

Research article

Time-dependent Axial Fluid Flow in an Annulus: A Semi-Analytical Approach

Jibrin Danjuma Yahaya¹ ^{*}, Bamidele David Michael¹ , Abraham Ayegba Alfa² , Isaac Adaji¹ ,
Aminu Ibrahim¹ , Raji Muhammed¹ , Agbata Benedict Celestine¹ 

¹Department of Mathematics and Statistics, Confluence University of Science and Technology, Osara, Nigeria.

²Department of Computer Science, Confluence University of Science and Technology, Osara, Nigeria.

^{*}Corresponding Author: Jibrin Danjuma Yahaya. Email: yahayajd@custech.edu.ng

Received: 14 July 2026; Accepted: 03 September 2026; Published: 14 September 2026

Citation: J.D. Yahaya, B.D Michael, A.A. Alfa, I. Adaji, A. Ibrahim, R. Muhammed, A.B. Celestine, Time-dependent Axial Fluid Flow in an Annulus: A Semi-Analytical Approach. *Ann. Commun. Math.* 9 (2026), 9. <https://doi.org/10.62072/acm.2026.09041>

Abstract: This study presents a semi-analytical solution for the start-up pressure-driven axial flow of an incompressible Newtonian fluid between two stationary concentric cylinders. The governing unsteady momentum equation is transformed using the Laplace technique, yielding closed-form expressions for the velocity and wall-shear distributions in terms of modified Bessel functions. The inverse transforms are evaluated by a Riemann-sum approximation, while exact steady-state solutions are derived for validation. For $\lambda = 0.2$, the velocities at $R = 0.4, 0.6$, and 0.8 reach 94.28%, 94.39%, and 95.06% of their steady values at $T = 0.2$, respectively. By $T = 0.4$, the maximum deviation from the steady solution decreases to 0.41%, and at $T = 10^4$ agreement is obtained to four decimal places. Increasing the radius ratio from 0.4 to 0.8 reduces the steady inner-wall shear from 0.3730 to 0.1042, a 72.06% decrease, while the outer-wall shear magnitude decreases from 0.2708 to 0.0967, corresponding to a 64.29% reduction. The main contribution is a compact, root-free benchmark formulation that combines the transient axial velocity, signed wall-shear histories and exact steady-state limits within a single computational framework.

Mathematics Subject Classification: 16U10, 16U80, 16U99, 16S34

Keywords: start-up annular flow; pressure-driven flow; Laplace transform; modified Bessel functions; Riemann-sum inversion; wall shear stress

1 Introduction

Pressure-driven flow through an annular passage is encountered in numerous engineering and scientific systems, including double-pipe heat exchangers, oil-well drilling, extrusion and coating equipment, lubrication systems, nuclear-reactor cooling channels, and biological transport devices. In many of these applications, the imposed pressure gradient changes with time, and the resulting fluid motion undergoes a finite development period before reaching the corresponding steady state. Accurate prediction of this start-up period is necessary because the instantaneous flow rate, velocity distribution, and wall shear can differ appreciably from their fully developed values.



The fundamental behaviour of pressure-driven viscous flows has been examined under a wide range of geometrical and constitutive conditions. Talon and Meiburg [1], for example, considered the stability of pressure-driven layered fluids with different viscosities, illustrating how material properties influence the development of viscous flows. For annular configurations, analytical solutions incorporating dynamic wall slip have been reported in [2,3]. Other studies have considered acoustic resonance, interfacial disturbance development, and wave propagation in annular systems [4,5]. These investigations demonstrate that annular flow is sensitive not only to the applied forcing but also to wall conditions, radius ratio, curvature, and fluid properties.

A related body of literature concerns Dean and Taylor–Dean flows, in which channel curvature, cylinder rotation, or centrifugal effects produce secondary motion and hydrodynamic instability. Elastic instability, order breaking, downstream decay of Dean vortices, and instability in doubly curved channels have been investigated in [6,7]. Exact and numerical solutions for pulsating, magnetohydrodynamic, and transitional Dean flows have also been presented in [8–12]. The relevance of Dean-flow mechanisms to microfluidic separation, biomedical transport, wall slip, non-Newtonian materials, and rotating micromixers has been demonstrated in [13–16]. More recently, unsteady magnetohydrodynamic Dean flow between concentric cylinders was analysed by [17]. Although these studies provide considerable insight into curvature-driven transport, their governing mechanisms differ from the purely axial, fully developed, pressure-driven start-up problem considered here.

Semi-analytical procedures provide an attractive alternative to purely numerical time integration because they retain the essential mathematical structure of the governing differential equations. Previous studies have applied Laplace transforms and Riemann-sum inversion to transient annular and curved-channel flows [18–21]. Such techniques are especially useful when the transformed governing equation admits a closed solution but its analytical inversion is complicated by Bessel functions and complex transform parameters. Unlike conventional eigenfunction expansions, the Riemann-sum method can be implemented without repeatedly calculating the roots of transcendental characteristic equations.

Despite the classical nature of annular Poiseuille flow, a compact formulation that simultaneously provides the transient velocity field, inner-wall shear, outer-wall shear, and an independent steady-state benchmark remains valuable. In particular, start-up wall loads cannot be inferred reliably from the steady solution during the early development period. A solution that does not require numerical time marching or explicit eigenvalue calculations can also serve as a verification benchmark for more complicated computational models.

The purpose of the present study is therefore to obtain a semi-analytical solution for the start-up axial flow of a Newtonian fluid in a concentric annulus subjected to a suddenly imposed constant pressure gradient. The velocity equation is transformed into the Laplace domain and solved in terms of modified Bessel functions. The inverse transform is evaluated numerically using a Riemann-sum approximation, and the transient wall shears are obtained by differentiating the transformed velocity field.

The scientific contributions of the work is associated with the following:

- (i) a closed Laplace-domain representation of the transient annular velocity satisfying both no-slip boundary conditions exactly;

- (ii) corresponding closed transformed expressions for the inner- and outer-wall shear histories;
- (iii) a root-free numerical inversion procedure that avoids explicit determination of Bessel-function eigenvalues; and
- (iv) quantitative verification of the transient results against exact steady-state velocity and wall-shear expressions.

2 Physical Interpretation and Basic Equations

Consider a long concentric annulus formed by two stationary cylinders of radii r_1 and r_2 , where $r_1 < r_2$. The axial coordinate is denoted by z' , while r is the radial coordinate measured from the common axis of the cylinders. The annular region is occupied by an incompressible Newtonian fluid of constant density ρ and dynamic viscosity μ .

The following assumptions are adopted:

- (i) the cylinders are sufficiently long for entrance and end effects to be neglected;
- (ii) the flow is laminar, axisymmetric, and fully developed in the axial direction;
- (iii) the only non-zero velocity component is the axial velocity $u(r, t)$;
- (iv) the fluid properties are constant;
- (v) the cylinder walls are impermeable, stationary, and satisfy the no-slip condition;
- (vi) gravitational and other body forces are neglected; and
- (vii) a constant axial pressure gradient is applied suddenly for $t > 0$.

Under these assumptions, continuity is identically satisfied, and the axial component of the Navier–Stokes equations becomes

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z'} + \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right). \quad (1)$$

Let the imposed pressure gradient be

$$-\frac{\partial p}{\partial z'} = \mathcal{G}, \quad \mathcal{G} > 0, \quad (2)$$

so that the pressure decreases in the positive axial direction. Equation (1) reduces to

$$\rho \frac{\partial u}{\partial t} = \mathcal{G} + \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right). \quad (3)$$

The start-up and no-slip conditions are

$$u(r, 0) = 0, \quad r_1 \leq r \leq r_2, \quad (4)$$

$$u(r_1, t) = 0, \quad u(r_2, t) = 0, \quad t > 0. \quad (5)$$

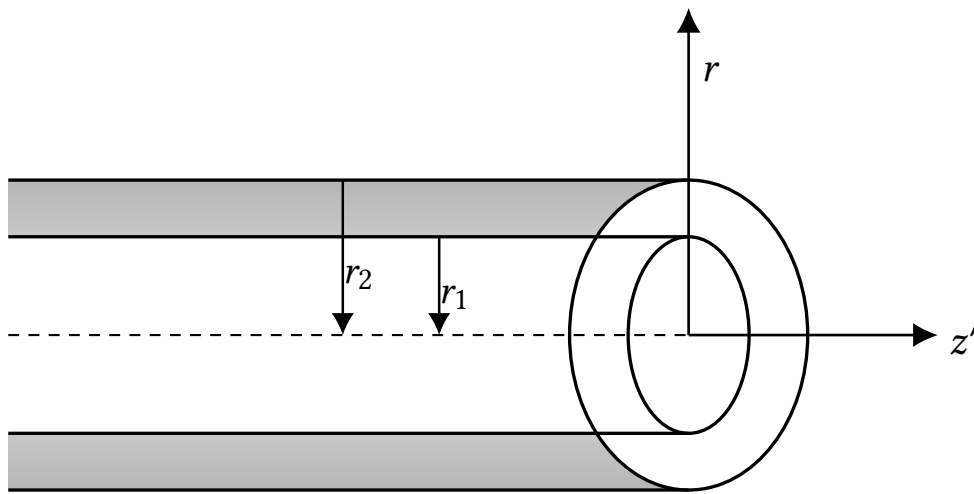


Figure 1: Geometry of the concentric annulus and coordinate system.

The dimensional wall shear stresses are defined by

$$\tau_{r_1} = \mu \frac{\partial u}{\partial r} \Big|_{r=r_1}, \quad \tau_{r_2} = \mu \frac{\partial u}{\partial r} \Big|_{r=r_2}. \quad (6)$$

The velocity increases away from the inner wall, giving a positive radial derivative there, whereas it decreases towards the outer wall, giving a negative radial derivative at $r = r_2$. Consequently, the two wall-shear measures have opposite signs.

Introduce the dimensionless variables

$$R = \frac{r}{r_2}, \quad \lambda = \frac{r_1}{r_2}, \quad T = \frac{\nu t}{r_2^2}, \quad U(R, T) = \frac{\mu u(r, t)}{Gr_2^2}, \quad \nu = \frac{\mu}{\rho}. \quad (7)$$

Here, λ is the radius ratio and $0 < \lambda < 1$. The dimensionless annular domain is $\lambda \leq R \leq 1$. Substitution of (7) into (3) gives

$$\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial R^2} + \frac{1}{R} \frac{\partial U}{\partial R} + 1. \quad (8)$$

The corresponding dimensionless initial and boundary conditions are

$$U(R, 0) = 0, \quad \lambda \leq R \leq 1, \quad (9)$$

$$U(\lambda, T) = 0, \quad U(1, T) = 0, \quad T > 0. \quad (10)$$

The dimensionless wall-shear measure is defined by

$$C_f(R, T) = \frac{\partial U}{\partial R}, \quad C_{f,\lambda}(T) = \frac{\partial U}{\partial R} \Big|_{R=\lambda}, \quad C_{f,1}(T) = \frac{\partial U}{\partial R} \Big|_{R=1}. \quad (11)$$

3 Semi-Analytical Solution

3.1 Laplace-domain velocity field

Define the Laplace transform with respect to T by [22]

$$\bar{U}(R, S) = \mathcal{L}\{U(R, T)\} = \int_0^\infty e^{-ST} U(R, T) dT, \quad \text{Re}(S) > 0. \quad (12)$$

Applying (12) to (8)–(10), and using $U(R, 0) = 0$, gives

$$\frac{d^2 \bar{U}}{dR^2} + \frac{1}{R} \frac{d\bar{U}}{dR} - S\bar{U} = -\frac{1}{S}, \quad (13)$$

with

$$\bar{U}(\lambda, S) = 0, \quad \bar{U}(1, S) = 0. \quad (14)$$

The homogeneous part of (13) is the modified Bessel equation of order zero. Its independent solutions are $I_0(R\sqrt{S})$ and $K_0(R\sqrt{S})$, and a constant particular solution is $1/S^2$. Application of the two transformed boundary conditions yields

$$\bar{U}(R, S) = \frac{1}{S^2} \left[1 + \frac{[K_0(\sqrt{S}) - K_0(\lambda\sqrt{S})]I_0(R\sqrt{S}) + [I_0(\lambda\sqrt{S}) - I_0(\sqrt{S})]K_0(R\sqrt{S})}{I_0(\sqrt{S})K_0(\lambda\sqrt{S}) - I_0(\lambda\sqrt{S})K_0(\sqrt{S})} \right]. \quad (15)$$

Substitution of $R = \lambda$ or $R = 1$ into (15) makes the expression in square brackets vanish, confirming that both no-slip conditions are satisfied exactly. The transient velocity in the physical time domain is formally given by

$$U(R, T) = \mathcal{L}^{-1}\{\bar{U}(R, S)\}. \quad (16)$$

Because (15) contains modified Bessel functions evaluated at complex arguments, direct analytical inversion is inconvenient. The inverse transform is therefore evaluated using the Riemann-sum procedure described in Section 3.4.

3.2 Exact steady-state solution

At steady state, $\partial U / \partial T = 0$, and the governing equation (8) becomes

$$\frac{d^2 U_s}{dR^2} + \frac{1}{R} \frac{dU_s}{dR} = -1, \quad (17)$$

subject to

$$U_s(\lambda) = 0, \quad U_s(1) = 0. \quad (18)$$

Integrating twice and applying the boundary conditions gives

$$U_s(R) = \frac{1}{4} \left[1 - R^2 - (1 - \lambda^2) \frac{\ln R}{\ln \lambda} \right]. \quad (19)$$

The transient solution must satisfy

$$\lim_{T \rightarrow \infty} U(R, T) = U_s(R). \quad (20)$$

Equation (19) provides an independent closed-form benchmark for assessing the large-time accuracy of the numerical Laplace inversion.

The location of the maximum steady velocity is obtained from $dU_s/dR = 0$, which gives

$$R_{\max} = \left[\frac{1 - \lambda^2}{2 \ln(1/\lambda)} \right]^{1/2}. \quad (21)$$

For $\lambda = 0.2$,

$$R_{\max} = 0.5461, \quad U_{s,\max} = 0.08523. \quad (22)$$

Thus, the maximum is displaced from the geometrical mid-gap position $R = 0.6$ towards the inner cylinder. This displacement is a consequence of the cylindrical-curvature term $R^{-1} \partial U / \partial R$.

3.3 Transformed wall-shear expressions

Differentiating (15) with respect to R , and using

$$\frac{dI_0(x)}{dx} = I_1(x), \quad \frac{dK_0(x)}{dx} = -K_1(x), \quad (23)$$

produces the transformed shear distribution. The inner-wall shear is

$$\bar{C}_{f,\lambda}(S) = \frac{[K_0(\sqrt{S}) - K_0(\lambda\sqrt{S})]I_1(\lambda\sqrt{S}) - [I_0(\lambda\sqrt{S}) - I_0(\sqrt{S})]K_1(\lambda\sqrt{S})}{(\sqrt{S})^3 [I_0(\sqrt{S})K_0(\lambda\sqrt{S}) - I_0(\lambda\sqrt{S})K_0(\sqrt{S})]}. \quad (24)$$

Similarly, the outer-wall shear is

$$\bar{C}_{f,1}(S) = \frac{[K_0(\sqrt{S}) - K_0(\lambda\sqrt{S})]I_1(\sqrt{S}) - [I_0(\lambda\sqrt{S}) - I_0(\sqrt{S})]K_1(\sqrt{S})}{(\sqrt{S})^3 [I_0(\sqrt{S})K_0(\lambda\sqrt{S}) - I_0(\lambda\sqrt{S})K_0(\sqrt{S})]}. \quad (25)$$

The time-domain wall shears are

$$C_{f,\lambda}(T) = \mathcal{L}^{-1}\{\bar{C}_{f,\lambda}(S)\}, \quad (26)$$

$$C_{f,1}(T) = \mathcal{L}^{-1}\{\bar{C}_{f,1}(S)\}. \quad (27)$$

Differentiation of the steady solution gives

$$C_{f,\lambda}^{(\text{st})} = \frac{1 - \lambda^2 - 2\lambda^2 \ln(1/\lambda)}{4\lambda \ln(1/\lambda)}, \quad (28)$$

$$C_{f,1}^{(\text{st})} = \frac{1 - \lambda^2 - 2 \ln(1/\lambda)}{4 \ln(1/\lambda)}. \quad (29)$$

Equation (28) is positive, whereas (29) is negative. The opposite signs reflect the increase of velocity away from the inner wall and its decrease towards the outer wall.

3.4 Riemann-sum approximation of Laplace inversion

For a transformed function $\bar{F}(S)$, the inverse Laplace transform is approximated as reported in [23,24]:

$$F(T) \approx \frac{e^{\epsilon T}}{T} \left[\frac{1}{2} \bar{F}(\epsilon) + \text{Re} \sum_{k=1}^N (-1)^k \bar{F} \left(\epsilon + \frac{ik\pi}{T} \right) \right], \quad (30)$$

where $i = \sqrt{-1}$, N is the truncation level, and $\epsilon > 0$ shifts the Bromwich contour into the right half of the complex plane. For the velocity solution, $\bar{F}(S) = \bar{U}(R, S)$, whereas $\bar{F}$ is replaced by $\bar{C}_{f,\lambda}$ or $\bar{C}_{f,1}$ when calculating the two wall shears.

The convergence parameter is selected such that $\epsilon T \approx 4.7$, and N is progressively increased until further increases produce no change at the reported numerical precision. This procedure avoids explicit eigenvalue calculations and directly evaluates the closed transformed expressions.

Following the across-gap sampling convention adopted by Orlandi et al.[25], representative radial stations are selected to describe the near-inner-wall, intermediate and near-outer-wall regions. For the baseline radius ratio $\lambda = 0.2$, $R = 0.2$ coincides with the inner wall and provides the exact boundary check

$$U(0.2, T) = 0. \quad (31)$$

while $R = 0.4, 0.6$, and 0.8 are used to quantify the interior velocity field. The additional location $R = 0.6$ is included because it lies close to $R_{\max} = 0.5461$ and therefore captures the largest velocities in the annulus.

The solution is assessed through four checks:

$$\bar{U}(\lambda, S) = 0, \quad \bar{U}(1, S) = 0, \quad (32)$$

$$\lim_{S \rightarrow \infty} S \bar{U}(R, S) = 0, \quad (33)$$

which confirms the initial condition, and

$$\lim_{S \rightarrow 0} S \bar{U}(R, S) = U_s(R), \quad (34)$$

which confirms the steady-state limit.

4 Results and Discussion

4.1 Temporal development of the velocity field

Figure 2 shows the evolution of the dimensionless axial velocity for the baseline radius ratio $\lambda = 0.2$. Every profile satisfies the no-slip conditions at $R = 0.2$ and $R = 1$. The velocity remains positive throughout the fluid region because the imposed pressure gradient acts continuously in the positive axial direction.

At the earliest reported times, the velocity is relatively small because the fluid has only recently been accelerated from rest. As T increases, the velocity rises monotonically and approaches the steady annular Poiseuille profile. The temporal development is governed by competition between the constant pressure forcing and viscous momentum diffusion. The pressure gradient supplies axial momentum throughout the gap, while viscosity transfers momentum towards the stationary walls, where it is dissipated through wall shear.

The profiles are asymmetric about the geometrical mid-gap location. For $\lambda = 0.2$, the exact steady maximum occurs at $R_{\max} = 0.5461$, rather than at the arithmetic midpoint $R = 0.6$. The maximum is shifted towards the inner cylinder because the term $R^{-1} \partial U / \partial R$ introduces a curvature-dependent radial diffusion that has no counterpart in a plane channel.

The numerical data in Table 1 quantify the approach to steady state. At $T = 0.2$, the calculated velocities are

$$U(0.4, 0.2) = 0.0692, \quad U(0.6, 0.2) = 0.0791, \quad U(0.8, 0.2) = 0.0539. \quad (35)$$

The corresponding steady-state values are 0.0734, 0.0838, and 0.0567. Hence, at $T = 0.2$, the transient solution has attained approximately 94.28%, 94.39%, and 95.06% of the local steady velocities at $R = 0.4$, 0.6, and 0.8, respectively. The remaining deviations are 5.72%, 5.61%, and 4.94%.

At $T = 0.4$, the corresponding velocities increase to 0.0731, 0.0836, and 0.0566. Their deviations from the steady solution are only 0.41%, 0.24%, and 0.18%, respectively. Therefore, by $T = 0.4$, more than 99.5% of the steady velocity has been established at each reported radial station.

At the near-peak station $R = 0.6$, the velocity increases from 0.0791 at $T = 0.2$ to 0.0836 at $T = 0.4$, corresponding to a 5.69% increase. Its limiting value is 0.0838, so the remaining increase after $T = 0.4$ is only 0.24%. At $T = 10^4$, the Riemann-sum and steady-state velocities agree to four decimal places at all reported locations.

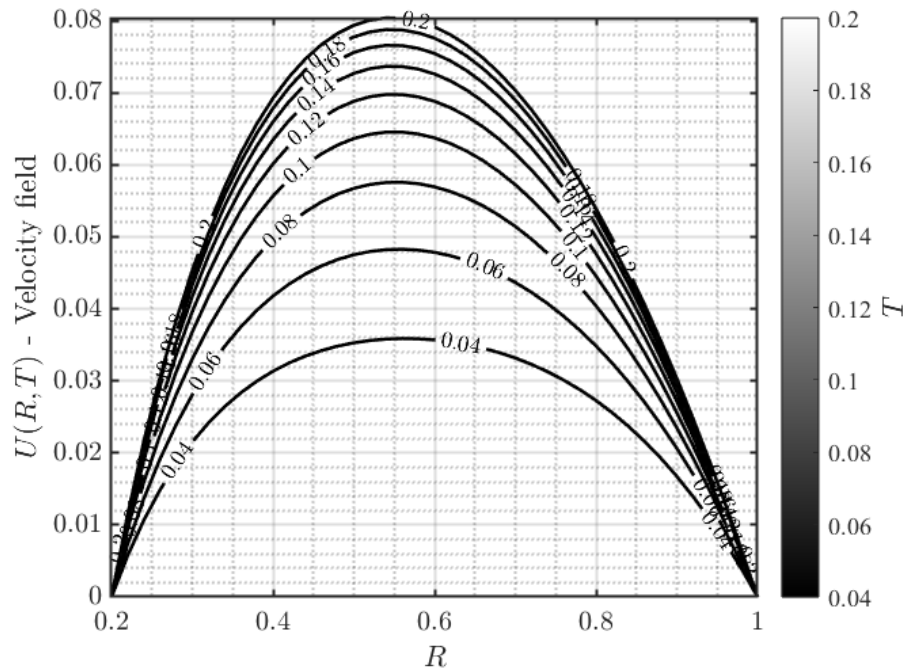


Figure 2: Effect of dimensionless time T on the velocity profile for $\lambda = 0.2$.

The slightly faster relative convergence observed near $R = 0.8$ should not be interpreted as stronger local acceleration. It arises partly because the local steady velocity is lower near the outer wall. In absolute terms, the largest velocity and the largest remaining transient correction occur in the interior of the annulus.

4.2 Inner-wall shear

Figure 3 shows the inner-wall shear $C_{f,\lambda}$ as a function of radius ratio and time. The inner-wall shear is positive because the velocity increases as the radial coordinate moves from the stationary inner cylinder into the fluid.

For a fixed radius ratio, the inner-wall shear generally increases in magnitude with time and approaches its steady value. Immediately after the pressure forcing is imposed, the fluid velocity is still small and the wall-normal velocity gradient has not fully developed. As momentum diffuses across the gap, the velocity gradient adjacent to the inner wall becomes progressively stronger until a balance is established between pressure forcing and viscous resistance. It is important to note that larger radius ratios produce weaker inner-wall shear because the gap becomes narrower. The reduction in shear indicates reduced velocity gradients at the wall.

The inner-wall shear decreases markedly as λ increases. According to Table 2, its steady value decreases from

$$C_{f,\lambda}^{(\text{st})} = 0.3730 \quad \text{at} \quad \lambda = 0.4 \quad (36)$$

Table 1: Axial velocity computed by the Riemann-sum approximation (RSA) and the exact steady-state solution for selected values of T and R at $\lambda = 0.2$.

T	R	RSA	Steady state
0.2	0.4	0.0692	0.0734
	0.6	0.0791	0.0838
	0.8	0.0539	0.0567
0.4	0.4	0.0731	0.0734
	0.6	0.0836	0.0838
	0.8	0.0566	0.0567
10^4	0.4	0.0734	0.0734
	0.6	0.0838	0.0838
	0.8	0.0567	0.0567

to

$$C_{f,\lambda}^{(\text{st})} = 0.1042 \quad \text{at} \quad \lambda = 0.8. \quad (37)$$

This represents a reduction of

$$\frac{0.3730 - 0.1042}{0.3730} \times 100 = 72.06\%. \quad (38)$$

The reduction occurs because increasing λ while keeping the outer radius fixed narrows the annular gap. The dimensional pressure-driven velocity scale then becomes increasingly controlled by the smaller gap width, and both the velocity and its dimensionless radial gradient decrease under the present outer-radius scaling.

At $T = 0.2$, the inner-wall values for $\lambda = 0.4, 0.6$, and 0.8 are $0.3716, 0.2221$, and 0.1043 , whereas the corresponding steady values are $0.3730, 0.2220$, and 0.1042 . The maximum relative difference is approximately 0.38% . At $T = 0.4$, all differences are approximately 0.10% or less. The very small instances in which the reported transient value exceeds the steady value by 10^{-4} should not be interpreted as a physically significant overshoot; they are comparable to the four-decimal-place rounding and truncation accuracy of the numerical inversion.

4.3 Outer-wall shear

Figure 4 presents the outer-wall shear $C_{f,1}$. It is negative because the velocity decreases from its interior maximum to zero at the outer cylinder. Its magnitude increases with time as the outer-wall viscous layer develops and subsequently approaches the steady value. Increasing radius ratio reduces shear asymmetry. The annulus approaches plane-channel behaviour when $\lambda \rightarrow 1$.

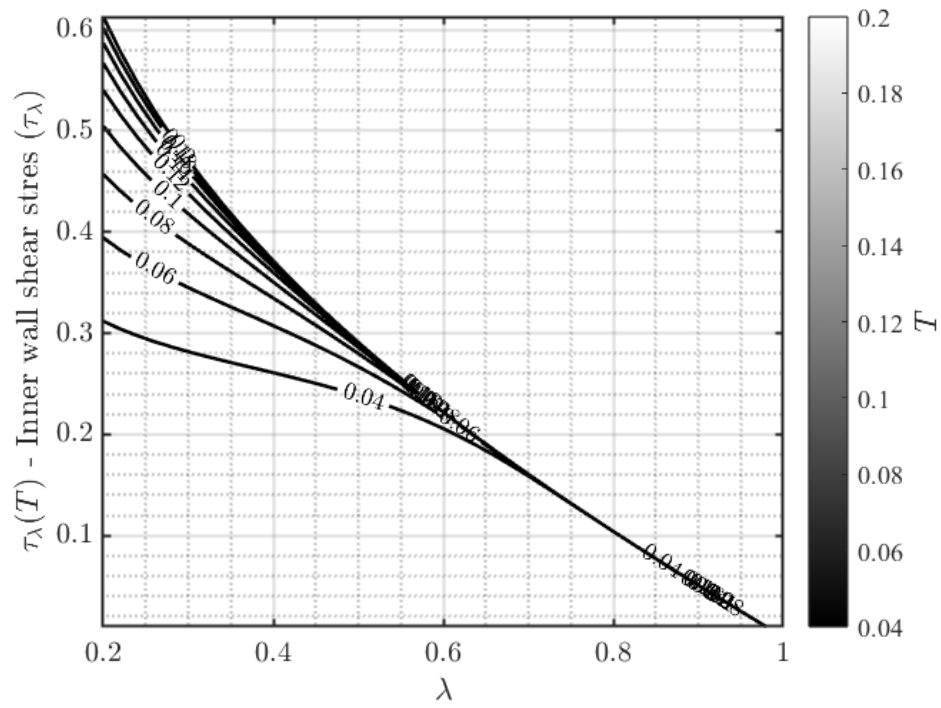


Figure 3: Effect of dimensionless time T on the inner-wall shear at $R = \lambda$.

At steady state, the magnitude of the outer-wall shear decreases from

$$\left| C_{f,1}^{(st)} \right| = 0.2708 \quad \text{at} \quad \lambda = 0.4 \quad (39)$$

to

$$\left| C_{f,1}^{(st)} \right| = 0.0967 \quad \text{at} \quad \lambda = 0.8. \quad (40)$$

The corresponding reduction is

$$\frac{0.2708 - 0.0967}{0.2708} \times 100 = 64.29\%. \quad (41)$$

At $T = 0.2$, the outer-wall values are -0.2700 , -0.1869 , and -0.0968 for $\lambda = 0.4$, 0.6 , and 0.8 , respectively. The maximum deviation from the corresponding steady value is approximately 0.30% . By $T = 0.4$, the maximum relative difference is approximately 0.10% .

For each radius ratio, the inner-wall shear magnitude is greater than the outer-wall shear magnitude. At $\lambda = 0.4$,

$$\frac{C_{f,\lambda}^{(st)} - \left| C_{f,1}^{(st)} \right|}{\left| C_{f,1}^{(st)} \right|} \times 100 = 37.74\%. \quad (42)$$

At $\lambda = 0.8$, the corresponding difference decreases to 7.76% . Thus, the shear asymmetry weakens as the gap becomes narrower. This behaviour is physically consistent with the fact that the annular geometry approaches the plane-channel limit as $\lambda \rightarrow 1$, in which the two walls have approximately equal curvature and comparable shear magnitudes.

Table 2: Inner- and outer-wall shear stresses computed by the Riemann-sum approximation (RSA) and the exact steady-state formula.

T	λ	Inner wall, $R = \lambda$		Outer wall, $R = 1$	
		RSA	Steady state	RSA	Steady state
0.2	0.4	0.3716	0.3730	-0.2700	-0.2708
	0.6	0.2221	0.2220	-0.1869	-0.1868
	0.8	0.1043	0.1042	-0.0968	-0.0967
0.4	0.4	0.3731	0.3730	-0.2710	-0.2708
	0.6	0.2222	0.2220	-0.1869	-0.1868
	0.8	0.1043	0.1042	-0.0968	-0.0967
10^4	0.4	0.3730	0.3730	-0.2708	-0.2708
	0.6	0.2220	0.2220	-0.1868	-0.1868
	0.8	0.1042	0.1042	-0.0967	-0.0967

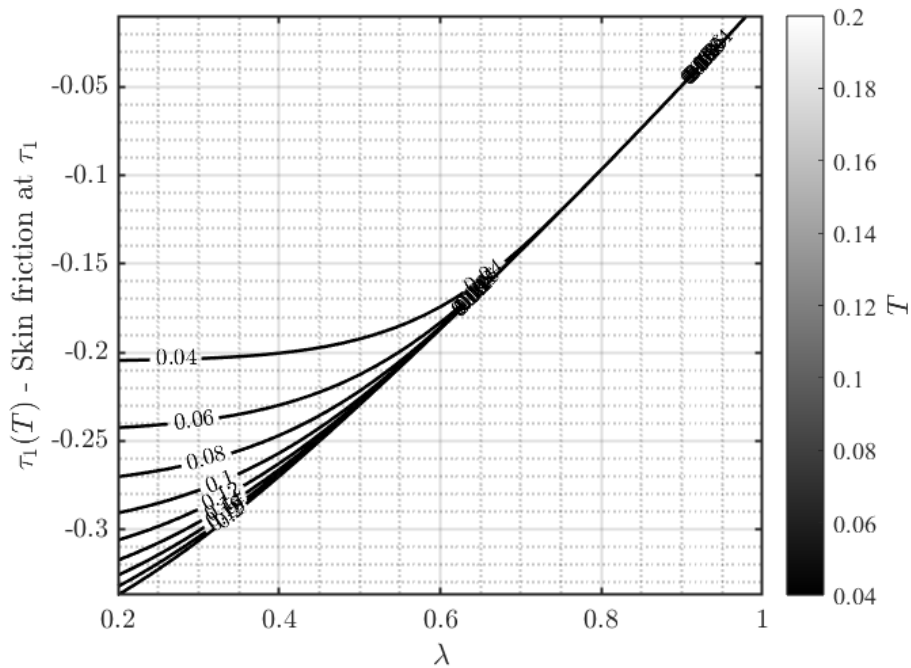


Figure 4: Effect of dimensionless time T on the outer-wall shear at $R = 1$.

4.4 Effect of radius ratio on the establishment time

The curves in Figs. 3 and 4 become increasingly close to their steady limits as λ increases. The characteristic transverse diffusion time based on the dimensional gap width $r_2 - r_1$ is proportional to

$$\frac{(r_2 - r_1)^2}{\nu}. \quad (43)$$

In terms of the present dimensionless variables, this time scale is proportional to $(1 - \lambda)^2$. Consequently, a narrow annulus with a large value of λ reaches its fully developed state more rapidly than a wide annulus when time is normalised using r_2^2/ν . This explains why the transient and steady wall-shear values are already nearly indistinguishable at $T = 0.2$ for $\lambda = 0.6$ and 0.8 .

The radius ratio therefore influences both the magnitude of the developed flow and the time required for its establishment. Larger values of λ produce smaller velocities and wall shears under the adopted scaling, but also shorten the viscous diffusion distance across the gap.

4.5 Validation of the semi-analytical procedure

The numerical results satisfy several independent consistency checks. First, the velocity is zero at both cylinder surfaces for every reported time. Second, the velocity remains non-negative throughout the gap

and increases from its initial value towards the positive steady solution. Third, the inner- and outer-wall shears have the expected positive and negative signs, respectively.

Most importantly, the transient velocity and wall-shear results converge to their independently derived steady-state expressions. At $T = 10^4$, all velocity and wall-shear entries in Tables 1 and 2 agree with their steady values to four decimal places. This agreement supports the correctness of the Laplace-domain solution, its radial differentiation, and the Riemann-sum inversion.

The method is particularly useful as a benchmark because it avoids spatial discretisation, time stepping, and explicit calculation of eigenvalues. Nevertheless, agreement with the steady solution verifies only the large-time limit. Additional experimental measurements or comparison with an independent transient numerical solver would be required for a complete external validation of the early-time solution.

5 Conclusion

The start-up pressure-driven axial flow of an incompressible Newtonian fluid between two stationary concentric cylinders has been investigated using a semi-analytical Laplace-transform approach. Closed expressions for the transformed velocity, inner-wall shear, and outer-wall shear were obtained in terms of modified Bessel functions. The time-domain solutions were recovered using a Riemann-sum approximation, while exact steady-state expressions were derived independently for validation.

The velocity increases monotonically from the initial state of rest and approaches the steady annular Poiseuille profile as viscous momentum diffusion proceeds across the gap. For $\lambda = 0.2$, the steady maximum occurs at $R = 0.5461$, with $U_{s,\max} = 0.08523$. The displacement of the maximum from the geometrical mid-gap location is caused by cylindrical curvature.

At $T = 0.2$, the velocities at $R = 0.4, 0.6$, and 0.8 are $0.0692, 0.0791$, and 0.0539 , representing $94.28\%, 94.39\%$, and 95.06% of their respective steady values. By $T = 0.4$, the maximum deviation from the steady velocity has decreased to 0.41% , and more than 99.5% of the steady state has been established at all three locations. At $T = 10^4$, the transient and steady solutions agree to four decimal places.

The wall-shear results show that the velocity gradient is positive at the inner cylinder and negative at the outer cylinder. Increasing the radius ratio from 0.4 to 0.8 reduces the steady inner-wall shear from 0.3730 to 0.1042 , corresponding to a 72.06% reduction. Over the same range, the magnitude of the steady outer-wall shear decreases from 0.2708 to 0.0967 , representing a 64.29% reduction. The difference between the two shear magnitudes decreases from 37.74% at $\lambda = 0.4$ to 7.76% at $\lambda = 0.8$, indicating that the flow becomes progressively more similar to plane-channel flow as the annular gap narrows.

The scientific contribution of the study is the construction of a unified, root-free semi-analytical benchmark for the transient velocity and both wall-shear histories. The method uses the closed Laplace-domain solution directly and avoids numerical time marching and the explicit calculation of Bessel-function eigenvalues. Its convergence to the exact steady-state solution demonstrates its suitability for verifying numerical simulations of transient annular flow.

The main limitations are the assumptions of a Newtonian fluid, constant properties, axisymmetric fully developed motion, perfectly concentric stationary cylinders, no-slip walls, and a suddenly imposed constant pressure gradient. The model excludes entrance effects, wall motion, eccentricity, wall slip, suction or

injection, porous resistance, magnetic fields, heat transfer, variable viscosity, non-Newtonian effects, and turbulent instability. Furthermore, the transient predictions have been verified against the analytical steady limit but have not yet been compared with experimental data.

Future investigations may extend the present formulation in several directions. Experimental studies involving concentric annular flow systems would provide additional validation of the transient predictions, particularly during the early development period where analytical comparison remains limited. Furthermore, comparison with high-resolution numerical simulations, including finite-difference, finite-volume, and spectral methods, would provide further assessment of the accuracy and computational efficiency of the proposed semi-analytical approach.

The present model may also be extended to more complex physical situations, including heat transfer effects, variable viscosity fluids, non-Newtonian fluids, time-dependent pressure gradients, moving cylinder boundaries, eccentric annular configurations, porous annular media, and magnetohydrodynamic effects. Such extensions would broaden the applicability of the method to industrial, biomedical, and energy-related transport systems.

In addition, future studies may investigate data-driven and machine-learning-assisted approaches combined with the semi-analytical framework to provide rapid prediction tools for transient annular flow behaviour under different operating conditions.

Nomenclature

Symbol	Description	Dimensions
r, z'	Radial and axial coordinates	L
t	Dimensional time	T
r_1, r_2	Inner- and outer-cylinder radii	L
$u(r, t)$	Dimensional axial velocity	LT^{-1}
R	Dimensionless radial coordinate, $R = r/r_2$	1
T	Dimensionless time, $T = \nu t/r_2^2$	1
$U(R, T)$	Dimensionless axial velocity	1
$U_s(R)$	Dimensionless steady-state axial velocity	1
p	Fluid pressure	$ML^{-1}T^{-2}$
$\mathcal{G}$	Constant axial pressure gradient	$ML^{-2}T^{-2}$
ρ	Fluid density	ML^{-3}
μ	Dynamic viscosity	$ML^{-1}T^{-1}$
ν	Kinematic viscosity, $\nu = \mu/\rho$	L^2T^{-1}
λ	Radius ratio, $\lambda = r_1/r_2$	1
S	Laplace-transform parameter	1
$\overline{U}(R, S)$	Laplace transform of $U(R, T)$	1
$I_0(\cdot), K_0(\cdot)$	Modified Bessel functions of order zero	1
$I_1(\cdot), K_1(\cdot)$	Modified Bessel functions of order one	1

Author Contributions: J.D. Yahaya: Conceptualization, Formal analysis, Methodology, Writing—Original Draft, Writing—Review and Editing, Visualization, Supervision, Software, Data curation, Project administration, Funding

acquisition, Investigation, Results analysis, Resources. *M.B. David*: Writing—Original Draft, Writing—Review and Editing, Visualization, Data curation, Project administration, Results analysis. *A.A. Alfa*: Conceptualization, Formal analysis, Writing—Review and Editing, Visualization, Software, Data curation, Results analysis, Resources. *I. Adaji*: Formal analysis, Methodology, Writing—Original Draft, Writing—Review and Editing, Visualization, Software, Data curation, Project administration, Investigation, Results analysis, Resources. *A. Ibrahim*: Formal analysis, Methodology, Writing—Original Draft, Writing—Review and Editing, Visualization, Software, Data curation, Investigation, Results analysis, Resources. *M. Raji*: Writing—Original Draft, Writing—Review and Editing, Visualization, Software, Data curation, Project administration, Investigation, Results analysis. *B.C. Agbata*: Formal analysis, Writing—Original Draft, Software, Data curation, Project administration, Investigation. All authors have read and approved the final version of the manuscript for publication.

Acknowledgement: Authors are thankful to anonymous reviewers for their fruitful suggestions, which have improved the earlier draft of this paper.

Funding Statement: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Data Availability Statement: Not applicable

Ethics Approval: Not applicable

Use of Generative-AI tools declaration: The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflicts of Interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

References

1. L. Talon and E. Meiburg, Plane Poiseuille flow of miscible layers with different viscosities: instabilities in the Stokes flow regime, *Journal of Fluid Mechanics* 686 (2011), 484–506. <https://doi.org/10.1017/jfm.2011.341>
2. M. El Farragui, O. Souhar and G. C. Georgiou, Analytical solutions of axial annular Newtonian flows with dynamic wall slip, *Advances in Computational Mechanics and Applications* (2024), 385–400. https://doi.org/10.1007/978-3-031-49791-9_27
3. A. E. K. Ali, A. F. Ghaleb, M. S. Abou-Dina and M. A. Helal, Laplace transform solution of the time-dependent annular Couette flow with dynamic wall slip, *Journal of the Brazilian Society of Mechanical Sciences and Engineering* 45(11) (2023), 586. <https://doi.org/10.1007/s40430-023-04498-y>
4. E. V. Kolesnik, D. K. Zaitsev, E. M. Smirnov, E. I. Shmelev, M. G. Maslov and A. V. Budnikov, Acoustic resonance in an annular cavity with axial transit flow, *Fluid Dynamics* 59(4) (2024), 777–788. <https://doi.org/10.1134/S0015462824602791>
5. S. A. Zdornikov, S. V. Isaenkov and A. V. Cherdantsev, Axial and azimuthal development of disturbance waves in annular flow in a horizontal pipe, *International Journal of Multiphase Flow* 172 (2024), 104704. <https://doi.org/10.1016/j.ijmultiphaseflow.2023.104704>
6. J. T. Ault, K. K. Chen and H. A. Stone, Downstream decay of fully developed Dean flow, *Journal of Fluid Mechanics* 777 (2015), 219–244. <https://doi.org/10.1017/jfm.2015.353>

7. J.-D. Debus, M. Mendoza and H. J. Herrmann, Dean instability in double-curved channels, *Physical Review E* 90(5) (2014), 053308. <https://doi.org/10.1103/PhysRevE.90.053308>
8. S. Tsangaris and N. W. Vlachakis, Exact solution for the pulsating finite gap Dean flow, *Applied Mathematical Modelling* 31(9) (2007), 1899–1906. <https://doi.org/10.1016/j.apm.2006.06.011>
9. O. A. Bèg, M. M. Hoque, M. Wahiduzzaman, M. M. Alam and M. Ferdows, Spectral numerical simulation of magneto-physiological laminar Dean flow, *Journal of Mechanics in Medicine and Biology* 14(4) (2014), 1450047. <https://doi.org/10.1142/S021951941450047X>
10. A. A. Aider, S. Skali-Lami and J. P. Brancher, Laminar–turbulent transition in Taylor–Dean flow, *Journal of Physics: Conference Series* 14(1) (2005), 118–127. <https://doi.org/10.1088/1742-6596/14/1/015>
11. N. Nivedita, P. Ligrani and I. Papautsky, Dean flow dynamics in low-aspect ratio spiral microchannels, *Scientific Reports* 7(2017), 44072. <https://doi.org/10.1038/srep44072>
12. A. A. Aider and L. Kadem, Transition to turbulence of the Dean and Taylor–Couette flows: similarities, *Multiphysics* (2011).
13. A. A. S. Bhagat, H. W. Hou, L. D. Li, C. T. Lim and J. Han, Dean flow fractionation (DFF) isolation of circulating tumor cells (CTC) from blood, *Proceedings of the 15th International Conference on Miniaturized Systems for Chemistry and Life Sciences* (2012), 524–526.
14. B. K. Jha and D. Gambo, Hydrodynamic effect of slip boundaries and exponentially decaying/growing time-dependent pressure gradient on Dean flow, *Journal of the Egyptian Mathematical Society* 29 (2021), 11. <https://doi.org/10.1186/s42787-021-00120-z>
15. M. Norouzi, B. Zare Vamerzani, M. Davoodi, N. Biglari and M. M. Shahmardan, An exact analytical solution for creeping Dean flow of Bingham plastics through curved rectangular ducts, *Rheologica Acta* 54(5) (2015), 391–402. <https://doi.org/10.1007/s00397-014-0807-x>
16. T. Kawabe, Y. Hayamizu, S. Yanase and T. Gondo et al., A micromixer using the Taylor–Dean flow: rotation effect of channel walls on the mixing, *Advanced Experimental Mechanics* 1 (2016), 47–51. https://doi.org/10.11395/aem.1.0_47
17. P. Dash, K. L. Ojha and G. C. Dash, Unsteady Dean flow between concentric circular cylinders in the presence of a uniform axial magnetic field, *Numerical Heat Transfer, Part B: Fundamentals* 86(10) (2025), 3562–3575. <https://doi.org/10.1080/10407790.2024.2362950>
18. Y. J. Danjuma, *Transient Flow Formation inside an Annulus: A Semi-Analytical Approach*, MoreBooks, (2019).
19. B. K. Jha and Y. J. Danjuma, Unsteady Dean flow formation in an annulus with partial slippage: a Riemann-sum approximation approach, *Results in Engineering* 5 (2020), 100078. <https://doi.org/10.1016/j.rineng.2019.100078>
20. B. K. Jha and J. D. Yahaya, Transient Dean flow in an annulus: a semi-analytical approach, *Journal of Taibah University for Science* 13(1) (2019), 169–176. <https://doi.org/10.1080/16583655.2018.1549529>
21. B. K. Jha and Y. J. Danjuma, Transient Dean flow in a channel with suction/injection: a semi-analytical approach, *Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering* 233(5) (2019), 1036–1044. <https://doi.org/10.1177/0954408919825718>
22. B. K. Jha and Y. J. Danjuma, Transient generalized Taylor–Couette flow of a dusty fluid: a semi-analytical approach, *Partial Differential Equations in Applied Mathematics* 5 (2022), 100400. <https://doi.org/10.1016/j.padiff.2022.100400>

23. J. K. Basant and J. D. Yahaya, Unsteady magnetohydrodynamic Dean flow with dynamic as well as static slips, *Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering* (2026), 09544089261430086. <https://doi.org/10.1177/09544089261430086>
24. B. K. Jha and Y. J. Danjuma, Impact of variable viscosity on unsteady Couette flow, *Heat Transfer* 54(3) (2025), 2032–2048. <https://doi.org/10.1002/htj.23263>
25. P. Orlandi, G. Soldati and S. Pirozzoli, Effects of curvature on turbulent flow in concentric annuli and curved channels, *Journal of Fluid Mechanics* 1009 (2025), A29. <https://doi.org/10.1017/jfm.2025.238>

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