpha_I(a_1)} \leq CM_J(e_2)e^{i\alpha_J(e_2)}$ and $CN_I(a_1)e^{i\beta_I(a_1)} \geq CN_J(e_2)e^{i\beta_J(e_2)}$ for each $a_1 \in P_1$ and e_2 being the identity element of P_2.

For simplicity, assume e_2 is the identity element in P_2. Suppose $CM_I(a_1) = 0.8e^{i1.2\pi}$ and $CM_J(e_2) = 0.5e^{i0.5\pi}$. We see that $0.8e^{i1.2\pi} \leq 0.5e^{i0.5\pi}$.

Similarly for non-membership: $CN_I(a_1) = 0.2e^{i0.8\pi}$ and $CN_J(e_2) = 0.5e^{i0.5\pi}$. We see that $0.2e^{i0.8\pi} \geq 0.5e^{i0.5\pi}$.

Since I and J satisfy the conditions, their Cartesian product $I \times J$, which is an IFMPG of $\langle P_1 \times P_2, \circ, e^{-1}\rangle$, should also satisfy the conditions.

Example 8: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup. With I a CIFMPG of P defined by*

$$I = \left\{ \begin{array}{l} (e \ : \ (0.9e^{1.8\pi i},\ 0.1e^{0.2\pi i}),\ (0.8e^{1.4\pi i},\ 0.2e^{0.6\pi i}),\ (0.6e^{1.2\pi i},\ 0.4e^{0.8\pi i})), \\ (\mu \ : \ (0.7e^{1.6\pi i},\ 0.3e^{0.4\pi i}),\ (0.6e^{1.2\pi i},\ 0.4e^{0.8\pi i}),\ (0.4e^{\pi i},\ 0.6e^{\pi i})) \end{array} \right\},$$

we have

$$I \times I = \left\{ \begin{array}{l} ((e,e) \ : \ (0.9e^{1.8\pi i},\ 0.1e^{0.2\pi i}),\ (0.8e^{1.4\pi i},\ 0.2e^{0.6\pi i}),\ (0.6e^{1.2\pi i},\ 0.4e^{0.8\pi i})), \\ ((e\mu) \ : \ (0.7e^{1.6\pi i},\ 0.3e^{0.4\pi i}),\ (0.6e^{1.2\pi i},\ 0.4e^{0.8\pi i}),\ (0.4e^{\pi i},\ 0.6e^{\pi i})), \\ ((\mu\mu) \ : \ (0.7e^{1.6\pi i},\ 0.3e^{0.4\pi i}),\ (0.6e^{1.2\pi i},\ 0.4e^{0.8\pi i}),\ (0.4e^{\pi i},\ 0.6e^{\pi i})), \\ ((\mu e) \ : \ (0.7e^{1.6\pi i},\ 0.3e^{0.4\pi i}),\ (0.6e^{1.2\pi i},\ 0.4e^{0.8\pi i}),\ (0.4e^{\pi i},\ 0.6e^{\pi i})) \end{array} \right\}.$$

4 Bipolar Complex Intuitionistic Fuzzy Multi-Polygroups

This section develops BCIFMPGs which represents a significant advancement in the field of fuzzy logic and algebraic structures. BCIFMPG introduces dual-polarity membership and non-membership values. This allows the representation of positive and negative impacts simultaneously, providing a comprehensive framework for modeling data that involves conflicting, hesitant, or dual-natured information. The inclusion of complex numbers further enhances its ability to handle intricate real-world uncertainties.

Definition 18: *Let $\langle P, \circ, e,^{-1} \rangle$ be a polygroup. Then, an IFMS I over P is a BCIFMPG of P if for all $a, b \in P$, the following conditions are satisfied:*

(i) $CM_I^+(c)e^{i\alpha_I^+(c)} \geq CM_I^+(a)e^{i\alpha_I^+(a)} \wedge CM_I^+(b)e^{i\alpha_I^+(b)}$,
$CM_I^-(c)e^{i\beta_I^-(c)} \leq CM_I^-(a)e^{i\beta_I^-(a)} \vee CM_I^-(b)e^{i\beta_I^-(b)}$,
$CN_I^+(c)e^{i\gamma_I^+(c)} \leq CN_I^+(a)e^{i\gamma_I^+(a)} \vee CN_I^+(b)e^{i\gamma_I^+(b)}$,
$CN_I^-(c)e^{i\delta_I^-(c)} \geq CN_I^-(a)e^{i\delta_I^-(a)} \wedge CN_I^-(b)e^{i\delta_I^-(b)}$.

(ii) $CM_I^+(a^{-1})e^{i\alpha_I^+(a^{-1})} \geq CM_I^+(a)e^{i\alpha_I^+(a)}$, $CM_I^-(a^{-1})e^{i\beta_I^-(a^{-1})} \leq CM_I^-(a)e^{i\beta_I^-(a)}$,
$CN_I^+(a^{-1})e^{i\gamma_I^+(a^{-1})} \leq CN_I^+(a)e^{i\gamma_I^+(a)}$, $CN_I^-(a^{-1})e^{i\delta_I^-(a^{-1})} \geq CN_I^-(a)e^{i\delta_I^-(a)}$.

It is an objective of the form: $I = \{(x,\ CM_I^+(c)e^{i\alpha_I^+(c)},\ CM_I^-(c)e^{i\beta_I^-(c)},\ CN_I^+(c)e^{i\gamma_I^+(c)},\ CN_I^-(c)e^{i\delta_I^-(c)}) : c \in a \circ b\}$ *for all* $a, b \in P$, *where* $CM_I^+, CN_I^+ : [0,1], CM_I^-, CN_I^- : [-1,0]$, *and* $0 \leq CM_I^+(c) + CN_I^+(c) \leq 1, -1 \leq CM_I^-(c) + CN_I^-(c) \leq 0$.

In addition, the phase term of bipolar positive complex membership function, bipolar negative complex membership function, bipolar positive complex non-membership degree, and negative complex non-membership degree belongs to $(0, 2\pi]$. By definition, the values of $CM_I^+(c)e^{i\alpha_I^+(c)}$ is the positive complex membership degree, $CM_I^-(c)e^{i\beta_I^-(c)}$ is the negative complex membership degree, $CN_I^+(c)e^{i\gamma_I^+(c)}$ is the positive complex non-membership degree and $CN_I^-(c)e^{i\delta_I^-(c)}$ is the negative complex non-membership degree in P, and their sum is within the unit circle in the complex plane, where $i = \sqrt{-1}$.

Example 9: *Let $P_1 = \{e, x\}$ and $\langle P, \circ, e^{-1}\rangle$ be the polygroup. Then,*

$$I = \left\{ \begin{array}{l} \big(e : \ (0.7e^{i1.7\pi}, -0.4e^{-i1.8\pi}),\ (0.3e^{i0.2\pi}, -0.3e^{-i0.2\pi})\big), \\ \big(x : \ (0.5e^{i1.2\pi}, -0.3e^{-i1.2\pi}),\ (0.3e^{i0.6\pi}, -0.6e^{-i0.3\pi})\big) \end{array} \right\}$$

is a BCIFMPG of P_1. Also $P_2 = \{e, x\}$ and $\langle P, \circ, e^{-1}\rangle$ be the polygroup, we have,

$$J = \left\{ \begin{array}{l} (e : \ \dots), \\ (x : \ \dots) \end{array} \right\}.$$

We show that I and J are BCIFMPG. Starting with I, we have

$$CM_I^+(ex)e^{i\alpha_I^+(ex)} = CM_I^+(x)e^{i\alpha_I^+(x)} = 0.5e^{i1.2\pi} \geq CM_I^+(e)e^{i\alpha_I^+(e)} \wedge CM_I^+(x)e^{i\alpha_I^+(x)} = 0.5e^{i1.2\pi}.$$

$$CM_I^-(ex)e^{i\beta_I^-(ex)} = CM_I^-(x)e^{i\beta_I^-(x)} = -0.3e^{-i1.2\pi} \leq CM_I^-(e)e^{i\beta_I^-(e)} \vee CM_I^-(x)e^{i\beta_I^-(x)} = -0.3e^{i1.2\pi}.$$

$$CN_I^+(ex)e^{i\gamma_I^+(ex)} = CN_I^+(x)e^{i\gamma_I^+(x)} = 0.3e^{i0.6\pi} \leq CN_I^+(e)e^{i\gamma_I^+(e)} \vee CN_I^+(x)e^{i\gamma_I^+(x)} = 0.3e^{i0.6\pi}.$$

$$CN_I^-(ex)e^{i\delta_I^-(ex)} = CN_I^-(x)e^{i\delta_I^-(x)} = -0.6e^{-i0.3\pi} \geq CN_I^-(e)e^{i\delta_I^-(e)} \wedge CN_I^-(x)e^{i\delta_I^-(x)} = -0.6e^{i0.5\pi}.$$

$$CM_I^+(xx)e^{i\alpha_I^+(xx)} = CM_I^+(e)e^{i\alpha_I^+(e)} = 0.7e^{i1.7\pi} \geq CM_I^+(x)e^{i\alpha_I^+(x)} \wedge CM_I^+(x)e^{i\alpha_I^+(x)} = 0.5e^{i1.2\pi}.$$

$$CM_I^-(xx)e^{i\beta_I^-(xx)} = CM_I^-(e)e^{i\beta_I^-(e)} = -0.4e^{-i1.8\pi} \leq CM_I^-(x)e^{i\beta_I^-(x)} \vee CM_I^-(x)e^{i\beta_I^-(x)} = -0.3e^{-i1.2\pi}.$$

$$CN_I^+(xx)e^{i\gamma_I^+(xx)} = CN_I^+(e)e^{i\gamma_I^+(e)} = 0.3e^{i0.2\pi} \leq CN_I^+(x)e^{i\gamma_I^+(x)} \vee CN_I^+(x)e^{i\gamma_I^+(x)} = 0.3e^{i0.6\pi}.$$

$$CN_I^-(xx)e^{i\delta_I^-(xx)} = CN_I^-(e)e^{i\delta_I^-(e)} = -0.3e^{-i0.2\pi} \geq CN_I^-(x)e^{i\delta_I^-(x)} \wedge CN_I^-(x)e^{i\delta_I^-(x)} = -0.6e^{i0.3\pi}.$$

$$CM_I^+(xe)e^{i\alpha_I^+(xe)} = CM_I^+(x)e^{i\alpha_I^+(x)} = 0.5e^{i1.2\pi} \geq CM_I^+(x)e^{i\alpha_I^+(x)} \wedge CM_I^+(e)e^{i\alpha_I^+(e)} = 0.5e^{i1.2\pi}.$$

$$CM_I^-(xe)e^{i\beta_I^-(xe)} = CM_I^-(x)e^{i\beta_I^-(x)} = -0.3e^{-i1.2\pi} \leq CM_I^-(e)e^{i\beta_I^-(e)} \vee CM_I^-(x)e^{i\beta_I^-(x)} = -0.3e^{-i1.2\pi}.$$

$$CN_I^+(xe)e^{i\gamma_I^+(xe)} = CN_I^+(x)e^{i\gamma_I^+(x)} = 0.3e^{i0.6\pi} \leq CN_I^+(e)e^{i\gamma_I^+(e)} \vee CN_I^+(x)e^{i\gamma_I^+(x)} = 0.3e^{i0.6\pi}.$$

$$CN_I^-(xe)e^{i\delta_I^-(xe)} = CN_I^-(x)e^{i\delta_I^-(x)} = -0.6e^{-i0.3\pi} \geq CN_I^-(e)e^{i\delta_I^-(e)} \wedge CN_I^-(x)e^{i\delta_I^-(x)} = -0.6e^{i0.3\pi}.$$

$$CM_I^+(ee)e^{i\alpha_I^+(ee)} = CM_I^+(e)e^{i\alpha_I^+(e)} = 0.7e^{i1.7\pi} \geq CM_I^+(e)e^{i\alpha_I^+(e)} \wedge CM_I^+(e)e^{i\alpha_I^+(e)} = 0.7e^{i1.7\pi},$$

$$CM_I^-(ee)e^{i\beta_I^-(ee)} = CM_I^-(e)e^{i\beta_I^-(e)} = -0.4e^{-i1.8\pi} \leq CM_I^-(e)e^{i\beta_I^-(e)} \vee CM_I^-(e)e^{i\beta_I^-(e)} = -0.4e^{-i1.8\pi}.$$

$$CN_I^+(ee)e^{i\gamma_I^+(ee)} = CN_I^+(e)e^{i\gamma_I^+(e)} = 0.3e^{i0.2\pi} \leq CN_I^+(e)e^{i\gamma_I^+(e)} \vee CN_I^+(e)e^{i\gamma_I^+(e)} = 0.3e^{i0.6\pi},$$

$$CN_I^-(ee)e^{i\delta_I^-(ee)} = CN_I^-(e)e^{i\delta_I^-(e)} = -0.3e^{-i0.2\pi} \geq CN_I^-(e)e^{i\delta_I^-(e)} \wedge CN_I^-(e)e^{i\delta_I^-(e)} = -0.3e^{i0.3\pi}.$$

$$CM_I^+(e^{-1})e^{i\alpha_I^+(e^{-1})} = CM_I^+(e)e^{i\alpha_I^+(e)} = 0.7e^{i1.7\pi},$$

$$CM_I^-(e^{-1})e^{i\beta_I^-(e^{-1})} = CM_I^-(e)e^{i\beta_I^-(e)} = -0.4e^{-i1.8\pi},$$

$$CN_I^+(e^{-1})e^{i\gamma_I^+(e^{-1})} = CN_I^+(e)e^{i\gamma_I^+(e)} = 0.3e^{i0.2\pi},$$

$$CN_I^-(e^{-1})e^{i\delta_I^-(e^{-1})} = CN_I^-(e)e^{i\delta_I^-(e)} = -0.3e^{-i0.2\pi}.$$

$$CM_I^+(x^{-1})e^{i\alpha_I^+(x^{-1})} = CM_I^+(x)e^{i\alpha_I^+(x)} = 0.5e^{i1.2\pi},$$

$$CM_I^-(x^{-1})e^{i\beta_I^-(x^{-1})} = CM_I^-(x)e^{i\beta_I^-(x)} = -0.3e^{-i1.2\pi},$$

$$CN_I^+(x^{-1})e^{i\gamma_I^+(x^{-1})} = CN_I^+(x)e^{i\gamma_I^+(x)} = 0.3e^{i0.6\pi},$$

$$CN_I^-(x^{-1})e^{i\delta_I^-(x^{-1})} = CN_I^-(x)e^{i\delta_I^-(x)} = -0.6e^{-i0.3\pi}.$$

Hence, I satisfies the conditions of BCIFMPG. Similarly, J is a BCIFMPG.

Definition 19: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup, and let I and J be two BCIFMPGs of P. Then J is said to be contained in I, denoted $J \subseteq I$, if for all $a, b \in P$ and $c \in a \circ b$, $CM_J^+(c)e^{i\alpha_J^+(c)} \leq CM_I^+(c)e^{i\alpha_I^+(c)}$, $CM_J^-(c)e^{i\beta_J^-(c)} \leq CM_I^-(c)e^{i\beta_I^-(c)}$, $CN_J^+(c)e^{i\gamma_J^+(c)} \geq CN_I^+(c)e^{i\gamma_I^+(c)}$, $CN_J^-(c)e^{i\delta_J^-(c)} \geq CN_I^-(c)e^{i\delta_I^-(c)}$.*

Example 10: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup and $P_1 = \{e, x\}$. Suppose I and J are BCIFMPGs of P_1 such that*

$$I = \left\{ \begin{array}{l} \left(e \ : \ (0.7e^{i1.7\pi},\ -0.4e^{-i1.8\pi},\ 0.3e^{i0.2\pi},\ -0.3e^{-i0.2\pi})\right), \\ \left(x \ : \ (0.5e^{i1.2\pi},\ -0.3e^{-i1.2\pi},\ 0.3e^{i0.6\pi},\ -0.6e^{-i0.3\pi})\right) \end{array} \right\},$$

$$J = \left\{ \begin{array}{l} \left(e \ : \ (0.6e^{i0.5\pi},\ -0.5e^{-i2\pi},\ 0.1e^{i0.3\pi},\ -0.6e^{-i0.6\pi})\right), \\ \left(x \ : \ (0.2e^{i1.1\pi},\ -0.4e^{-i1.6\pi},\ 0.1e^{i0.3\pi},\ -0.6e^{-i0.6\pi})\right) \end{array} \right\}.$$

Then $J \subseteq I$.

Definition 20: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup and let I and J be two BCIFMPGs of P. Then $I \circ J$ is defined as the bipolar complex intuitionistic fuzzy count function for $c \in a \circ b$ and for all $a, b \in P$ as follows:*
$CM_{I\cap J}^+(c)e^{i\alpha_{I\cap J}^+(c)} = CM_I^+(a)e^{i\alpha_I^+(a)} \wedge CM_J^+(b)e^{i\alpha_J^+(b)}$,
$CM_{I\cap J}^-(c)e^{i\beta_{I\cap J}^-(c)} = CM_I^-(a)e^{i\beta_I^-(a)} \vee CM_J^-(b)e^{i\beta_J^-(b)}$,
$CN_{I\cap J}^+(c)e^{i\gamma_{I\cap J}^+(c)} = CN_I^+(a)e^{i\gamma_I^+(a)} \vee CN_J^+(b)e^{i\gamma_J^+(b)}$,
$CN_{I\cap J}^-(c)e^{i\delta_{I\cap J}^-(c)} = CN_I^-(a)e^{i\delta_I^-(a)} \wedge CN_J^-(b)e^{i\delta_J^-(b)}$.

Remark 4: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup and let I and J be two IFMSs of P. If I and J are BCIFMPGs of P, then $I \cap J$ is a BCIFMPG of P.*

Example 11: *By using Example 4.2, we have*

$$I \cap J = \left\{ \begin{array}{l} \left(e \ : \ (0.6e^{i0.5\pi}, -0.4e^{-i1.8\pi},\ 0.3e^{i0.2\pi}, -0.6e^{-i0.6\pi})\right), \\ \left(x \ : \ (0.2e^{i1.1\pi}, -0.3e^{-i1.2\pi},\ 0.3e^{i0.6\pi}, -0.6e^{-i0.6\pi})\right) \end{array} \right\}.$$

Remark 5: *The union of two BCIFMPGs may not necessarily be a BCIFMPG.*

Definition 21: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup. Let*

$$I = \left\{ \left(x,\ CM_I^+(c)e^{i\alpha_I^+(c)},\ CM_I^-(c)e^{i\beta_I^-(c)},\ CN_I^+(c)e^{i\gamma_I^+(c)},\ CN_I^-(c)e^{i\delta_I^-(c)}\right)\ :\ \ c \in a \circ b \right\}$$

be a BCIFMPG of P. Then the complement of I, denoted I^c, is defined by

$$I^c = \left\{ \left(x,\ CM_I^-(c)e^{i\beta_I^-(c)},\ CM_I^+(c)e^{i\alpha_I^+(c)},\ CN_I^-(c)e^{i\delta_I^-(c)},\ CN_I^+(c)e^{i\gamma_I^+(c)}\right)\ :\ \ c \in a \circ b \right\}.$$

Remark 6: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup and let I be an IFMS of P. If I is a BCIFMPG of P, then I^c is also a BCIFMPG of P.*

Example 12: *By using Example 4.2, the complement of I is*

$$I^c = \left\{ \begin{array}{l} \left(e\ :\ \left(-0.4e^{-i1.8\pi},\ 0.7e^{i1.7\pi},\ -0.3e^{-i0.2\pi},\ 0.3e^{i0.2\pi}\right)\right), \\ \left(x\ :\ \left(-0.3e^{-i1.2\pi},\ 0.5e^{i1.2\pi},\ -0.6e^{-i0.3\pi},\ 0.3e^{i0.6\pi}\right)\right) \end{array} \right\}.$$

Similarly, the complement of J is

$$J^c = \left\{ \begin{array}{l} \left(e\ :\ \left(-0.5e^{-i2\pi},\ 0.6e^{i0.5\pi},\ -0.6e^{-i0.6\pi},\ 0.1e^{i0.3\pi}\right)\right), \\ \left(x\ :\ \left(-0.4e^{-i1.6\pi},\ 0.2e^{i1.1\pi},\ -0.6e^{-i0.6\pi},\ 0.1e^{i0.3\pi}\right)\right) \end{array} \right\}.$$

Definition 22: *Let $\langle P, \circ, e^{-1}\rangle$ be a polygroup, and let I and J be two BCIFMPGs of P. Then J is said to be equal to or similar to I, denoted $J = I$, if the following hold for all $a, b \in P$ and $c \in a \circ b$:*

$$CM_J^+(c)e^{i\alpha_J^+(c)} \leq CM_I^+(c)e^{i\alpha_I^+(c)}, \qquad CN_J^+(c)e^{i\gamma_J^+(c)} \geq CN_I^+(c)e^{i\gamma_I^+(c)},$$
$$CM_J^-(c)e^{i\beta_J^-(c)} \geq CM_I^-(c)e^{i\beta_I^-(c)}, \qquad CN_J^-(c)e^{i\delta_J^-(c)} \leq CN_I^-(c)e^{i\delta_I^-(c)},$$

and

$$CM_I^+(c)e^{i\alpha_I^+(c)} \leq CM_J^+(c)e^{i\alpha_J^+(c)}, \qquad CN_I^+(c)e^{i\gamma_I^+(c)} \geq CN_J^+(c)e^{i\gamma_J^+(c)},$$
$$CM_I^-(c)e^{i\beta_I^-(c)} \geq CM_J^-(c)e^{i\beta_J^-(c)}, \qquad CN_I^-(c)e^{i\delta_I^-(c)} \leq CN_J^-(c)e^{i\delta_J^-(c)}.$$

5 Conclusion

This study explored the theoretical foundations and algebraic structures of complex intuitionistic fuzzy multi-polygroups (CIFMPGs) and bipolar complex intuitionistic fuzzy multi-polygroups (BCIFMPGs). The integration of IFSs, complex numbers, and multi-polygroups provided a robust framework for representing uncertainty, ambiguity, and dual-polarity information. The integration of bipolar and complex-valued functions within the IFMPG framework has significantly advanced the capability to model complex and ambivalent data. The proposed BCIFMPG structure enhances the modeling of real-world scenarios where dual-polarity and hesitancy play critical roles. By offering greater precision and adaptability than traditional fuzzy systems, BCIFMPGs will pave the way for improved decision-making processes in domains such as pattern recognition, robotics, and data analysis. Future studies should delve into the advanced algebraic properties of BCIFMPGs, such as their subgroup structures, homomorphisms, and isomorphisms.

Author Contributions: All authors contributed to the concepts, mathematical development, analysis, and revision of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Acknowledgement: The authors gratefully acknowledge the editor and the reviewers for their valuable comments and suggestions, which helped improved the quality and presentation of this work. The authors also acknowledge the contributions of the researchers whose works are cited.

Funding Statement: This research did not receive external funding.

Data Availability Statement: Not applicable.

Ethics Approval: Not applicable. This study did not involve human participants or animals.

Use of Generative-AI tools declaration: The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflicts of Interest: The authors declare no competing interests.

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