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Research article

On 2-Movable Connected Domination of Some Graphs

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Abstract: A connected dominating set C in G is a 2-movable connected dominating set of G if for every pair $x, y \in C$, $C \setminus \{x, y\}$ is a connected dominating set in G, or there exists $u, v \in V(G) \setminus C$ such that u and v are adjacent to x and y, respectively, and $(C \setminus \{x, y\}) \cup \{u, v\}$ is a connected dominating set of G. The minimum cardinality of a 2-movable connected dominating set of G, denoted by $\gamma^2_{mc}(G)$ is the 2-movable connected domination number of G. A 2-movable connected dominating set with cardinality $\gamma^2_{mc}(G)$ is called a minimum 2-movable connected dominating set or a γ^2_{mc}-set of G. In this paper, the researcher obtained the 2-movable connected domination number of the Complete Graph K_n, Join Graph $G + H$, and Generalized Wheel Graph $W_{m,n}$.

Mathematics Subject Classification: 05C69
Keywords: domination in graphs; connected domination; 2-movable domination; 2-movable connected domination

1 Introduction

All graphs considered in this paper are connected, finite, simple, and undirected. Let $G = (V, E)$ be a connected, finite, simple, and undirected graph. The graph G has a vertex set $V = V(G)$ and an edge set $E = E(G)$.

One of the most actively studied areas in graph theory is domination in graphs, which has received increased attention in recent decades. In 1962, Oystein Ore introduced the terms *"dominating set"* and *"domination number"*, which are commonly used today. Later, in 1977, Cockayne and Hedetniemi provided the formal definition of the domination number, which is denoted by $\gamma(G)$. [6] A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality equal to $\gamma(G)$ is called a γ-set of G. Moreover, introduced by Sampathkumar E. and Walikar H. [9] in 1979, a dominating set D is said to be a connected dominating set if the subgraph

$\langle D\rangle$ induced by D is connected in G. The minimum cardinality of the connected dominating sets is the connected domination number and is denoted by $\gamma_c(G)$.

Expanding on this idea, Lomarda J. and Canoy S. [7] introduced in 2018 the idea of 1-movable connected dominating set. A connected dominating set C in G is a 1-movable connected dominating set of G if for every $v \in C$ either $C \setminus \{v\}$ is a connected dominating set, or there exists a vertex $u \in (V(G) \setminus C) \cap N(v)$ such that $(C \setminus \{v\}) \cup \{u\}$ is a connected dominating set of G. The minimum cardinality of a 1-movable connected dominating set of G, denoted by $\gamma^1_{mc}(G)$, is the 1-movable connected domination number of G. A 1-movable connected dominating set with cardinality $\gamma^1_{mc}(G)$ is called a minimum 1-movable connected dominating set or a γ^1_{mc}-set of G.

Moreover, in 2024, Pedrano A. and Paluga R. [8] explored the 2-movable domination in the join and corona of graphs. A non-empty set $S \subseteq V(G)$ is a 2-movable dominating set of G if S is a dominating set and for every pair $x, y \in S$, $S \setminus \{x, y\}$ is a dominating set in G, or there exist $u, v \in V(G) \setminus S$ such that u and v are adjacent to x and y, respectively, and $(S \setminus \{x, y\}) \cup \{u, v\}$ is a dominating set in G. The 2-movable domination number of G, denoted by $\gamma^2_m(G)$, is the minimum cardinality of a 2-movable dominating set of G. A 2-movable dominating set with cardinality equal to $\gamma^2_m(G)$ is called a γ^2_m-set of G.

Combining the concepts of 2-movable domination and connected domination, this paper explores the notion of a 2-movable connected domination of some graphs.

2 Basic Concepts

Definition 2.1: [6] A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality equal to $\gamma(G)$ is called a γ-set of G.

Definition 2.2: [7] Let $G = (V(G), E(G))$ be a graph. A dominating set $S \subseteq V(G)$ is called a *connected dominating set* of G if the subgraph $\langle S\rangle$ induced by S is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$ is the smallest cardinality of a connected dominating set of G. A connected dominating set S of G with $|S| = \gamma_c(G)$ is called a γ_c-set.

Definition 2.3: A connected dominating set C in G is a 2-*movable connected dominating set* of G if for every pair $x, y \in C$, $C \setminus \{x, y\}$ is a connected dominating set in G, or there exists $u, v \in V(G) \setminus C$ such that u and v are adjacent to x and y, respectively, and $(C \setminus \{x, y\}) \cup \{u, v\}$ is a connected dominating set of G. The minimum cardinality of a 2-movable connected dominating set of G, denoted by $\gamma^2_{mc}(G)$ is the 2-*movable connected domination number* of G. A 2-movable connected dominating set with cardinality $\gamma^2_{mc}(G)$ is called a *minimum* 2-*movable connected dominating set* or a γ^2_{mc}-set of G.

3 Related Results

Theorem 3.1: *[8] Let G and H be graphs of order at least 2. Then,* $\gamma^2_m(G + H) = 2$.

Corollary 3.2: *[8] Let H be a connected graph of order at least 4. Then,* $\gamma^2_m(K_1 + H) = \gamma(H)$.

Remark 3.3: *[7] Every connected dominating set contains every cut-vertex.*

Theorem 3.4: *[7] A connected nontrivial graph G has a 1-movable connected dominating set if and only if G has no cut-vertices.*

Lemma 3.5: *[7] $\gamma_{mc}^1(K_n) = 1$ for all $n \geq 2$.*

Corollary 3.6: *[7] Let H be a connected graph of order $n \geq 2$. Then,*

$$\gamma_{mc}^1(K_1 + H) = \gamma_c(H).$$

Corollary 3.7: *[7] Let $m \geq 2$ and $n \geq 2$ be positive integers. Then,*

$$\gamma_{mc}^1(K_{m,n}) = 4.$$

Theorem 3.8: *[4] For $n \geq 3$,*

$$\gamma(P_n) = \gamma(C_n) = \lceil \tfrac{n}{3} \rceil.$$

Proposition 3.9: *[9] The minimum of the cardinalities of the connected dominating sets of a graph G is called the* connected domination number *of G, and is denoted by $\gamma_c(G)$. The connected domination numbers of some standard graphs are given as follows:*

(i) $\gamma_c(K_n) = 1$.

(ii) $\gamma_c(K_n + G) = 1$, *for any graph G.*

(iii)

$$\gamma_c(K_{m,n}) = \begin{cases} 1, & \text{if } m = 1 \text{ or } n = 1, \\ 2, & \text{if } m, n \geq 2. \end{cases}$$

(iv) $\gamma_c(C_n) = n - 2$.

(v) *For any tree T of order n,*

$$\gamma_c(T) = n - e,$$

where e is the number of pendant vertices (i.e., vertices of degree 1) in T.

4 Results and Discussion

This section presents the result on the 2-movable connected domination of K_n, $G + H$, and $W_{m,n}$.

4.1 Some Results on the 2-movable Connected Domination Number of Graphs

The next remark follows directly from Definition 2.3:

Remark 4.1: For any connected graph G of order $n \geq 4$,

$$2 \leq \gamma_{mc}^2(G) \leq n.$$

Theorem 4.2: For any connected graph G of order $n \geq 4$,

$$\gamma(G) \leq \gamma_{mc}^2(G) \text{ and } \gamma_c \leq \gamma_{mc}^2(G).$$

Proof: Suppose S is a 2-movable connected dominating set of G. It follows that S is a connected dominating set and hence S is a dominating set of G. Therefore, by Definition 2.3, we have

$$\gamma(G) \leq \gamma_{mc}^2(G)$$

and

$$\gamma_c(G) \leq \gamma_{mc}^2(G).$$

□

4.2 2-Movable Connected Domination Number of the Complete Graph K_n

Theorem 4.3: Let K_n be a complete graph where $n \geq 4$. Then

$$\gamma_{mc}^2(K_n) = 2.$$

Proof: Let K_n be a complete graph where $n \geq 4$. Suppose $S = \{x, y\}$. Then, we must show that S is a connected dominating set. Observe that every vertex in a complete graph is adjacent to $n-1$ vertices. This implies that $N(x) = V(K_n) \setminus \{x\}$ and $N(y) = V(K_n) \setminus \{y\}$. Thus,

$$\begin{aligned} N(S) &= N(x) \cup N(y) \\ &= \big(V(K_n) \setminus \{x\}\big) \cup \big(V(K_n) \setminus \{y\}\big) \\ &= V(K_n). \end{aligned}$$

Hence,

$$\begin{aligned} N[S] &= S \cup N(S) \\ &= \{x, y\} \cup \big(N(x) \cup N(y)\big) \\ &= \{x, y\} \cup \Big(\big(V(K_n) \setminus \{x\}\big) \cup \big(V(K_n) \setminus \{y\}\big)\Big) \\ &= V(K_n). \end{aligned}$$

Therefore, S is a dominating set. Note that $xy \in E(K_n)$. Thus, S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set of K_n.

Now, consider the vertices $x, y \in S$. We want to show that S is a 2-movable connected dominating set. Note that S is a connected dominating set. Then, there exist vertices $x', y' \in V(K_n) \setminus S$ such that x' and y' are adjacent to x and y, respectively. Thus, we need to show that $S' = (S \setminus \{x, y\}) \cup \{x', y'\} = \{x', y'\}$ is a connected dominating set of K_n. Recall that every vertex in a complete graph is adjacent to $n-1$ vertices. This means that $N(x') = V(K_n)\setminus\{x'\}$ and $N(y') = V(K_n)\setminus\{y'\}$. Then,

$$\begin{aligned} N(S') &= N(x') \cup N(y') \\ &= \big(V(K_n)\setminus\{x'\}\big) \cup \big(V(K_n)\setminus\{y'\}\big) \\ &= V(K_n). \end{aligned}$$

Thus,

$$\begin{aligned} N[S'] &= S' \cup N(S') \\ &= \{x', y'\} \cup \big(N(x') \cup N(y')\big) \\ &= \{x', y'\} \cup \Big(\big(V(K_n)\setminus\{x'\}\big) \cup \big(V(K_n)\setminus\{y'\}\big)\Big) \\ &= V(K_n). \end{aligned}$$

Hence, S' is a dominating set. Note that $x'y' \in E(K_n)$. Then, S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. Thus, S is a 2-movable connected dominating set. Furthermore, S is the minimum 2-movable connected dominating set of K_n. Hence, $\gamma_{mc}^2(K_n) \leq 2$. And so, by Remark 4.1, $\gamma_{mc}^2(K_n) = 2$. □

4.3 2-Movable Connected Domination Number of Join Graphs $G + H$

Theorem 4.4: Let G and H be a nontrivial graphs of order at least 2. Then,

$$\gamma_{mc}^2(G+H) = 2.$$

Proof: Let G and H be a nontrivial graphs with $|V(G)| \geq 2$ and $|V(H)| \geq 2$, where $V(G+H) = V(G) \cup V(H)$ and $|V(G+H)| \geq 4$. Let $V(G) = \{u_1, u_2, u_3, \dots, u_m\}$ and $V(H) = \{v_1, v_2, v_3, \dots, v_n\}$. Suppose that $S = \{u, v\}$ where $u \in V(G)$ and $v \in V(H)$. We need to show that S is a connected dominating set of $G+H$. Observe that every vertex in G is adjacent to v and every vertex in H is adjacent to u. Then, $V(G) \subseteq N[v]$ and $V(H) \subseteq N[u]$. Thus,

$$\begin{aligned} V(G+H) &= V(G) \cup V(H) \\ &\subseteq N[v] \cup N[u] \\ &= N[S]. \end{aligned}$$

Therefore, S is a dominating set. Notice that $uv \in E(G+H)$. Then, S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

Now, consider the vertices $u, v \in S$. We want to show that S is 2-movable connected dominating set. Since S is a connected dominating set, there exists vertices $u' \in V(G)$ and $v' \in V(H)$ such that $u' \neq u$ and $v' \neq v$. Thus, we will show that $S' = (S \setminus \{u, v\}) \cup \{u', v'\} = \{u', v'\}$ is a connected dominating set of $G+H$.

Notice that every vertex in G is adjacent to v' and every vertex in H is adjacent to u'. This implies that $V(G) \subseteq N[v']$ and $V(H) \subseteq N[u']$. Hence,

$$\begin{aligned} V(G+H) &= V(G) \cup V(H) \\ &\subseteq N[v'] \cup N[u'] \\ &= N[S']. \end{aligned}$$

This means that S' is a dominating set. Notice that $u'v' \in E(G+H)$. Then, S' is a connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. Hence, S is a 2-movable connected dominating set. Also, S is the minimum 2-movable connected dominating set of $G+H$. Therefore, $\gamma^2_{mc}(G+H) \leq 2$. By Remark 4.1, $\gamma^2_{mc}(G+H) = 2$. □

Theorem 4.5: Let W_n be a wheel graph where $n \geq 3$. Then,

$$\gamma^2_{mc}(W_n) = \begin{cases} 2, & \text{if } n = 3, 4, \\ 3, & \text{if } n = 5, \\ 4, & \text{if } n = 6, \\ n+1, & \text{if } n \geq 7. \end{cases}$$

Proof: Let $W_n \cong C_n + K_1$. Suppose that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_1) = \{u\}$. Then, $V(W_n) = V(C_n) \cup V(K_1)$. To prove the theorem, we consider the following cases:

Case 1: $n = 3$ or $n = 4$.

Subcase 1.1: $n = 3$.

Consider the wheel graph W_3, defined by $V(W_3) = V(C_3) \cup V(K_1)$, where $V(C_3) = \{v_1, v_2, v_3\}$ and $V(K_1) = \{u\}$, as shown in Figure 1. Also, note that $E(W_3) = E(C_3) \cup \{uv_1, uv_2, uv_3\}$ where $E(C_3) = \{v_1v_2, v_2v_3, v_1v_3\}$. Let $S = \{v_1, v_3\}$. Then, we will show that S is a connected dominating set.

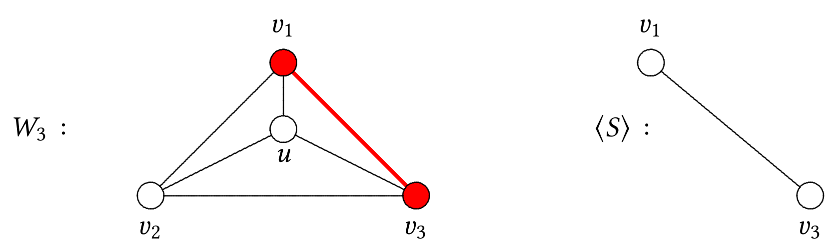

Figure 1: Wheel Graph W_3 and its connected dominating set $S = \{v_1, v_3\}$ with induced subgraph $\langle S \rangle$.

Observe that $N(v_1) = \{u, v_2, v_3\}$ and $N(v_3) = \{u, v_1, v_2\}$. Then,

$$N(S) = N(v_1) \cup N(v_3) = \{u, v_2, v_3\} \cup \{u, v_1, v_2\} = \{u, v_1, v_2, v_3\}$$

Thus,

$N[S] = S \cup N(S) = \{v_1, v_3\} \cup \{u, v_1, v_2, v_3\} = \{u, v_1, v_2, v_3\} = V(W_3).$

Hence, S is a dominating set. Observe that $v_1v_3 \in E(C_3)$. This implies that S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

Additionally, we must show that S is a 2-movable connected dominating set of W_3. Consider the vertices $v_1, v_3 \in S$. Then, we must show that S is a 2-movable connected dominating set. Since S is connected dominating set, there exists vertices $u, v_2 \in V(W_3) \setminus S$, such that u and v_2 is adjacent to v_1 and v_3, respectively. Now, we need to show that $S' = (S \setminus \{v_1, v_3\}) \cup \{u, v_2\} = \{u, v_2\}$ is also a connected dominating set. Note further that $N(u) = \{v_1, v_2, v_3\}$ and $N(v_2) = \{u, v_1, v_3\}$. Then,

$$N(S') = N(u) \cup N(v_2) = \{v_1, v_2, v_3\} \cup \{u, v_1, v_3\} = \{u, v_1, v_2, v_3\}.$$

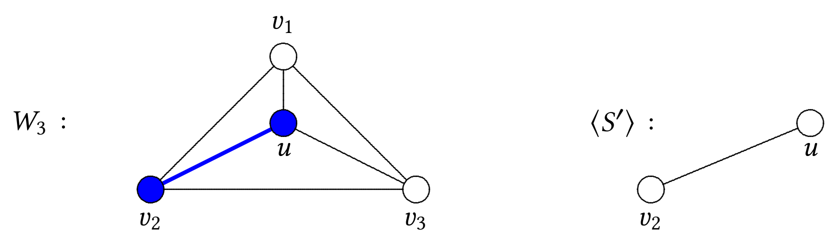

Figure 2: Wheel Graph W_3 and its connected dominating set $S' = \{u, v_2\}$ with induced subgraph $\langle S' \rangle$.

Thus,

$$N[S'] = S' \cup N(S') = \{u, v_2\} \cup \{u, v_1, v_2, v_3\} = \{u, v_1, v_2, v_3\} = V(W_3).$$

Hence, S' is a dominating set. Notice that $uv_2 \in E(W_3)$. This implies that S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. In effect, $S = \{v_1, v_2\}$ is a 2-movable connected dominating set of W_3. Moreover, S is the minimum 2-movable connected dominating set of W_3, and so, S is a γ_{mc}^2-set of W_3. Therefore, $\gamma_{mc}^2(W_3) = 2$.

Subcase 1.2: $n = 4$.

Consider the wheel graph W_4, where $V(W_4) = V(C_4) \cup V(K_1)$ with $V(C_4) = \{v_1, v_2, v_3, v_4\}$ and $V(K_1) = \{u\}$, as presented in Figure 3. Also, $E(W_4) = E(C_4) \cup \{uv_1, uv_2, uv_3, uv_4\}$ where $E(C_4) = \{v_1v_2, v_2v_3, v_3v_4, v_1v_4\}$. Let $S = \{v_1, v_4\}$. Then, we have to show that S is a connected dominating set.

Observe that $N(v_1) = \{u, v_2, v_4\}$ and $N(v_4) = \{u, v_1, v_3\}$. Then,

$N(S) = N(v_1) \cup N(v_4) = \{u, v_2, v_4\} \cup \{u, v_1, v_3\} = \{u, v_1, v_2, v_3, v_4\}.$

Thus,

$N[S] = S \cup N(S) = \{v_1, v_4\} \cup \{u, v_1, v_2, v_3, v_4\} = \{u, v_1, v_2, v_3, v_4\} = V(W_4).$

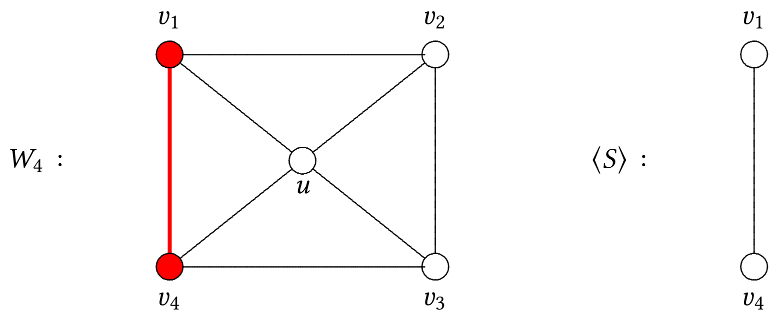

Figure 3: Wheel Graph W_4 and its connected dominating set $S = \{v_1, v_4\}$ with induced subgraph $\langle S\rangle$.

Hence, S is a dominating set. Furthermore, notice that $v_1v_4 \in E(C_4)$. Then, S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

In addition, we need to show that S is a 2-movable connected dominating set of W_4. Take the vertices $v_1, v_4 \in S$. Note that S is a connected dominating set. Then, there exists vertices $u, v_3 \in V(W_4) \setminus S$, such that u and v_3 is adjacent to v_1 and v_4, respectively. Now, we must show that $S' = (S \setminus \{v_1, v_4\}) \cup \{u, v_3\} = \{u, v_3\}$ is a connected dominating set.

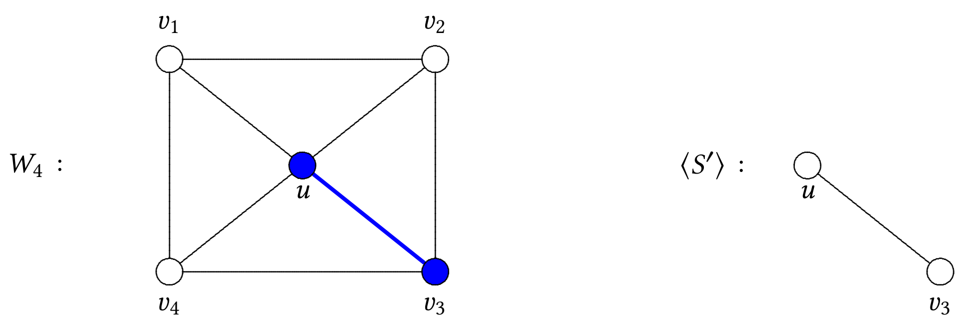

Figure 4: Wheel Graph W_4 and its connected dominating set $S' = \{u, v_3\}$ with induced subgraph $\langle S'\rangle$.

Observe further that $N(u) = \{v_1, v_2, v_3, v_4\}$ and $N(v_3) = \{u, v_2, v_4\}$. Then,

$$N(S') = N(u) \cup N(v_3) = \{v_1, v_2, v_3, v_4\} \cup \{u, v_2, v_4\} = \{u, v_1, v_2, v_3, v_4\}.$$

Thus,

$$N[S'] = S' \cup N(S') = \{u, v_3\} \cup \{u, v_1, v_2, v_3, v_4\} = \{u, v_1, v_2, v_3, v_4\} = V(W_4)$$

Hence, S' is a dominating set. Furthermore, notice that $uv_3 \in E(W_4)$. This implies that S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. In effect, S is a 2-movable connected dominating set of W_4. Also, S is the minimum 2-movable connected dominating set of W_4. Therefore, $\gamma_{mc}^2(W_4) = 2$.

Case 2: $n = 5$

Consider the wheel graph W_5, where $V(W_5) = V(C_5) \cup V(K_1)$ with $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and $V(K_1) = \{u\}$, as shown in Figure 5. Also, note that $E(W_5) = E(C_5) \cup \{uv_1, uv_2, uv_3, uv_4, uv_5\}$ where the edge set of C_5 is $E(C_5) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let $S = \{u, v_1, v_4\}$.

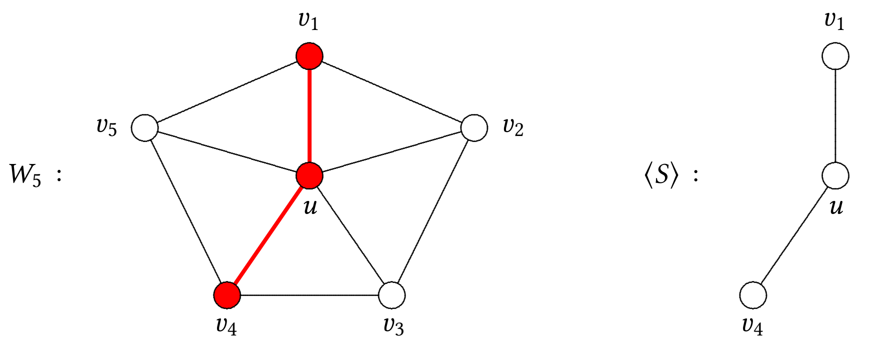

Figure 5: Wheel Graph W_5 and its connected dominating set $S = \{u, v_1, v_4\}$ with induced subgraph $\langle S \rangle$.

Observe that $N(u) = \{v_1, v_2, v_3, v_4, v_5\}$, $N(v_1) = \{u, v_2, v_5\}$ and $N(v_4) = \{u, v_3, v_5\}$. Then,

$$\begin{aligned} N(S) &= N(u) \cup N(v_1) \cup N(v_4) \\ &= \{v_1, v_2, v_3, v_4, v_5\} \cup \{u, v_2, v_5\} \cup \{u, v_3, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\}. \end{aligned}$$

Thus,

$$\begin{aligned} N[S] &= S \cup N(S) \\ &= \{u, v_1, v_4\} \cup \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= V(W_5). \end{aligned}$$

Hence, S is a dominating set. Notice that $uv_1, uv_4 \in E(W_5)$. This implies that S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

Now, we must show that S is a 2-movable connected dominating set of W_5. Consider the vertices $u, v_1 \in S$. Recall that S is a connected dominating set. Then, there exists vertices $v_3, v_5 \in V(W_5) \setminus S$, such that v_3 and v_5 is adjacent to u and v_1, respectively. We will show that $S_1 = (S \setminus \{u, v_1\}) \cup \{v_3, v_5\} = \{v_3, v_4, v_5\}$ is also a connected dominating set.

Note that $N(v_3) = \{u, v_2, v_4\}$, $N(v_4) = \{u, v_3, v_5\}$, and $N(v_5) = \{u, v_1, v_4\}$. Then,

$$\begin{aligned} N(S_1) &= N(v_3) \cup N(v_4) \cup N(v_5) \\ &= \{u, v_2, v_4\} \cup \{u, v_3, v_5\} \cup \{u, v_1, v_4\} \end{aligned}$$

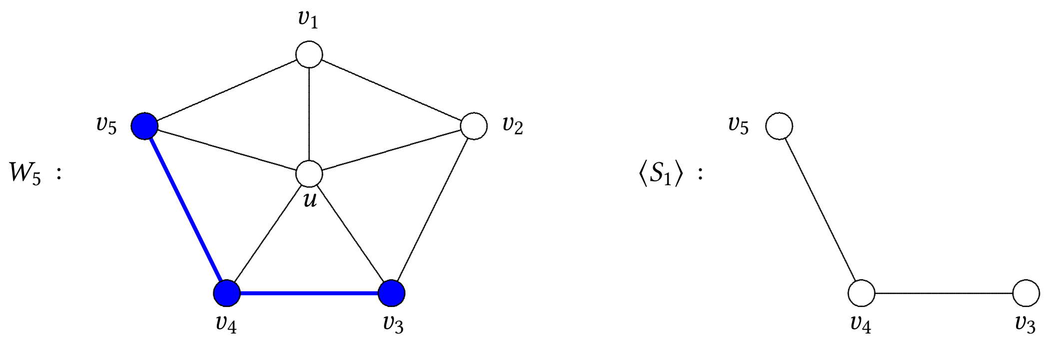

Figure 6: Wheel Graph W_5 and its connected dominating set $S_1 = \{v_3, v_4, v_5\}$ with induced subgraph $\langle S_1 \rangle$.

$$= \{u, v_1, v_2, v_3, v_4, v_5\}.$$

Thus,

$$\begin{aligned} N[S_1] &= S_1 \cup N(S_1) \\ &= \{v_3, v_4, v_5\} \cup \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= V(W_5). \end{aligned}$$

Hence, S_1 is a dominating set. Observe that $v_3v_4, v_4v_5 \in E(C_5)$. This implies that S_1 is connected. Since S_1 is a dominating set and S_1 is connected, it follows that S_1 is a connected dominating set.

Furthermore, consider the vertices $u, v_4 \in S$. Since S is a connected dominating set, there exists vertices $v_2, v_5 \in V(W_5) \setminus S$, such that v_2 and v_5 is adjacent to u and v_4, respectively. Now, $S_2 = (S \setminus \{u, v_4\}) \cup \{v_2, v_5\} = \{v_1, v_2, v_5\}$ must also be a connected dominating set.

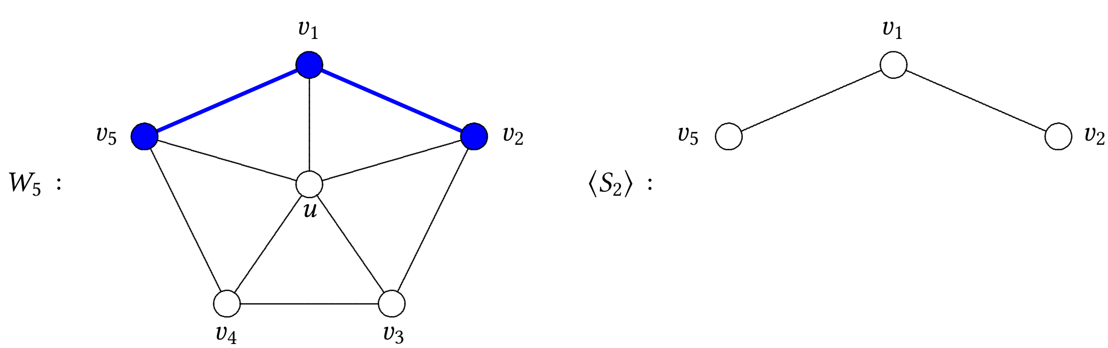

Figure 7: Wheel Graph W_5 and its connected dominating set $S_2 = \{v_1, v_2, v_5\}$ with induced subgraph $\langle S_2 \rangle$.

Observe that $N(v_1) = \{u, v_2, v_5\}$, $N(v_2) = \{u, v_1, v_3\}$ and $N(v_5) = \{u, v_1, v_4\}$. Then,

$$\begin{aligned}N(S_2) &= N(v_1) \cup N(v_2) \cup N(v_5)\\ &= \{u, v_2, v_5\} \cup \{u, v_1, v_3\} \cup \{u, v_1, v_4\}\\ &= \{u, v_1, v_2, v_3, v_4, v_5\}.\end{aligned}$$

Thus,

$$\begin{aligned}N[S_2] &= S_2 \cup N(S_2)\\ &= \{v_1, v_2, v_5\} \cup \{u, v_1, v_2, v_3, v_4, v_5\}\\ &= \{u, v_1, v_2, v_3, v_4, v_5\}\\ &= V(W_5).\end{aligned}$$

Hence, S_2 is a dominating set. Notice that $v_1v_2, v_1v_5 \in E(C_5)$. This implies that S_2 is connected. Since S_2 is a dominating set and S_2 is connected, it follows that S_2 is a connected dominating set.

Moreover, consider the vertices $v_1, v_4 \in S$. Note that S is a connected dominating set. Then, there exists vertices $v_2, v_3 \in V(W_5) \setminus S$, such that v_2 and v_3 is adjacent to v_1 and v_4, respectively. Now, $S_3 = (S \setminus \{v_1, v_4\}) \cup \{v_2, v_3\} = \{u, v_2, v_3\}$ must also be a connected dominating set.

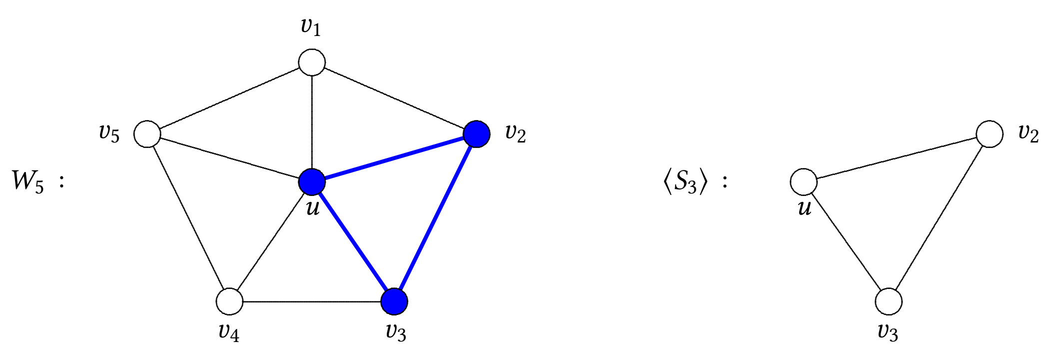

Figure 8: Wheel Graph W_5 and its 2-movable connected dominating set $S_3 = \{u, v_2, v_3\}$ with induced subgraph $\langle S_3 \rangle$.

Note that $N(u) = \{v_1, v_2, v_3, v_4, v_5\}$, $N(v_2) = \{u, v_1, v_3\}$, and $N(v_3) = \{u, v_2, v_4\}$. Then,

$$\begin{aligned}N(S_3) &= N(u) \cup N(v_2) \cup N(v_3)\\ &= \{v_1, v_2, v_3, v_4, v_5\} \cup \{u, v_1, v_3\} \cup \{u, v_2, v_4\}\\ &= \{u, v_1, v_2, v_3, v_4, v_5\}.\end{aligned}$$

Thus,

$$N[S_3] = S_3 \cup N(S_3)$$

$$= \{u, v_2, v_3\} \cup \{u, v_1, v_2, v_3, v_4, v_5\}$$
$$= \{u, v_1, v_2, v_3, v_4, v_5\}$$
$$= V(W_5).$$

Hence, S_3 is a dominating set. Observe that $uv_2, uv_3 \in E(W_5)$. This implies that S_3 is connected. Since S_3 is a dominating set and S_3 is connected, it follows that S_3 is a connected dominating set. Therefore, S is a 2-movable connected dominating set of W_5. In fact, there is no other set $S \subseteq V(W_5)$ that will satisfy the condition of a 2-movable connected dominating set. Thus, S is the minimum 2-movable connected dominating set of W_5. Therefore, $\gamma_{mc}^2(W_5) = 3$.

Case 3: $n = 6$.

Consider the wheel graph W_6, that is, defined by $V(W_6) = V(C_6) \cup V(K_1)$ with $V(C_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V(K_1) = \{u\}$, as presented in Figure 9. Also, $E(W_6) = E(C_6) \cup \{uv_1, uv_2, uv_3, uv_4, uv_5, uv_6\}$ where the edge set of C_6 is $E(C_6) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$. Let $S = \{u, v_1, v_3, v_5\}$.

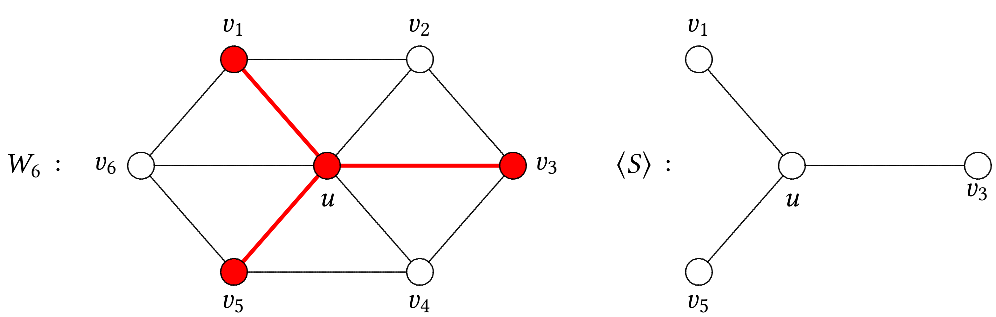

Figure 9: Wheel Graph W_6 and its connected dominating set $S = \{u, v_1, v_3, v_5\}$ with induced subgraph $\langle S \rangle$.

Observe that $N(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $N(v_1) = \{u, v_2, v_6\}$, $N(v_3) = \{u, v_2, v_4\}$, and $N(v_5) = \{u, v_4, v_6\}$. Then,

$$N(S) = N(u) \cup N(v_1) \cup N(v_3) \cup N(v_5)$$
$$= \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{u, v_2, v_6\} \cup \{u, v_2, v_4\} \cup \{u, v_4, v_6\}$$
$$= \{u, v_1, v_2, v_3, v_4, v_5, v_6\}.$$

Thus,

$$N[S] = S \cup N(S)$$
$$= \{u, v_1, v_3, v_5\} \cup \{u, v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$= \{u, v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$= V(W_6).$$

Hence, S is a dominating set. Observe that $uv_1, uv_3, uv_5 \in E(W_6)$. Then, S is connected. Since S is a dominating set and S is a connected, it follows that S is a connected dominating set.

Now, we will show that S is a 2-movable connected dominating set of W_6. Consider the vertices $u, v_1 \in S$. Since S is a connected dominating set, there exists vertices $v_4, v_6 \in V(W_6) \setminus S$, such that v_4 and v_6 is adjacent to u and v_1, respectively. Thus, we will show that $S' = (S \setminus \{u, v_1\}) \cup \{v_4, v_6\} = \{v_3, v_4, v_5, v_6\}$ is a connected dominating set.

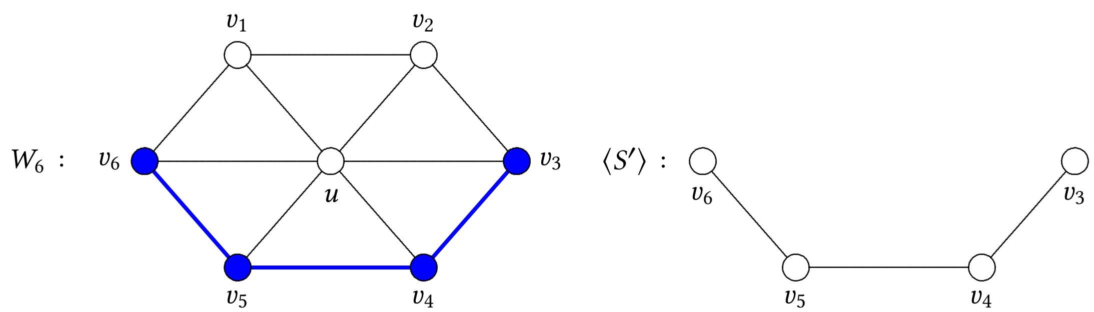

Figure 10: Wheel Graph W_6 and its connected dominating set $S' = \{v_3, v_4, v_5, v_6\}$ with induced subgraph $\langle S' \rangle$.

Note that $N(v_3) = \{u, v_2, v_4\}$, $N(v_4) = \{u, v_3, v_5\}$, $N(v_5) = \{u, v_4, v_6\}$, and $N(v_6) = \{u, v_1, v_5\}$. Then,

$$\begin{aligned} N(S') &= N(v_3) \cup N(v_4) \cup N(v_5) \cup N(v_6) \\ &= \{u, v_2, v_4\} \cup \{u, v_3, v_5\} \cup \{u, v_4, v_6\} \cup \{u, v_1, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5, v_6\}. \end{aligned}$$

Thus,

$$\begin{aligned} N[S'] = S' \cup N(S') &= \{v_3, v_4, v_5, v_6\} \cup \{u, v_1, v_2, v_3, v_4, v_5, v_6\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5, v_6\} \\ &= V(W_6). \end{aligned}$$

Hence, S' is a dominating set. Observe that $v_3v_4, v_4v_5, v_5v_6 \in E(C_6)$. This implies that S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. In fact, the same argument follows if we consider for every pair $x, y \in S$. In effect, S is a 2-movable connected dominating set of W_6. Also, note that there is no other set $S \subseteq V(W_6)$ that will satisfy the condition of a 2-movable connected dominating set. Therefore, S is the minimum 2-movable connected dominating set of W_6. Consequently, $\gamma^2_{mc}(W_6) = 4$.

Case 4: $n \geq 7$.

Consider the wheel graph W_n, defined by, $V(W_n) = V(C_n) \cup V(K_1)$ with $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $V(K_1) = \{u\}$, as illustrated in Figure 11. Additionally, $E(W_n) = E(C_n) \cup \{uv_1, uv_2, uv_3, \ldots, uv_n\}$ where the edge set of C_n is $E(C_n) = \{v_1v_2, v_2, v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\}$. Let $S = V(W_n)$.

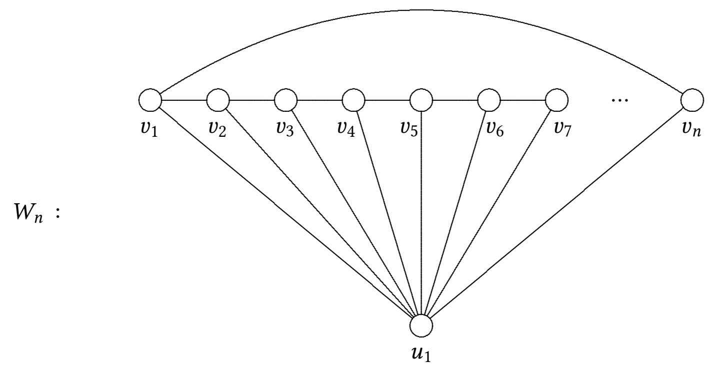

Figure 11: Wheel graph W_n.

Since $S = V(W_n)$, it follows that S is a dominating set. Similarly, S is connected. Therefore, S is a connected dominating set of W_n, where $n \geq 7$. Further, $|S| = n + 1$. We want to show that S is a 2-movable connected dominating set. To prove this, we will consider the following subcases:

Subcase 4.1: $S_1 = S \setminus \{u, v_i\} = \{v_j | \ 1 \leq j \leq n \ \text{ and } j \neq i\}$.

Note that S is a connected dominating set. Consider the vertices $u, v_i \in S$ for $1 \leq i \leq n$. We need to show that $S_1 = S \setminus \{u, v_i\} = \{v_j | \ 1 \leq j \leq n \ \text{ and } j \neq i\}$ is a connected dominating set. First, we have to show that S_1 is a dominating set such that $N[S_1] = V(W_n)$. Observe that, for $j = 1$,

$$N[v_1] = \{u, v_1, v_2, v_n\},$$

for $j = n$,

$$N[v_n] = \{u, v_1, v_{n-1}, v_n\},$$

and for $2 \leq j \leq n - 1$,

$$N[v_j] = \{u, v_{j-1}, v_j, v_{j+1}\}.$$

Hence,

$$\begin{aligned} N[S_1] &= N[v_1] \cup N[v_n] \cup \bigcup_{j=2}^{n-1} N[v_j] \\ &= \{u, v_1, v_2, v_n\} \cup \{u, v_1, v_{n-1}, v_n\} \cup \{u, v_{j-1}, v_j, v_{j+1}\} \\ &= \{u, v_1, v_2, v_3, \ldots, v_{j-1}, v_j, v_{j+1} \ldots, v_{n-1}, v_n\} \\ &= V(W_n). \end{aligned}$$

Thus, S_1 is a dominating set. Furthermore, observe that each vertex $v_j \in S_1$ belongs to $V(C_n)$. Observe further that $V(C_n) \setminus \{v_i\} \cong V(P_{n-1})$. Then, S_1 is connected. Since S_1 is a dominating set and S_1 is connected, it follows that S_1 is a connected dominating set.

Subcase 4.2: $S_2 = S \setminus \{v_i, v_{i+1}\} = \{u, v_j | \ 1 \leq j \leq n$ and $j \neq i, i+1\}$.

Note that S is a connected dominating set. Take the vertices $v_i, v_{i+1} \in S$. Then, we need to show that $S_2 = S \setminus \{v_i, v_{i+1}\} = \{u, v_j | \ 1 \leq j \leq n$ and $j \neq i, i+1\}$ is a connected dominating set. To prove that S_2 is a dominating set, we have to show that $N[S_2] = V(W_n)$. Observe that $u \in S_2$ and the closed neighborhood of u is

$$N[u] = \{u, v_1, v_2, v_3, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{n-1}, v_n\} = V(W_n).$$

Hence, S_2 is a dominating set. To prove that S_2 is a connected dominating set, observe further that $u \in S_2$ and $uv_j \in E(W_n)$. Therefore, S_2 is connected. Since S_2 is a dominating set and S_2 is connected, it follows that S_2 is a connected dominating set.

Subcase 4.3: $S_3 = S \setminus \{v_i, v_j\} = \{u, v_k | \ 1 \leq k \leq n$ with $k \neq i, j$ and $i \neq j\}$.

Recall that S is a connected dominating set. Take the vertices $v_i, v_j \in S$. We must show that $S_3 = S \setminus \{v_i, v_j\} = \{u, v_k | \ 1 \leq k \leq n$ with $k \neq i, j$ and $i \neq j\}$ is a connected dominating set. To prove that S_3 is a dominating set, we need to show that $N[S_3] = V(W_n)$. Notice that $u \in S_3$. Then, the closed neighborhood of u is

$$N[u] = \{u, v_1, v_2, v_3, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{n-1}, v_n\} = V(W_n).$$

Therefore, S_3 is a dominating set. Observe that $u \in S_3$ and $uv_k \in E(W_n)$. Hence, S_3 is connected. Since S_3 is a dominating set and connected, it follows that S_3 is a connected dominating set.

Considering the subcases above, we can conclude that S is a 2-movable connected dominating set. Now, we want to show that S is a minimum 2-movable connected dominating set of W_n where $n \geq 7$, that is, for every set $T \subset V(W_n)$ with $|T| = |S| - 1$ is not a connected dominating set. We will consider the following cases:

Case 1: $T_1 = S \setminus \{u\} = \{v_i \mid 1 \leq i \leq n\}$

Let $T_1 = S \setminus \{u\} = \{v_i \mid 1 \leq i \leq n\}$ be a connected dominating set. Then, by Remark 3.3, T_1 contains every cut-vertex. Consider the following subcases to show that T_1 is not a 2-movable connected dominating set:

Subcase 1.1: $T_1' = T_1 \setminus \{v_i, v_{i+1}\} = \{v_j \mid 1 \leq j \leq n, \ j \neq i, i+1\}$

Take $v_i, v_{i+1} \in T_1$. Then, $T_1' = T_1 \setminus \{v_i, v_{i+1}\} = \{v_j \mid 1 \leq j \leq n, \ j \neq i, i+1\}$ is a dominating set. Also, T_1' is a connected. Since T_1' is a dominating set and T_1' is connected, it follows that T_1' is a connected dominating set.

Subcase 1.2: $T_1' = T_1 \setminus \{v_i, v_j\} = \{v_k \mid 1 \leq k \leq n, \ k \neq i, j\}$

Consider $v_i, v_j \in T_1$. Then, $T_1' = T_1 \setminus \{v_i, v_j\} = \{v_k \mid 1 \leq k \leq n, \ k \neq i, j\}$ is a dominating set. However, T_1' is not connected. Since T_1' is a dominating set and T_1' is not connected, it follows that T_1' is not a connected dominating set. This is a contradiction to our assumption that T_1 is a connected dominating set.

Considering the subcases above, we can conclude that T_1 is not a 2-movable connected dominating set of W_n where $n \geq 7$.

Case 2: $T_2 = S \setminus \{v_1\} = \{u, v_i | \ 2 \leq v_i \leq n\}$.

Let $T_2 = S \setminus \{v_1\} = \{u, v_i | \ 2 \leq v_i \leq n\}$ be a connected dominating set. Then, by Remark 3.3, T_2 contains every cut-vertex. Consider the following subcases to show that T_2 is not a 2-movable connected dominating set:

Subcase 2.1: $T_2' = T_2 \setminus \{u, v_i\} = \{v_j | \ 2 \leq j \leq n, \ j \neq i\}$

Subcase 2.1.1:

Consider $u, v_i \in T_2$ where $3 \leq i \leq n-1$. Then, observe that the set $T_2' = T_2 \setminus \{u, v_i\} = \{v_j \mid 2 \leq j \leq n, \ j \neq i\}$ is a dominating set. However, T_2' is not connected. Since T_2' is a dominating set and T_2' is not connected, it follows that T_2' is not a connected dominating set. This is a contradiction to our assumption that T_2 is a connected dominating set.

Subcase 2.1.2:

Consider $u, v_2 \in T_2$. Then, $T_2' = T_2 \setminus \{u, v_2\} = \{v_i | \ 3 \leq i \leq n\}$ is a dominating set. Also, T_2' is connected. Since T_2' is a dominating set and T_2' is connected, it follows that T_2' is not a connected dominating set.

Subcase 2.1.3:

Consider $u, v_n \in T_2$. Then, $T_2' = T_2 \setminus \{u, v_n\} = \{v_i | \ 2 \leq i \leq n-1\}$ is a dominating set. Also, T_2' is connected. Since T_2' is a dominating set and T_2' is connected, it follows that T_2' is a connected dominating set.

Subcase 2.2: $T_2' = T_2 \setminus \{v_i, v_{i+1}\} = \{u, v_j | \ 2 \leq j \leq n$ and $j \neq i, i+1\}$

Take $v_i, v_{i+1} \in T_2$. Then, $T_2' = T_2 \setminus \{v_i, v_{i+1}\} = \{u, v_j | \ 2 \leq j \leq n, \ j \neq i, i+1\}$ is a dominating set. Notice that $uv_j \in E(W_n)$ where $2 \leq j \leq n$ and $j \neq i, i+1$. Then, T_2' is connected. Since T_2' is a dominating set and T_2' is connected, it follows that T_2' is a connected dominating set.

Subcase 2.3: $T_2' = T_2 \setminus \{v_i, v_j\} = \{u, v_k | \ 2 \leq k \leq n$ where $k \neq i, j\}$

Take $v_i, v_j \in T_2$. Then, $T_2' = T_2 \setminus \{v_i, v_j\} = \{u, v_k | \ 2 \leq k \leq n$ where $k \neq i, j\}$ is a dominating set. Furthermore, notice that $uv_k \in E(W_n)$ where $2 \leq k \leq n$, and $k \neq i, j$. Then, T_2' is connected. Since T_2' is a dominating set and T_2' is connected, it follows that T_2' is a connected dominating set.

Considering the subcases above, we can conclude that T_2 is not a 2-movable connected dominating set of the wheel graph W_n where $n \geq 7$.

Case 3: $T_3 = S \setminus \{v_n\} = \{u, v_i | \ 1 \leq i \leq n-1\}$

Suppose $T_3 = S \setminus \{v_n\} = \{u, v_i | \ 1 \leq i \leq n-1\}$ be a connected dominating set. Then, by Remark 3.3, T_3 contains every cut-vertex. Consider the following subcases to show that T_2 is not a 2-movable connected dominating set:

Subcase 3.1: $T_3' = T_3 \setminus \{u, v_i\} = \{v_j | \ 2 \leq j \leq n-2, \ j \neq i\}$

Subcase 3.1.1:

Consider $u, v_i \in T_3$ where $2 \leq j \leq n-2$. Note that T_3 is a connected dominating set. Then, observe that $T_3' = T_3 \setminus \{u, v_i\} = \{v_j | \ 2 \leq j \leq n-2, \ j \neq i\}$ is a dominating set. However, T_3' is not connected. Since T_3' is a dominating set and T_3' is not connected, it follows that T_3' is not a connected dominating set. Thus, this is contradiction to our assumption that T_3 is a connected dominating set.

Subcase 3.1.2:

Consider $u, v_1 \in T_3$. Then, $T_3' = T_3 \setminus \{u, v_1\} = \{v_j | \ 2 \leq j \leq n-1\}$ is a dominating set. Also, T_3' is connected. Since T_3' is a dominating set and T_3' is connected, it follows that T_3' is a connected dominating set.

Subcase 3.1.3:

Consider $u, v_{n-1} \in T_3$. Then, $T_3' = T_3 \setminus \{u, v_{n-1}\} = \{v_j | \ 1 \leq j \leq n-2\}$ is a dominating set. Similarly, T_3' is connected. Since T_3' is a dominating set and T_3' is connected, it follows that T_3' is a connected dominating set.

Subcase 3.2: $T_3' = T_3 \setminus \{v_i, v_{i+1}\} = \{u, v_j | \ 1 \leq j \leq n-1 \text{ and } j \neq i, i+1\}$

Take $v_i, v_{i+1} \in T_3$. Recall that T_3 is a connected dominating set. Then, observe that $T_3' = T_3 \setminus \{v_i, v_{i+1}\} = \{u, v_j | \ 1 \leq j \leq n-1, \ j \neq i, i+1\}$ is a dominating set. Also, T_3 is connected. Since T_3' is a dominating set and T_3' is connected, it follows that T_3' is a connected dominating set.

Subcase 3.3: $T_3' = T_3 \setminus \{v_i, v_j\} = \{u, v_k | \ 1 \leq k \leq n-1 \text{ where } k \neq i, j\}$

Consider $v_i, v_j \in T_3$. Note that T_3 is a connected dominating set. Then, observe further that $T_3' = T_3 \setminus \{v_i, v_j\} = \{u, v_k | \ 1 \leq k \leq n-1 \text{ where } k \neq i, j\}$ is a dominating set. Also, T_3' is connected. Since T_3' is a dominating set and T_3' is connected, it follows that T_3' is a connected dominating set.

Considering the subcases above, T_3 is not a 2-movable connected dominating set of the wheel graph W_n where $n \geq 7$.

Case 4: $T_4 = S \setminus \{v_i\} = \{u, v_j | \ 1 \leq j \leq n, \ j \neq i\}$ where $2 \leq i \leq n-1$.

Let $T_4 = S \setminus \{v_i\} = \{u, v_j \mid 1 \leq j \leq n, \ j \neq i\}$, where $2 \leq i \leq n-1$, be a connected dominating set. Then, by Remark 3.3, T_4 contains every cut-vertex. Consider the following subcases to show that T_4 is not a 2-movable connected dominating set:

Subcase 4.1: $T_4' = T_4 \setminus \{u, v_j\} = \{v_k \mid 1 \leq k \leq n, \ k \neq i, j\}$

Subcase 4.1.1:

Take $u, v_{i-1} \in T_4$. Then, $T_4' = T_4 \setminus \{u, v_{i-1}\} = \{v_k \mid 1 \leq k \leq n, \ k \neq i, i-1\}$ is a dominating set. Also, T_4' is connected. Since T_4' is a dominating set and T_4' is connected, it follows that T_4' is a connected dominating set.

Subcase 4.1.2:

Take $u, v_{i+1} \in T_4$. Then, $T_4' = T_4 \setminus \{u, v_{i+1}\} = \{v_k \mid 1 \leq k \leq n, \ k \neq i, i+1\}$ is a dominating set. Also, T_4' is connected. Since T_4' is a dominating set and T_4' is connected, it follows that T_4' is a connected dominating set.

Subcase 4.1.3:

Take $u, v_{n-1} \in T_4$. Then, $T_4' = T_4 \setminus \{u, v_{n-1}\} = \{v_k \mid 1 \leq k \leq n, k \neq i, n-1\}$ is a dominating set. However, T_4' is not connected. Since T_4' is a dominating set and T_4' is not connected, it follows that T_4' is not a connected dominating set.

Subcase 4.1.4:

Take $u, v_n \in T_4$. Then, $T_4' = T_4 \setminus \{u, v_n\} = \{v_k \mid 1 \leq k \leq n, \ k \neq i, n\}$ is a dominating set. But, T_4' is not connected. Since T_4' is a dominating set and T_4' is not connected, it follows that T_4' is not a connected dominating set.

Subcase 4.1.5:

Consider $u, v_j \in T_4$. Then, $T_4' = T_4 \setminus \{u, v_j\} = \{v_k \mid 1 \leq k \leq n,\ k \neq i, j\}$ where $j \neq \{i-1, i, i+1, n-1, n\}$, is a dominating set. However, T_4' is not connected. Since T_4' is a dominating set and T_4' is not connected, it follows that T_4' is not a connected dominating set.

Subcase 4.2: $T_4' = T_4 \setminus \{v_j, v_{j+1}\} = \{u, v_k \mid 1 \leq k \leq n,\ k \neq i, j, j+1\}$

Consider $v_j, v_{j+1} \in T_4$. Recall that T_4 is a connected dominating set. Then, observe that $T_4' = T_4 \setminus \{v_j, v_{j+1}\} = \{u, v_k \mid 1 \leq k \leq n,\ k \neq i, j, j+1\}$ is a dominating set. Also, T_4' is a connected set. Since T_4' is a dominating set and T_4' is connected, it follows that T_4' is a connected dominating set.

Subcase 4.3: $T_4' = T_4 \setminus \{v_j, v_k\} = \{u, v_r \mid 1 \leq r \leq n,\ r \neq i, j, k\}$

Take $v_j, v_k \in T_4$. Then, $T_4' = T_4 \setminus \{v_j, v_k\} = \{u, v_r \mid 1 \leq r \leq n,\ r \neq i, j, k\}$ is a dominating set. Also, T_4' is a connected set. Since T_4' is a dominating set and T_4' is connected, it follows that T_4' is a connected dominating set.

From the subcases above, T_4 is not a 2-movable connected dominating set of the wheel graph W_n where $n \geq 7$.

Considering the cases above, we can conclude that T is not a 2-movable connected dominating set. Therefore, for all $n \geq 7$, the 2-movable connected domination number of the wheel graph W_n, is $\gamma_{mc}^2(W_n) = |S| = n+1$. Moreover, S is the minimum 2-movable connected dominating set of W_n for $n \geq 7$. □

Corollary 4.6: Let $K_{m,n}$ be a bipartite graph with $m \geq 2$ and $n \geq 2$. Then,

$$\gamma_{mc}^2(K_{m,n}) = 2.$$

4.4 2-Movable Connected Domination Number of Generalized Graph $W_{m.n}$

Theorem 4.7: Let $W_{m,n}$ be a generalized wheel graph with $m \geq 2$ and $n \geq 2$. Then,

$$\gamma_{mc}^2(W_{m,n}) = 2.$$

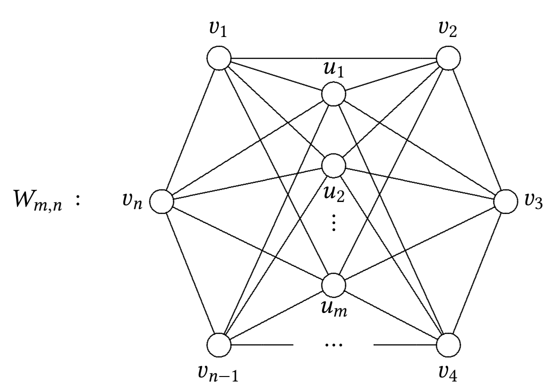

Figure 12: Generalized Wheel Graph $W_{m,n}$.

Proof: Let $V(W_{m,n}) = V(C_n) \cup V(\overline{K_m})$ be a generalized wheel graph $W_{m,n}$ where $V(\overline{K_m}) = \{u_1, u_2, u_3, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$, as presented in Figure 12. Let $S = \{u, v\}$ where $u \in V(\overline{K_m})$ and $v \in V(C_n)$. We must show that S is a connected dominating set. Observe that for every $u_i \in V(\overline{K_m})$ and $v_j \in V(C_n)$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, we have $uv_j, u_iv \in E(W_{m,n})$. Thus, $V(C_n) \subseteq N[u]$ and $V(\overline{K_m}) \subseteq N[v]$, implying that

$$V(W_{m,n}) = V(C_n) \cup V(\overline{K_m}) \subseteq N[u] \cup N[v] = N[S].$$

Therefore, S is a dominating set of $W_{m,n}$. Notice that $uv \in E(W_{m,n})$. This implies that S is connected. Since S is dominating set and S is connected, it follows that S is a connected dominating set.

Now, we will show that S is a 2-movable connected dominating set of $W_{m,n}$. Consider the vertices $u, v \in S$. Since S is a connected dominating set, there exists vertices $v' \in V(C_n)$ where $v' \neq v$, and there exists $u' \in V(\overline{K_m})$ where $u' \neq u$ such that v' and u' is adjacent to u and v, respectively. Then, we will show that $S' = (S \setminus \{u, v\}) \cup \{v', u'\} = \{v', u'\}$ is a connected dominating set of $W_{m,n}$. Notice that for every $v_j \in V(C_n)$ and $u_i \in V(\overline{K_m})$, $u_iv', v_ju' \in E(W_{m,n})$. Thus, $V(\overline{K_m}) \subseteq N[v']$ and $V(C_n) \subseteq N[u']$. Hence,

$$V(W_{m,n}) = V(C_n) \cup V(\overline{K_m}) \subseteq N[u'] \cup N[v'] = N[S'].$$

Therefore, S' is a dominating set. Observe that $u'v' \in E(W_{m,n})$. Then, S' is connected. Since S' is a dominating set and S' connected, it follows that S' is a connected dominating set. In effect, S is a 2-movable connected dominating set. Also, S is the minimum 2-movable connected dominating set of $W_{m,n}$. Thus, $\gamma_{mc}^2(W_{m,n}) \leq 2$. Consequently, by Remark 4.1, $\gamma_{mc}^2(W_{m,n}) = 2$. □

4.5 2-Movable Connected Domination Number of Nontrivial Graphs

Theorem 4.8: Let G be a connected nontrivial graph. Then, a graph G has a 2-movable connected dominating set if and only if G has no cut-vertices.

Proof: Let G be a connected nontrivial graph and suppose that G has a 2-movable connected dominating set, say S. Since S is a 2-movable connected dominating set, it follows that S is a connected and S is a dominating set. Assume further that G has two cut-vertices, say, x and y. Then, by Remark 3.3, $x, y \in S$. Since $x, y \in S$, it follows that $S \setminus \{x, y\}$ and $(S \setminus \{x, y\}) \cup \{u, v\}$, where $u, v \in V(G) \setminus S$, such that u is adjacent to x and v is adjacent to y are not connected dominating set of G. This implies that S is not a 2-movable connected dominating set. And so, this is a contradiction to our assumption that G has two cut-vertices. Thus, G has no cut-vertices.

Conversely, suppose that G has no cut-vertices. Let $S = V(G)$. Since $S = V(G)$, it follows that S is a dominating set. And so, S is connected. Thus, S is a connected dominating set. Assume $x, y \in S$. Note that G has no cut-vertices. Then, $S \setminus \{x, y\}$ is a connected dominating set of G. Therefore, S is a 2-movable connected dominating set of G. □

Theorem 4.9: Let G be a connected graph. Then, there exists a graph G such that

(i) $\gamma_m^2(G) < \gamma_{mc}^2(G)$
(ii) $\gamma_m^2(G) = \gamma_{mc}^2(G)$

Proof: To prove the theorem, consider the following cases:

Case 1: Let G be a connected graph such that $G \cong K_1 + P_5$. Then, by Corollary 3.2, $\gamma_m^2(K_1 + P_5) = \gamma(P_5)$. Note that, by Theorem 3.8, $\gamma(P_5) = \lceil \frac{5}{3} \rceil = 2$. Therefore, $\gamma_m^2(K_1 + P_5) = \gamma(P_5) = 2$.

On the other hand, consider the join graph $K_1 + P_5$ where $V(K_1) = \{u\}$ and $V(P_5) = \{v_1, v_2, v_3, v_4, v_5\}$, as presented in Figure 13. Suppose $S = \{v_2, v_3, v_4\}$. Then, we must show that S is a connected dominating set.

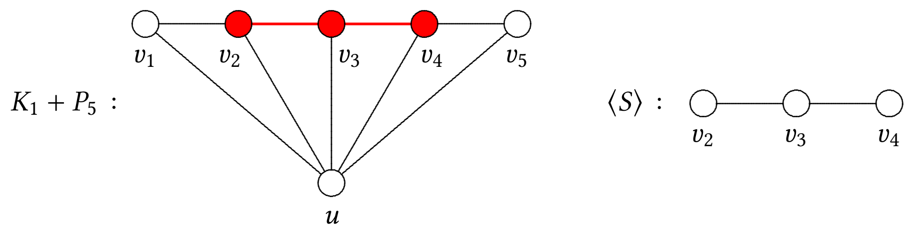

Figure 13: Join graph $K_1 + P_5$ and its connected dominating set $S = \{v_2, v_3, v_4\}$ with induced subgraph $\langle S \rangle$.

Note that $N(v_2) = \{u, v_1, v_3\}$, $N(v_3) = \{u, v_2, v_4\}$ and $N(v_4) = \{u, v_3, v_5\}$. Then,

$$\begin{aligned} N(S) &= N(v_2) \cup N(v_3) \cup N(v_4) \\ &= \{u, v_1, v_3\} \cup \{u, v_2, v_4\} \cup \{u, v_3, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\} \end{aligned}$$

Thus,

$$\begin{aligned} N[S] &= S \cup N(S) \\ &= \{v_2, v_3, v_4\} \cup \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= V(K_1 + P_5). \end{aligned}$$

Hence, S is a dominating set. Observe that $v_2v_3 \in E(P_5)$ and $v_3v_4 \in E(P_5)$. This implies that S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

Now, we will show that S is a 2-movable connected dominating set. Consider the vertices $v_2, v_3 \in S$. Recall that S is a connected dominating set. Then, there exists vertices $v_1, u \in V(K_1 + P_5) \setminus S$, such that v_1 and u is adjacent to v_2 and v_3, respectively. Thus, we will show that $S' = (S \setminus \{v_2, v_3\}) \cup \{v_1, u\} = \{u, v_1, v_4\}$ is a connected dominating set.

Note that $N(u) = \{v_1, v_2, v_3, v_4, v_5\}$, $N(v_1) = \{u, v_2\}$, and $N(v_4) = \{u, v_3, v_5\}$. Then,

$$\begin{aligned} N(S') &= N(u) \cup N(v_1) \cup N(v_4) \\ &= \{v_1, v_2, v_3, v_4, v_5\} \cup \{u, v_2\} \cup \{u, v_3, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\} \end{aligned}$$

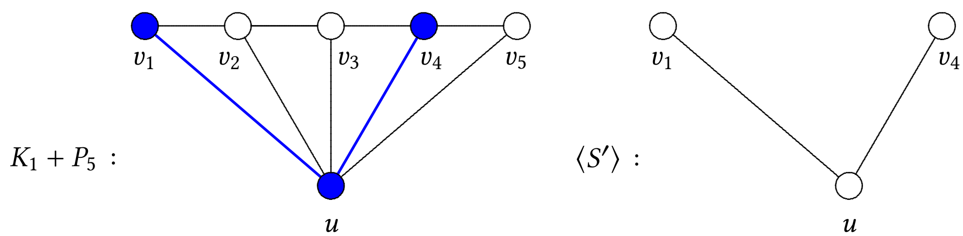

Figure 14: Join graph $K_1 + P_5$ and its connected dominating set $S' = \{u, v_1, v_4\}$ with induced subgraph $\langle S' \rangle$.

Thus,

$$\begin{aligned} N[S'] &= S' \cup N(S') \\ &= \{u, v_1, v_4\} \cup \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= \{u, v_1, v_2, v_3, v_4, v_5\} \\ &= V(K_1 + P_5). \end{aligned}$$

Hence, S' is a dominating set. Observe that $uv_1, uv_4 \in E(K_1 + P_5)$. This implies that S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. In fact, the same argument follows if we consider for every pair $x, y \in S$. In effect, S is a 2-movable connected dominating set of $K_1 + P_5$. Moreover, S is the minimum 2-movable connected dominating set of $K_1 + P_5$. Hence $\gamma^2_{mc}(K_1 + P_5) = 3$.

Accordingly, there exists a graph G such that $\gamma^2_m(G) < \gamma^2_{mc}(G)$.

Case 2: Let G be a connected graph such that $G \cong P_4 + K_4$. By Theorem 3.1, $\gamma^2_m(P_4 + K_4) = 2$. On the other hand, by Theorem 4.4, $\gamma^2_{mc}(P_4 + K_4) = 2$.

Accordingly, there exists a graph G such that $\gamma^2_m(G) = \gamma^2_{mc}(G)$. $\square$

Theorem 4.10: Let G be a connected graph. Then, there exists a graph G such that

(i) $\gamma^2_{mc}(G) < \gamma^1_{mc}(G)$
(ii) $\gamma^1_{mc}(G) < \gamma^2_{mc}(G)$
(iii) $\gamma^1_{mc}(G) = \gamma^2_{mc}(G)$

Proof: To prove the theorem, consider the following cases:

Case 1. Let G be a connected graph such that $G \cong K_{4,n}$, where $n \geq 3$ in Figure 15.

Let $V(K_{4,n}) = V(A) \cup V(B)$ where $V(A) = \{u_1, u_2, u_3, u_4\}$ as well as $V(B) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $S = \{u, v\}$ where $u \in V(A)$ and $v \in V(B)$. Observe that $V(A) \subseteq N[v]$ and $V(B) \subseteq N[u]$. Then,

$$V(K_{4,n}) = V(A) \cup V(B) \subseteq N[v] \cup N[u] = N[S].$$

Thus, S is a dominating set. Furthermore, notice that $uv \in E(K_{4,n})$. This implies that S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

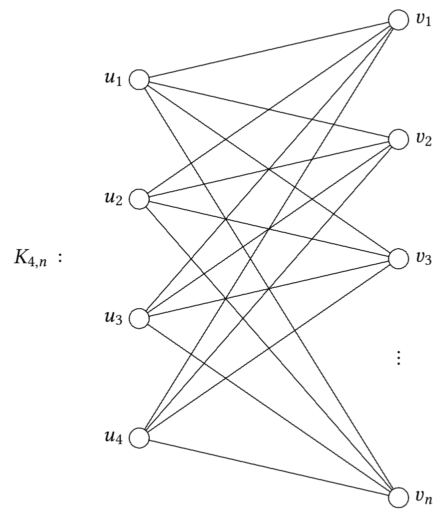

Figure 15: Complete Bipartite Graph $K_{4,n}$.

Now, note that $|V(A)| = 4$ and $|V(B)| \geq 3$. Since S is a connected dominating set, there exists vertices $u' \in V(A)$ and $v' \in V(B)$, such that u' is adjacent to v and v' is adjacent to u. Thus, $S' = (S \setminus \{u, v\}) \cup \{v', u'\} = \{v', u'\}$ must be a connected dominating set. Notice further that $V(A) \subseteq N[v']$ and $V(B) \subseteq N[u']$. This implies that

$$V(K_{4,n}) = V(A) \cup V(B) \subseteq N[v'] \cup N[u'] = N[S'].$$

Thus, S' is a connected dominating set. Additionally, observe that $u'v' \in E(K_{4,n})$. Hence, S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set of $K_{4,n}$. Therefore, $\gamma^2_{mc}(K_{4,n}) \leq 2$. Consequently, by Remark 4.1, $\gamma^2_{mc}(K_{4,n}) = 2$.

On the other hand, by Corollary 3.7, $\gamma^1_m(K_{4,n}) = 4$. Accordingly, there exists a graph G such that $\gamma^2_{mc}(G) < \gamma^1_{mc}(G)$.

Case 2: Let G be a connected graph such that $G \cong K_n$ where $n \geq 4$. By Lemma 3.5, $\gamma^1_{mc}(K_n) = 1$. In contrast, by Theorem 4.3, $\gamma^2_{mc}(K_n) = 2$.

Accordingly, there exists a graph G such that $\gamma^1_{mc}(G) < \gamma^2_{mc}(G)$.

Case 3: Let G be a connected graph such $G \cong K_1 + P_4$. By Corollary 3.6, $\gamma^1_{mc}(K_1 + P_4) = \gamma_c(P_4)$. Then, by Proposition 3.9, $\gamma_c(P_4) = 2$. Therefore, $\gamma^1_{mc}(K_1 + P_4) = \gamma_c(P_4) = 2$.

On the other hand, consider the join graph $K_1 + P_4$ with $V(K_1) = \{u\}$ and $V(P_4) = \{v_1, v_2, v_3, v_4\}$, as seen in Figure 16. Additionally, the edge set of $K_1 + P_4$ is $E(K_1 + P_4) = E(P_4) \cup \{uv_1, uv_2, uv_3, uv_4\}$ where $E(P_4) = \{v_1v_2, v_2v_3, v_3v_4\}$. Suppose $S = \{u, v_4\}$. Then, we must show that S is a connected dominating set.

Note that $N(u) = \{v_1, v_2, v_3, v_4\}$ and $N(v_4) = \{u, v_3\}$. Then,

$$\begin{aligned} N(S) = N(u) \cup N(v_4) &= \{v_1, v_2, v_3, v_4\} \cup \{u, v_3\} \\ &= \{u, v_1, v_2, v_3, v_4\}. \end{aligned}$$

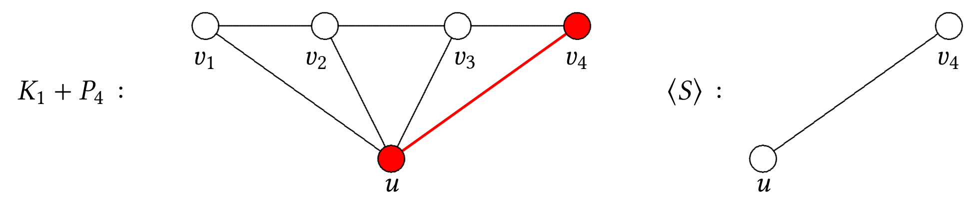

Figure 16: Join Graph $K_1 + P_4$ and its connected dominating set $S = \{u, v_4\}$ with induced subgraph $\langle S \rangle$.

Thus,

$$\begin{aligned} N[S] = S \cup N(S) &= \{u, v_4\} \cup \{u, v_1, v_2, v_3, v_4\} \\ &= \{u, v_1, v_2, v_3, v_4\} \\ &= V(K_1 + P_4). \end{aligned}$$

Hence, S is a dominating set. Observe that $uv_4 \in E(K_1 + P_4)$. Then, S is connected. Since S is a dominating set and S is connected, it follows that S is a connected dominating set.

Furthermore, we will show that S is a 2-movable connected dominating set. Consider the vertices $u, v_4 \in S$. Since S is a connected dominating set, there exists vertices $v_2, v_3 \in V(K_1 + P_4) \setminus S$, such that v_2 and v_3 is adjacent to u and v_4, respectively. Now, we must show that $S' = (S \setminus \{u, v_4\}) \cup \{v_2, v_3\} = \{v_2, v_3\}$ is a connected dominating set.

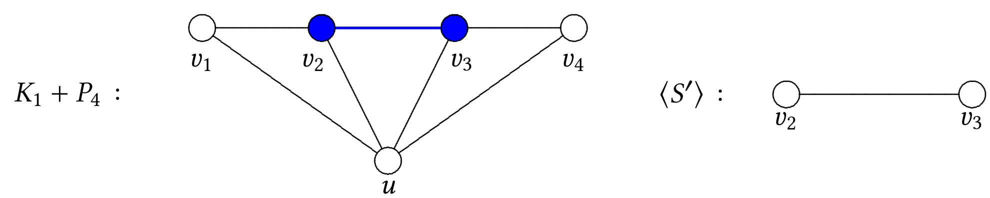

Figure 17: Join Graph $K_1 + P_4$ and its connected dominating set $S' = \{v_2, v_3\}$ with induced subgraph $\langle S' \rangle$.

Note further that $N(v_2) = \{u, v_1, v_3\}$ and $N(v_3) = \{u, v_2, v_4\}$. Then,

$$\begin{aligned} N(S') = N(v_2) \cup N(v_3) &= \{u, v_1, v_3\} \cup \{u, v_2, v_4\} \\ &= \{u, v_1, v_2, v_3, v_4\}. \end{aligned}$$

Thus,

$$\begin{aligned} N[S'] = S' \cup N(S') &= \{v_2, v_3\} \cup \{u, v_1, v_2, v_3, v_4\} \\ &= \{u, v_1, v_2, v_3, v_4\} \\ &= V(K_1 + P_4). \end{aligned}$$

Hence, S' is a dominating set. Observe that $v_2v_3 \in E(P_4)$. This means that S' is connected. Since S' is a dominating set and S' is connected, it follows that S' is a connected dominating set. In effect, S is a 2-movable connected dominating set of $K_1 + P_4$. Moreover, S is the minimum 2-movable connected dominating set of $K_1 + P_4$. Thus, $\gamma^2_{mc}(K_1 + P_4) = 2$.

Accordingly, there exist a graph G such that $\gamma^1_{mc}(G) = \gamma^2_{mc}(G)$. $\square$

5 Conclusions

In this study, the researcher investigated the 2-movable connected domination of the the complete graph K_n, join graph $G + H$, generalized wheel graph $W_{m,n}$. The author would like to recommend to future researchers certain graph operations such as, cartesian product, corona product, and lexicographic product, that were not considered in this study.

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References

1. G. Alexanderson, About the Cover: Euler and the Königsberg Bridges: A Historical View, *Bulletin (New Series) of the American Mathematical Society 43* **(2006)**, 567–573. https://doi.org/10.1090/S0273-0979-06-01130-X
2. J. Blair, R. Gera and S. Horton, Movable dominating sensor sets in networks, *Journal of Combinatorial Mathematics and Combinatorial Computing 77* **(2011)**, 103–123.
3. G. Chartrand and P. Zhang, *Chromatic Graph Theory*, Chapman & Hall/CRC, Boca Raton, **(2008)**.
4. A. Frendrup, M. A. Henning, B. Randerath and P. D. Vestergaard, An upper bound on the domination number of a graph with minimum degree 2, *Discrete Mathematics 309(4)* **(2009)**, 639–646. https://doi.org/10.1016/j.disc.2007.12.080
5. F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Boston, **(1969)**.
6. R. G. Hinampas Jr. and S. R. Canoy Jr., 1-Movable domination in graphs, *Applied Mathematical Sciences 8(172)* **(2014)**, 8565–8571. https://doi.org/10.12988/ams.2014.410859
7. J. Lomarda and S. R. Canoy Jr., 1-Movable connected dominating sets in graphs, *Applied Mathematical Sciences 9* **(2015)**, 507–514. https://doi.org/10.12988/ams.2015.411941
8. A. C. Pedrano and R. N. Paluga, On 2-movable domination in the join and corona of graphs, *Advances and Applications in Discrete Mathematics 42(2)* **(2025)**, 89–96. https://doi.org/10.17654/0974165825007

9. E. Sampathkumar and H. B. Walikar, The connected domination number of a graph, *Journal of Mathematical Physics and Sciences 13* **(1979)**, 607–613.