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Research article

Generalization of an Inequality by Laforgia and Natalini

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Abstract: In this short paper, we establish a generalization of an inequality involving the Rieman zeta function. The inequality was established by Laforgia and Natalini in 2006 and subsequently improved by other researchers. The main tools used to prove our results are the logarithmic convexity property of the Riemann zeta function and monotonicity properties of certain functions containing the Riemann zeta function.

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1 Introduction

The Riemann zeta function is arguably one of the most frequently encountered functions in mathematical analysis. It is defined by any of the following equivalent forms.

$$\begin{aligned}\zeta(z) &= \sum_{k=1}^{\infty} \frac{1}{k^z}, \quad z > 1, &(1)\\ &= \frac{1}{1-2^{-z}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^z}, \quad z > 1, &(2)\\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{r^{z-1}}{e^r - 1} dr, \quad z > 1, &(3)\\ &= \frac{2^{z-1}}{\Gamma(z+1)} \int_0^{\infty} \frac{r^z}{(\sinh(r))^2} dr, \quad z > 1, &(4)\end{aligned}$$

where $\Gamma(z)$ is the gamma function. Beyond the field of mathematical analysis, its applications extend to other disciplines such as analytic number theory, statistics, probability theory, mathematical physics and engineering. Due to the pivotal role it plays, many researchers have investigated its properties along different directions. In the past decades, the attention has shifted to analytical properties and inequalities involving the function. Consequently, many interesting inequalities involving the function have been discovered. We refer to [2–8,11,13,16] and the references cited therein.

In 2006, Lafogia and Natalini [14] established the remarkable inequality

$$\frac{z}{z+1} \leq \frac{\zeta(z)\zeta(z+2)}{\zeta^2(z+1)}, \quad z > 1, \tag{5}$$

by utilizing the integral representation (3), a generalized form of the Cauchy-Schwarz integral inequality, and the recurrence relation of the gamma function. Thereafter, and due to its importance, the inequality (5) attracted the attention of other renowned researchers, leading to the establishment of an improved result which is given as

$$1 \leq \frac{\zeta(z)\zeta(z+2)}{\zeta^2(z+1)}, \quad z > 1. \tag{6}$$

The inequality (6) was independently discovered under different circumstances by different authors. In 2008, Cerone and Dragomir obtained it in (2.4) of [9]. In 2013, Ahsan et al. obtained it in (2) of [1]. In 2018, Srivastava et al. obtained it in Theorem 5 of [17]. Then in 2023, Cerone and Dragomir recovered it in (3.5) of [10]. Furthermore, the inequality (6) has been generalized in Proposition 3.1 of [10] and Theorem 2.1 of [15].

In this work, the ultimate objective is to provide a generalization of the results of Laforgia and Natalini as in (5) as well as its improved version in (6), by using a different and very simple technique.

2 Results

Theorem 1: *Let $z > 1$, $a \geq 0$ and $c > k \geq 0$. Then the inequality*

$$\frac{z+k}{z+a+k} \leq \frac{\zeta(z+k)\zeta(z+a+c)}{\zeta(z+a+k)\zeta(z+c)} \tag{7}$$

holds.

Proof: It is known in [12, Lemma 2.1] that the function $\zeta(z)$ is strictly logarithmically convex on $(1,\infty)$. This implies that the function $\zeta'(z)/\zeta(z)$ is strictly increasing on $(1,\infty)$. For $z > 1$ and $c > k \geq 0$, let $\mathcal{P}(z)$ be defined as

$$\mathcal{P}(z) = (z+k)\frac{\zeta(z+c)}{\zeta(z+k)}.$$

By direct computation and some rearrangement, we obtain

$$\frac{\mathcal{P}'(z)}{\mathcal{P}(z)} = \frac{1}{z+k} + \frac{\zeta'(z+c)}{\zeta(z+c)} - \frac{\zeta'(z+k)}{\zeta(z+k)} > 0$$

which implies that $\mathcal{P}'(z) > 0$. Thus, $\mathcal{P}(z)$ is strictly increasing on $(1, \infty)$. Let $a \geq 0$. Then by the monotonicity of $\mathcal{P}(z)$, $z \leq z + a$, implies that $\mathcal{P}(z) \leq \mathcal{P}(z + a)$ and that is

$$(z + k)\frac{\zeta(z + c)}{\zeta(z + k)} \leq (z + a + k)\frac{\zeta(z + a + c)}{\zeta(z + a + k)}$$

which after rearranging gives the desired inequality (7). $\square$

Remark 1: *In particular, if $k = 0$ and $a = c = 1$, then inequality* (7) *reduces to the inequality* (5) *by Laforgia and Natalini.*

We proceed to provide a generalization of the improved result (6) as well.

Theorem 2: *Let $z > 1$, $a \geq 0$ and $c > k \geq 0$. Then the inequality*

$$1 \leq \frac{\zeta(z + k)\zeta(z + a + c)}{\zeta(z + a + k)\zeta(z + c)} \tag{8}$$

holds.

Proof: We follow the same procedure as in the proof of Theorem 1. Similarly, for $z > 1$ and $c > k \geq 0$, let $\mathcal{Q}(z)$ be defined as

$$\mathcal{Q}(z) = \frac{\zeta(z + c)}{\zeta(z + k)}.$$

Then simple computation yields

$$\frac{\mathcal{Q}'(z)}{\mathcal{Q}(z)} = \frac{\zeta'(z + c)}{\zeta(z + c)} - \frac{\zeta'(z + k)}{\zeta(z + k)} > 0$$

since $\zeta'(z)/\zeta(z)$ is strictly increasing on $(1, \infty)$. Hence $\mathcal{Q}(z)$ is strictly increasing on $(1, \infty)$. Then for $z \leq z + a$ where $a \geq 0$, we have $\mathcal{Q}(z) \leq \mathcal{Q}(z + a)$ which gives the inequality (8) when rearranged. $\square$

Remark 2: *In particular, if $k = 0$ and $a = c = 1$, then inequality* (8) *reduces to the inequality* (6).

Remark 3: *In [14, (2.9)], Laforgia and Natalini obtain the inequality*

$$\frac{\zeta(z)\zeta(z + 2)}{\zeta^2(z + 1)} \geq \frac{\Gamma^2(z + 1)}{\Gamma(z)\Gamma(z + 2)} \tag{9}$$

for $z > 1$. *It is well known in the literature that, for positive real numbers, the gamma function is logarithmically convex. This implies that*

$$\frac{\Gamma(z+a)\Gamma(z+c)}{\Gamma(z)\Gamma(z+a+c)} \leq 1 \tag{10}$$

for $z > 0$, $c \geq 0$ *and* $a \geq 0$. *If* $a = c = 1$, *then combining* (6) *and* (10) *gives*

$$\frac{\zeta(z)\zeta(z+2)}{\zeta^2(z+1)} \geq 1 \geq \frac{\Gamma^2(z+1)}{\Gamma(z)\Gamma(z+2)} \tag{11}$$

for $z > 1$, *which is an improvement of* (9).

3 Conclusion

In this short note, we have established a generalization of an inequality by Laforgia and Natalini. We achieved this by utilizing the logarithmic convexity property of the Riemann zeta function and monotonicity properties of certain functions containing the Riemann zeta function. Unlike the work of Laforgia and Natalini, the present work does not rely on the Cauchy-Schwarz inequality or the recurrence relation of the gamma function. A future work could consider extending the results of this paper to Hurwitz zeta function.

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