Annals of Communications in Mathematics
Volume 9, Number 3 (2026), 6
DOI: 10.62072/acm.2026.09038
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

Research article

Outer-connected convex Roman domination in graphs

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Received: 01 July 2026; Accepted: 16 August 2026; Published: 30 August 2026

Citation: L. F. Casinillo, Outer-connected convex Roman domination in graphs. Ann. Commun. Math. 9 **(2026)**, 6. https://doi.org/10.62072/acm.2026.09038

Abstract: Let $G = (V(G), E(G))$ be a connected graph and let $\alpha = (V_0, V_1, V_2)$ be a CvRDF on G. A function α is an outer-connected convex Roman dominating function (OCCvRDF) on G provided that for every $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, $V_1 \cup V_2$ is a convex set on G and $\langle V_0 \rangle$ is a connected subgraph. The weight of OCCvRDF α is denoted by $\widetilde{\omega}_G^{cCvR}(\alpha)$ and is defined as $\widetilde{\omega}_G^{cCvR}(\alpha) = \sum_{z \in V(G)} \alpha(z) = |V_1| + 2|V_2|$. The outer-connected convex Roman domination number of graph G is denoted by $\widetilde{\gamma}_{cCvR}(G)$ and is defined as the minimum weight of an OCCvRDF α on G, which can be written as $\widetilde{\gamma}_{cCvR}(G) = min\{\widetilde{\omega}_G^{cCvR}(\alpha) : \alpha \ is \ an$ OCCvRDF on $G\}$. Each OCCvRDF α on G satisfying the equation $\widetilde{\omega}_G^{cCvR}(\alpha) = \widetilde{\gamma}_{cCvR}(G)$ is called a $\widetilde{\gamma}_{cCvR}$-function on G. In this paper, we introduce a new study of convex Roman domination in graphs, called the outer-connected convex Roman dominating function, and give some of its graph-theoretic properties.

Mathematics Subject Classification: 05C69
Keywords: Outer-connected domination, Roman domination, convex Roman domination, outer-connected convex Roman domination

1 Introduction

Graph theory is a branch of discrete mathematics that has been actively researched and has experienced tremendous growth and recognition in the literature. Graph theory has several concepts, and one of its interesting topics is graph domination, which has many real-life applications, particularly in computer science and operations research [1,2,7,8]. During the year 2004, Cockayne et al. [4] introduced the concept of Roman domination in graphs, a labeling version of dominating sets, which is based on a defense strategy during the 4th century A.D. by Emperor Constantine to protect the Roman Empire [9,10]. The defense strategy uses an optimum number of legions (1 legion consists of a large number of soldiers) that protects the entire region and ensures that if a legion transfers to another location, there is one legion remaining that guards the original place. In that case, it means that a place with no legion must be adjacent to a location with two legions, which implies that the entire region is guarded. Hence, a Roman dominating function in graphs labels 0, 1, or 2 to the nodes of a graph, in which the nodes with label 0 are adjacent to at least 1 node with label 2 (two legions).

Nowadays, the topic of Roman domination in graphs has become a research interest of many discrete mathematicians and computer scientists, and several variations have been published. In 2023, an intriguing variation of Roman domination was introduced by Fortosa and Canoy [6], called the convex Roman dominating function, requiring the dominating set to be convex. Way back in 2007, outer-connected domination in graphs was developed by Cyman [5], which restricted the non-dominating set to be a connected subgraph. In fact, outer-connected dominating sets have also become a motivating parameter in domination theory. With these two parameters, the author is inspired to unite the two existing topics to introduce a new variation of Roman domination called the outer-connected convex Roman dominating function. Thus, this study's new idea is that the dominating set (labeled 1 or 2) is a convex set and the non-dominating set (labeled 0) is a connected subgraph. Hence, this newly introduced concept has a stronger defense strategy as opposed to the existing parameters of Roman domination due to its connectivity and compact dominating sets. This concept can be linked to the study of computer science, which deals with network security and focuses on optimal resource allocation, requiring the placement of resources in a connected location.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple and connected graph. The set $V(G)$ is a collection of vertices on G called the vertex set, and the set $E(G)$ is a collection of edges on G called the edge set. The cardinality of $V(G)$ is the order of G and is denoted by $|V(G)|$. The cardinality of $E(G)$ is the size of G and is denoted by $|E(G)|$. Let $a \in V(G)$. The *open neighborhood* of a is the set $N_G(a) = \{b \in V(G) : ab \in E(G)\}$ and the *closed neighborhood* of a is the set $N_G[a] = N_G(a) \cup \{a\}$. The degree of a is the number of edges incident to a and is denoted by $deg_G(a)$. Let $O_n \subseteq V(G)$. The collection $N_G(O_n) = N(O_n) = \bigcup_{x \in O_n} N_G(x)$ is called the *open neighborhood* of vertex set O_n in G. The set $N_G[O_n] = N[O_n] = N(O_n) \cup O_n$ is called the *closed neighborhood* of vertex set O_n in G. Let $a, b \in V(G)$. Then the distance from a to b is defined as the length of the shortest walk (path) from a to b in a graph G and is denoted by $d_G(a, b)$. If no such walk from a to b in G, then we defined the distance as $d_G(a, b) = \infty$. An a-b path with distance $d_G(a, b) \geq 1$ is called *a-b geodesic*. Let $I_G[a, b]$ be a collection of closed intervals that consists of all vertices on G that lie on an a-b geodesic, and let $A \subseteq V(G)$. The union of all collections $I_G[a, b]$ where $a, b \in A$ is denoted by $I_G[A]$, in that case, we have $\cup_{a,b \in A} I_G[a, b] = I_G[A]$. Then $c \in I_G[A]$ if and only if c lie in some a-b geodesic for any $a, b \in A$. A set A is *convex* on G if and only if the equality $I_G[A] = A$ holds [6]. If for all $a, b \in V(G)$, there exists an a-b path, then G is a connected graph. Hence, the vertex set $V(G)$ is a convex set if G is a connected graph.

A collection $D \subseteq V(G)$ is called a *dominating set* of G if for every $a \in V(G) \setminus D$, there exists $b \in D$ such that $d_G(a, b) = 1$ [8]. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set D on G, that is, $\gamma(G) = min\{D | D \text{ is a dominating set on } G\}$. If $|D| = \gamma(G)$, then D is a γ-set in G. A collection $C \subseteq V(G)$ is called an *outer-connected dominating set* of G if for every $a \in V(G) \setminus C$, there exists $b \in C$ such that $d_G(a, b) = 1$ and $\langle V(G) \setminus C \rangle$ is a connected subgraph of G [5]. The smallest cardinality of an outer-connected dominating set C on G is called the *outer-connected domination number* of G and is denoted by $\widetilde{\gamma}_c(G)$. If $|C| = \widetilde{\gamma}_c(G)$, then C is a $\widetilde{\gamma}_c$-set on G. A collection $C_c \subseteq V(G)$ is called an *outer-connected convex dominating set* of G if for every $a \in V(G) \setminus C_c$, there exists $b \in C_c$ such that $d_G(a,b) = 1$, C_c is a convex set, and $\langle V(G) \setminus C_c \rangle$ is a connected subgraph of G. The smallest cardinality of an outer-connected convex dominating set C_c on G is called the *outer-connected convex domination number* of G and is denoted by $\widetilde{\gamma}_{cCv}(G)$. If $|C_c| = \widetilde{\gamma}_{cCv}(G)$, then C_c is a $\widetilde{\gamma}_{cCv}$-set on G.

Let $\alpha : V(G) \to \{0,1,2\}$ be a function on G. Then set the following: $V_0 = \{v \in V(G) : \alpha(v) = 0\}$, $V_1 = \{v \in V(G) : \alpha(v) = 1\}$ and $V_2 = \{v \in V(G) : \alpha(v) = 2\}$. Hence, we can represent α as follows: $\alpha = (V_0, V_1, V_2)$. A function α is called a *Roman dominating function* (RDF) if for each $a \in V_0$, there exists $b \in V_2$ such that $d_G(a,b) = 1$ [4]. The *weight* of an RDF α is given as $\omega_G^R(\alpha) = \sum_{z \in V(G)} \alpha(z)$. The *Roman domination number* of G is denoted by $\gamma_R(G)$ and is defined as the smallest weight of α on G. In that case, we have $\gamma_R(G) = min\{\omega_G^R(\alpha) : \alpha\ is\ an\ \text{RDF}\ on\ G\}$. Every α on G with $\omega_G^R(\alpha) = \gamma_R(G)$ is a γ_R-function on G. A function α is a convex Roman dominating function (CvRDF) on G if for every $a \in V_0$, there exists $b \in V_2$ such that $d_G(a,b) = 1$, and $V_1 \cup V_2$ is a convex set on G [6]. The weight of α is denoted by $\omega_G^{CvR}(\alpha)$ and is defined as the sum $\omega_G^{CvR}(\alpha) = \sum_{z \in V(G)} \alpha(z)$. In that case, we have that $\omega_G^{CvR}(f) = |V_1| + 2|V_2|$. The convex Roman domination number of G is denoted by $\gamma_{CvR}(G)$ and is defined as the minimum weight of an CvRDF on G, that is, $\gamma_{CvR}(G) = min\{\omega_G^{CvR}(\alpha) : \alpha\ is\ an\ \text{CvRDF}\ on\ G\}$. Every CvRDF α on G with $\omega_G^{CvR}(\alpha) = \gamma_{CvR}(G)$ is a γ_{CvR}-function on G.

A function α is an outer-connected convex Roman dominating function (OCCvRDF) on G provided that for each $a \in V_0$, there exists $b \in V_2$ such that $d_G(a,b) = 1$, $V_1 \cup V_2$ is a convex set and $\langle V_0 \rangle$ is a connected subgraph of G. The weight of OCCvRDF α is denoted by $\widetilde{\omega}_G^{cCvR}(\alpha)$ and is defined as the sum $\widetilde{\omega}_G^{cCvR}(\alpha) = \sum_{z \in V(G)} \alpha(z) = |V_1| + 2|V_2|$. The outer-connected convex Roman domination number of graph G, denoted by $\widetilde{\gamma}_{cCvR}(G)$, is defined as the minimum weight of an OCCvRDF α on G. In that case, we have that $\widetilde{\gamma}_{cCvR}(G) = min\{\widetilde{\omega}_G^{cCvR}(\alpha) : \alpha\ is\ an\ \text{OCCvRDF}\ on\ G\}$. Every OCCvRDF α on G with $\widetilde{\omega}_G^{cCvR}(\alpha) = \widetilde{\gamma}_{cCvR}(G)$ is a $\widetilde{\gamma}_{cCvR}$-function on G. In Figure 1, we can consider the connected graph G of order $n = 8$. Let $\alpha = (V_0, V_1, V_2)$ be an RDF on G such that $V_1 = \{v_1\}$ and $V_2 = \{v_5, v_6\}$. Then $V(G) \setminus (V_1 \cup V_2) = V_0 = \{v_2, v_3, v_4, v_7, v_8\}$. In that case, it is clear that α is a $\widetilde{\gamma}_{cCvR}$-function on G. Thus, we have that $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |\{v_1\}| + 2|\{v_5, v_6\}| = 1 + 2(2) = 5$.

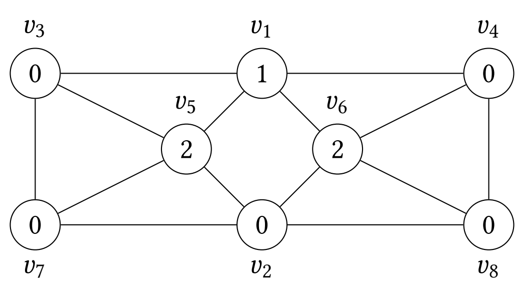

Figure 1: A graph G with $\widetilde{\gamma}_{cCvR}(G) = 5$

For some definitions of terms in this study that are not introduced in this section, readers may consult the following studies: [3,7,8]. This paper introduces a new restricted version of convex Roman domination, called outer-connected convex Roman domination in graphs, and initially investigates some of its combinatorial properties.

3 Results

This section presents some combinatorial properties of the newly introduced variation of convex Roman domination called the outer-connected convex Roman dominating function in connected graphs.

Theorem 1: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. If $\widetilde{\gamma}_{cCvR}(G) < n$, then $|V_2| < |V_0| \geq 2$, $I_G[V_1 \cup V_2] = V_1 \cup V_2$, and for every $x, y \in V_0$, there exists an x-y path on $\langle V_0 \rangle$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then α is an OCCvRDF on G and so, α is a CvRDF and also an RDF on G. Assume that $\widetilde{\gamma}_{cCvR}(G) < n$. Then $\widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| < n = |V(G)| = |V_0| + |V_1| + |V_2|$. In that case, we get $|V_2| < |V_0|$. Since $\widetilde{\gamma}_{cCvR}(G) < n$, $V_0 \neq \emptyset$. Let $v \in V_0$. Since α is an RDF on G, there exists $u \in V_2$ such that $uv \in E(G)$. In that case, $|V_2| \geq 1$. Now, since $|V_2| < |V_0|$, it follows that $|V_0| \geq 2$. Since α is an CvRDF on G, it implies that for every $u, v \in V_1 \cup V_2$, there exists a u-v walk and it is a u-v geodesic, that is, $d_{\langle V_1 \cup V_2 \rangle}(u, v)$ is defined. In that case, we have that $V_1 \cup V_2 = \cup_{u,v \in V_1 \cup V_2} I_G[u, v] = I_G[V_1 \cup V_2]$. Further, since α is an OCCvRDF on G, it follows that $\langle V_0 \rangle$ is a connected subgraph of G. Hence, for all $x, y \in V_0$, there exists x-y path on $\langle V_0 \rangle$. This completes the proof. $\square$

Corollary 1: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then $\widetilde{\gamma}_{cCvR}(G) < n$ if and only if $0 < |V_2| < |V_0|$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Suppose that $\widetilde{\gamma}_{cCvR}(G) < n$. Then $|V_0| \neq 0$. By definition of OCCvRDF on G, it follows that $|V_2| > 0$. In view of Theorem 1, we have that $|V_2| < |V_0|$. Therefore, we conclude that $0 < |V_2| < |V_0|$. Conversely, we suppose that $0 < |V_2| < |V_0|$. Then we have that $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. This proves the assertion. $\square$

Theorem 2: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then $|V_0| = 0$ if and only if $|V_2| = 0$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G. First, we assume that $|V_0| = 0$. Suppose, for the contrary, $|V_2| > 0$. Let $a \in V_2$. Then we set $M_0 = V_0$, $M_1 = V_1 \cup \{a\}$, and $M_2 = V_2 \setminus \{a\}$. Then it is clear that $\alpha' = (M_0, M_1, M_2)$ is an OCCvRDF on G. Thus, we obtain $\widetilde{\omega}_G^{cCvR}(\alpha') = |M_1| + 2|M_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \widetilde{\omega}_G^{cCvR}(\alpha) = \widetilde{\gamma}_{cCvR}(G)$. This is a contradiction. Therefore, we say that $|V_2| = 0$. Secondly, we assume that $|V_2| = 0$. Since α is a OCCvRDF on G, it is forced that $|V_0| = 0$. This proves the assertion. $\square$

Corollary 2: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. If $|V_0| = 0$, then $\widetilde{\gamma}_{cCvR}(G) = n$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Assume that $|V_0| = 0$. By Theorem 2, $|V_2| = 0$. Then it follows that $V_1 = V(G)$. Accordingly, we obtain $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| = |V(G)| = n$. Which proves the assertion. □

Theorem 3: *Let G be a connected graph. If $\alpha = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G, then $V_1 \cup V_2$ is an outer-connected convex dominating set on G.*

Proof: Assume that $\alpha = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G. Then it follows that α is an OCCvRDF on G. It also implies that α is an RDF on G, and so, $V_1 \cup V_2$ is a dominating set on G. Let $v \in V_0$. Then there exists $u \in V_2$ such that $d_G(u, v) = 1$, $V_1 \cup V_2$ is a convex set on G, and $\langle V_0 \rangle$ is a connected subgraph. Since $V(G) \setminus (V_1 \cup V_2) = V_0$ and $V_1 \cup V_2$ is a dominating set on G, it means that $V_1 \cup V_2$ is an outer-connected convex dominating set on G. □

Corollary 3: *Let G be a connected graph. If $\alpha = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G, then $V_1 \cup V_2$ is not always a $\widetilde{\gamma}_{cCv}$-set on G.*

Proof: To prove Corollary 3, we let $G = C_4 = [x_1, x_2, x_3, x_4, x_1, ...]$ and let $\alpha = (V_0, V_1, V_2)$ be an RDF on G. Let $V_0 = \{x_1\}$, $V_1 = \{x_3, x_4\}$ and $V_2 = \{x_2\}$. Then $\langle V_0 \rangle$ is a trivial connected subgraph of G. Also, we have that $V_1 \cup V_2$ is a convex set. By construction, it follows that $\alpha = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G. Let C be an outer-connected convex dominating set. Then it is easy to see that $C = \{x_2, x_3\}$. Consequently, we get $|C| < |V_1 \cup V_2|$. Therefore, it is not always the case that $V_1 \cup V_2$ is a $\widetilde{\gamma}_{cCv}$-set on G. □

Theorem 4: *Let G be a connected graph of order $n \in \mathbb{N}$. Then the following compound inequality holds:*

$$1 \leq max\{\gamma_R(G), \widetilde{\gamma}_{cCv}(G)\} \leq \widetilde{\gamma}_{cCvR}(G) \leq min\{2\widetilde{\gamma}_{cCv}(G), n\}.$$

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. By Theorem 3, $V_1 \cup V_2$ is an outer-connected convex dominating set on G. Hence, we have $\widetilde{\gamma}_{cCv}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\gamma}_{cCvR}(G)$. Additionally, since all OCCvRDF on G is an RDF on G, $\gamma_R(G) \leq \widetilde{\gamma}_{cCvR}(G)$. So, we have $1 \leq max\{\widetilde{\gamma}_{cCv}(G), \gamma_R(G)\} \leq \widetilde{\gamma}_{cCvR}(G)$. Let $|V_0| = 0$. By Theorem 2, we get $|V_2| = 0$. Consequently, we obtain $V_1 = V(G)$. It is easy to see that $\alpha = (\emptyset, V_1 = V(G), \emptyset)$ is an OCCvRDF on G. This follows that $\widetilde{\gamma}_{cCvR}(G) \leq \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Accordingly, we get $\widetilde{\gamma}_{cCvR}(G) \leq n$. Let C be a $\widetilde{\gamma}_{cCv}$-set on G. Then we have $\widetilde{\gamma}_{cCv}(G) = |C|$. Let $M_0 = V(G) \setminus C$, $M_1 = \emptyset$, and $M_2 = C$. Then $\alpha^* = (M_0, M_1, M_2)$ is an OCCvRDF on G, that is, $\langle M_0 = V(G) \setminus C \rangle$ is a connected subgraph and M_2 is convex set on G. Hence, we have $\widetilde{\gamma}_{cCvR}(G) \leq \widetilde{\omega}_G^{cCvR}(\alpha^*) = |M_1| + 2|M_2| = 2|C| = 2\widetilde{\gamma}_{cCv}(G)$. In that case, we end up with $\widetilde{\gamma}_{cCvR}(G) \leq min\{n, 2\widetilde{\gamma}_{cCv}(G)\}$. And the desired compound inequality holds. □

Theorem 5: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then $|V_0| = |V_2|$ if and only if $\widetilde{\gamma}_{cCvR}(G) = n$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Assume that $|V_0| = |V_2|$. Then we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. As for the converse, we assume that $\widetilde{\gamma}_{cCvR}(G) = n$. Suppose, for the contrary, $|V_0| \neq |V_2|$. Then we obtain either $|V_0| < |V_2|$ or $|V_0| > |V_2|$. If $|V_0| < |V_2|$, then we have that $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$, a contradiction. In addition, if we consider $|V_0| > |V_2|$, then we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$, a contradiction to our assumption. Therefore, $|V_0| = |V_2|$. □

Corollary 4: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. If $|V_0| = 1$, then $\widetilde{\gamma}_{cCvR}(G) = n$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then α is an OCCvRDF on G. Assume that $|V_0| = 1$. Since α is an OCCvRDF on G, it implies that $V_2 \neq \emptyset$. Consider $|V_2| > 1$. Then we have that $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$, a contradiction to Theorem 4. Consequently, it implies that $|V_2| = 1$ and so, $|V_0| = |V_2|$. By Theorem 5, it suffices to conclude that $\widetilde{\gamma}_{cCvR}(G) = n$. □

Allowing $V_1 = \emptyset$, we have the following results.

Theorem 6: *Let G be a connected graph and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then $|V_1| = 0$ if and only if V_2 is a $\widetilde{\gamma}_{cCv}$-set on G.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Assume that $|V_1| = 0$. By Theorem 3, V_2 is an outer-connected convex dominating set on G. Suppose V_2 is not a $\widetilde{\gamma}_{cCv}$-set on G. Then there exists $V_2' \subseteq V(G)$ such that V_2' is a $\widetilde{\gamma}_{cCv}$-set on G. Accordingly, we have that $|V_2'| < |V_2|$. Let $M_0 = V(G) \setminus V_2'$, $M_1 = \emptyset$, and $M_2 = V_2'$. Then it is easy to see that $\alpha' = (M_0, M_1, M_2)$ is an OCCvRDF on G. Hence, we obtain $\widetilde{\omega}_G^{cCvR}(\alpha') = |M_1| + 2|M_2| = 2|V_2'| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{cCvR}(\alpha) = \widetilde{\gamma}_{cCvR}(G)$, a contradiction to the definition of $\widetilde{\gamma}_{cCvR}$-function. Hence, we conclude that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. As for the converse, we assume that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Suppose that $|V_1| \neq 0$. Then it follows that $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| > 2|V_2|$. Let $V_0'' = V_0 \cup V_1$, $V_1'' = \emptyset$, and $V_2'' = V_2$. Then we have that $V_0'' \subseteq N_G^2(V_2'')$ and $V_2 = V_1'' \cup V_2''$ is a outer-connected convex dominating set on G. Accordingly, it is clear that $\alpha'' = (V_0'', V_1'', V_2'')$ is an OCCvRDF on G. So, we get $\widetilde{\omega}_G^{cCvR}(\alpha'') = |V_1''| + 2|V_2''| = 2|V_2| < |V_1| + 2|V_2| = \widetilde{\omega}_G^{cCvR}(\alpha) = \widetilde{\gamma}_{cCvR}(G)$. This is a contradiction. Therefore, we end up with $|V_1| = 0$. □

Corollary 5: *Let G be a connected graph and let $\alpha = (V_0, V_1 = \emptyset, V_2)$ be a $\widetilde{\gamma}_{cCvR}(G)$-function on G. Then $\widetilde{\gamma}_{cCvR}(G) = 2\widetilde{\gamma}_{cCv}(G)$.*

Proof: Let $\alpha = (V_0, V_1 = \emptyset, V_2)$ be a $\widetilde{\gamma}_{cCvR}(G)$-function on G. Then by Theorem 6, V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Thus, $\widetilde{\gamma}_{cCv}(G) = |V_2|$. Therefore, we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = 2\widetilde{\gamma}_{cCv}(G)$. This proves the assertion. □

Theorem 7: *Let G be a connected graph and $\alpha = (V_0, V_1 = \emptyset, V_2)$ be an OCCvRDF on G. If V_2 is a $\widetilde{\gamma}_{cCv}$-set on G, then α is $\widetilde{\gamma}_{cCvR}$-function on G.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be an OCCvRDF on G such that $|V_1| = 0$. Assume that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Suppose that α is not a $\widetilde{\gamma}_{cCvR}$-function on G. Then it follows that there exists a function $\alpha' = (M_0, M_1 = \emptyset, M_2)$ such that α' is a $\widetilde{\gamma}_{cCvR}$-function on G. So, we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha') = |M_1| + 2|M_2| = 2|M_2| < \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2|$. Thus, $|M_2| < |V_2|$. This is a contradiction since V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Therefore, α is a $\widetilde{\gamma}_{cCvR}$-function on G. □

Theorem 8: *Let G be a connected graph and $\alpha = (V_0, V_1, V_2)$ be a OCCvRDF on G. Then every cut-vertex $c \in V(G)$, $c \in V_1 \cup V_2$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a OCCvRDF on G. Assume for a moment that there exists a cut vertex $c \in V(G)$ such that $c \notin V_1 \cup V_2$. Let C_1 and C_2 be the components of $G - \{c\}$. Let $c_1 \in (V_1 \cup V_2) \cap V(C_1)$ and $c_2 \in (V_1 \cup V_2) \cap V(C_2)$. Since c is a cut vertex, it implies that every c_1-c_2 path must contain c. In that case, it means that $\langle (V_1 \cup V_2) \setminus \{c\} \rangle$ is not a connected subgraph on G and so, $V_1 \cup V_2$ is not a convex set on G, a contradiction. Hence, we have that $c \in V_1 \cup V_2$. Without loss of generality, suppose that $|V(C_1)| = 1$. Then it implies that $V(C_1) \subseteq V_0$. Consequently, $c \in V_2$. Therefore, the conclusion follows. □

Theorem 9: *If $G = P_n$ where $n \in \mathbb{N}$, then $\widetilde{\gamma}_{cCvR}(G) = n$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on $G = P_n$ where $n \in \mathbb{N}$. Let $n \leq 2$. Then it is easy to see that $\widetilde{\gamma}_{cCvR}(G) = n$. Let $n \geq 3$ and $G = P_n = [p_1, p_2, ..., p_n]$. Since $C = \{p_2, p_3, ..., p_{n-1}\}$ is a cut-vertex set on G, by Theorem 8, $C \subseteq V_1 \cup V_2$. Additionally, since $p_1 p_n \notin E(G)$, it follows that $\{p_1, p_n\} \subseteq V_1 \cup V_2$. Now, since α is a $\widetilde{\gamma}_{cCvR}$-function on G, it implies that $\{p_1, p_2, ..., p_n\} = V_1$ and $V_0 = \emptyset = V_2$. Consequently, we obtain $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. □

Theorem 10: *Let G be a connected graph. Then the two statements hold:*

(*i*) $\widetilde{\gamma}_{cCvR}(G) = 1$ *if and only if* $G = K_1$.

(*ii*) $\widetilde{\gamma}_{cCvR}(G) = 2$ *if and only if* $G \in \{K_2, K_1 + H\}$ *where H is any connected graph of order greater than 1.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}(G)$-function on G. Assume that $\widetilde{\gamma}_{cCvR}(G) = 1$. Suppose for a moment that $G \neq K_1$. Then we have $|V(G)| \geq 2$. Let $|V_0| \neq 0$. By Theorem 2, it implies that $|V_2| \neq 0$. Hence, $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| > 1$. This is a contradiction to the assumption. Now, let $|V_0| = 0$. By Theorem 2, we have that $|V_2| = 0$. So, we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |V_1| = |V(G)| > 1$, a contradiction to the assumption. Therefore, it suffices to say that $G = K_1$. On the other hand, assume that $G = K_1$. Since α is a $\widetilde{\gamma}_{cCvR}(G)$-function on G, it is easy to see that $|V_0| = 0 = |V_2|$. Therefore, by Corollary 2, we have $\widetilde{\gamma}_{cCvR}(G) = 1$ and so, (*i*) is satisfied.

Now, assume that $\widetilde{\gamma}_{cCvR}(G) = 2$. This follows that $\widetilde{\omega}_G^{cCvR}(\lambda) = |V_1| + 2|V_2| = 2$. Then we have $|V_2| \leq 1$. Let $|V_2| = 0$. By Theorem 2, it implies that $|V_0| = 0$. Hence, $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 2$. Since G is connected, it follows that $G = K_2$. Let $|V_2| = 1$. Then we have that $|V_1| = 0$. Let $V_2 = \{u\}$. Then V_2 is a trivial convex set. Set $G = K_1 + \langle V(G) \setminus \{u\} \rangle$ where $V(K_1) = \{u\}$. Since V_2 is an outer-connected convex dominating set on G by Theorem 3, we let $\langle V_0 = V(G) \setminus \{u\} \rangle$ be a connected subgraph of G. Suppose

$|V(G)\setminus\{u\}| = 1$. Then we get $G = K_1 + K_1 = K_2$. If $|V(G)\setminus\{u\}| \geq 2$, then we let $H = V(G)\setminus\{u\}$. Therefore, we concluded that $G \in \{K_2, K_1 + H\}$ where H is any connected graph of order greater than 1. Conversely, assume that $G \in \{K_2, K_1 + H\}$ where H is any connected graph of order greater than 1. If $G = K_2$, then we can have that $|V_0| = 1 = |V_2|$, and $V_1 = \emptyset$. Hence, we obtain $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = 2$. Now, let $G = K_1 + H$ where H is any connected graph of order greater than 1. So, we let $V_0 = H$, $V_1 = \emptyset$, and $V_2 = V(K_1)$. Then $\alpha = (V_0, V_1, V_2)$ is an OCCvRDF on G. By construction, it is clear thatbα is a $\widetilde{\gamma}_{cCvR}$-function on G. Therefore, we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V(K_1)| = 2$ and hence, (ii) is shown. □

Theorem 11: *Let G be a connected graph and $\alpha = (V_0, V_1, V_2)$ be an OCCvRDF on G. If for any $x, y \in V_1 \cup V_2$, $d_G(x, y) > 1$, then there exists an x-y path such that every v in that x-y path, $v \in V_1 \cup V_2$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be an OCCvRDF on G. Assume that for all $x, y \in V_1 \cup V_2$, $d_G(x, y) > 1$. Suppose that for all x-y path, every v in that x-y path, $v \notin V_1 \cup V_2$. Then it implies that $v \in V_0$. So, we get $d_{\langle V_1 \cup V_2\rangle}(x, y) = \infty$. Hence, $\langle (V_1 \cup V_2)\setminus\{v\}\rangle$ is a disconnected subgraph of G, a contradiction to the convexity of the set $V_1 \cup V_2$. Therefore, it means that there exists an x-y path such that every v in that x-y path, $v \in V_1 \cup V_2$. □

Theorem 12: *If $G = C_n$ where $n \in \mathbb{N}\setminus\{1, 2\}$, then*

$$\widetilde{\gamma}_{cCvR}(G) = \begin{cases} 2, & \text{if } n = 3, \\ n, & \text{if } n \geq 4. \end{cases}$$

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on $G = C_n$ where $n \in \mathbb{N}$. Let $n = 3$. Then $G = K_1 + H$ where $H = K_2$. In view of Theorem 10(ii), it is clear that $\widetilde{\gamma}_{cCvR}(G) = 2$. Now, let $n \geq 4$ and $G = C_n = [c_1, c_2, ..., c_n, c_1, ...]$. Then we can set $\{c_2, c_{n-1}\} \subseteq V_2$. Invoking the idea of Theorem 11, it implies that $\{c_3, c_4, ..., c_{n-2}\} \subseteq V_1 \cup V_2$. Since $\langle\{c_1, c_n\}\rangle$ is a connected subgraph, we have that $\{c_1, c_n\} = V_0$. Since α is a $\widetilde{\gamma}_{cCvR}$-function on G, it means that $V_1 = \{c_3, c_4, ..., c_{n-2}\}$. Consequently, we get $|V_0| = 2 = |V_2|$. By Theorem 5, we therefore get $\widetilde{\gamma}_{cCvR}(G) = n$. □

Theorem 13: *If $G = K_n$ where $n \in \mathbb{N}$, then*

$$\widetilde{\gamma}_{cCvR}(G) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n \geq 2. \end{cases}$$

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on $G = K_n$. Let $n = 1$. Then it is clear that $\widetilde{\gamma}_{cCvR}(G) = 1$. Let $n = 2$. Then by Theorem 9, it follows that $\widetilde{\gamma}_{cCvR}(G) = 2$. Let $n \geq 3$. Then it follows that $G = K_1 + H$ where H is a connected graph of order greater than 1, that is, $H = K_{n-1}$ where $n \geq 3$. Hence, by Theorem 10(ii), we therefore have $\widetilde{\gamma}_{cCvR}(G) = 2$. □

Theorem 14: *Let $G \neq K_1 + H$ where H is a connected graph with $|V(H)| = n \geq 2$ and $\alpha = (V_0, V_1 = \emptyset, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then $\widetilde{\gamma}_{cCvR}(G) = 4$ if and only if there exists a dominating set $C = \{u, v\}$ such that C is a convex set and $\langle V(G) \setminus C\rangle$ is a connected subgraph of G.*

Proof: Let $\alpha = (V_0, V_1 = \emptyset, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on $G \neq K_1 + H$ where H is a connected graph with $|V(H)| = n \geq 2$. Assume that $\widetilde{\gamma}_{cCvR}(G) = 4$. Then we have $\widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 4$. Since $V_1 = \emptyset$, we get $|V_2| = 2$. Since α is a $\widetilde{\gamma}_{cCvR}$-function on G, by Theorem 6, it implies that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Thus, V_2 is an outer-connected convex dominating set on G. Let $V_2 = \{u, v\}$. Then $C = \{u, v\}$ is a dominating set on G and C is a convex set on G. Let $V_0 = V(G) \setminus C$. Then $\langle V(G) \setminus C\rangle$ is a connected subgraph of G. Conversely, assume that there exists a dominating set $C = \{u, v\}$ such that C is a convex set and $\langle V(G) \setminus C\rangle$ is a connected subgraph of G. Then it means that C is an outer-connected convex dominating set on G. Since α is a $\widetilde{\gamma}_{cCvR}$-function on G and $V_1 = \emptyset$, by Theorem 6, it implies that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Thus, $2 = |C| \geq |V_2|$. If we let $|V_2| = 0$, then $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2(0) = 0$, a contradiction to the definition of OCCvRDF on G. Next, we let $|V_2| = 1$. Then by Theorem 10(ii), we obtain $G = K_1 + H$ where H is a connected graph of order greater than 1. This is a contradiction to our assumption. Consequently, it follows that $|V_2| \geq 2$ and hence, $|V_2| = 2$. Therefore, it suffices to have $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = 2(2) = 4$. $\square$

Theorem 15: *If $G = K_{m,n}$ where m and n are natural numbers greater than 1, then $\widetilde{\gamma}_{cCvR}(G) = 4$.*

Proof: Let $G = K_{m,n} = (V^1, V^2, E)$ where V^1 and V^2 are vertex sets with $|V^1| = m \geq 2$ and $|V^2| = n \geq 2$, and E is the edge set of G. Moreover, let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G and let $V^1 = \{u_1, u_2, ..., u_m\}$, and $V^2 = \{v_1, v_2, ..., v_n\}$. Note that for any vertex $u_j \in V^1$ where $j \in \{1, 2, 3, ..., m\}$, then $d_G(u_j, v_i) = 1$ for any $i \in \{1, 2, 3, ..., n\}$. Thus, it follows that $V^2 \subseteq N_G(u_j)$ for each $j \in \{1, 2, 3, ..., m\}$. By similar argument, we have $V^1 \subseteq N_G(v_k)$ for each $i \in \{1, 2, 3, ..., n\}$. Also, we note that for all $j \in \{1, 2, 3, ..., m\}$ and for all $i \in \{1, 2, 3, ..., n\}$, $\{u_j, v_i\}$ is a convex set on G. Then we can set $V_2 = \{u_j, v_i\}$ for some $j \in \{1, 2, 3, ..., m\}$ and $i \in \{1, 2, 3, ..., n\}$. Hence, we obtain $V(G) \subseteq N_G(\{u_j, v_k\})$. Furthermore, $\langle V(G) \setminus V_2\rangle$ is a connected subgraph on G. Since $\alpha = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G, it means that $V_1 = \emptyset$ and $V_0 = V(G) \setminus V_2$. Invoking Theorem 14, we therefore have $\widetilde{\gamma}_{cCvR}(G) = 4$. $\square$

Theorem 16: *Let G be a connected graph of order $n \in \mathbb{N}$ and $\alpha = (V_0, V_1 = \emptyset, V_2)$ be an OCCvRDF on G. Then $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\gamma}_{cCv}(G) + 1$ if and only if there exists $z \in V(G)$ such that $deg_G(z) = n - \widetilde{\gamma}_{cCv}(G)$.*

Proof: Let $\alpha = (V_0, V_1 = \emptyset, V_2)$ be an OCCvRDF on G. Assume that $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\gamma}_{cCv}(G) + 1$. Then we get $\widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = \widetilde{\gamma}_{cCv}(G) + 1$. Since $V_1 = \emptyset$, by Theorem 6, we have that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. Hence, $\widetilde{\gamma}_{cCv}(G) = |V_2|$. Thus, we obtain $2|V_2| = |V_2| + 1$. Consequently, we have $|V_2| = 1$. Let $V_2 = \{z\}$. Since $V_2 = \{z\}$ is a $\widetilde{\gamma}_{cCv}$-set on G, it means that $V(G) = N_G[z]$ and $\langle V(G) \setminus \{z\}\rangle$ is a connected subgraph of G. Therefore, $deg_G(z) = |N_G(z)| = |V(G) \setminus \{z\}| = |V(G)| - |\{z\}| = n - |V_2| = n - \widetilde{\gamma}_{cCv}(G)$.

Conversely, assume that there exists $z \in V(G)$ such that $deg_G(z) = n - \widetilde{\gamma}_{cCv}(G)$. Let $V_2 = \{z\}$. Then $V_2 = \{z\}$ is a convex set on G and $\langle V(G) \setminus V_2\rangle$ is a connected subgraph of G. Since $V_1 = \emptyset$

and V_2 is a convex dominating set, by Theorem 6, it follows that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. So, we get $|V_2| = \widetilde{\gamma}_{cCv}(G) = n - deg_G(z)$. Since V_2 is a $\widetilde{\gamma}_{cCv}$-set on G and $\langle V(G) \setminus V_2 \rangle$ is a connected subgraph, it implies that V_2 is an outer-connected convex dominating set on G. Hence, $V_0 = V \setminus V_2$. Now, since V_2 is a $\widetilde{\gamma}_{cCv}$-set on G and $V_1 = \emptyset$, by Theorem 6, $\alpha = (V_0, V_1 = \emptyset, V_2)$ is a $\widetilde{\gamma}_{cCvR}$-function on G. Therefore, $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = |V_2| + |V_2| = \widetilde{\gamma}_{cCv}(G) + 1$. This proves the assertion. □

Theorem 17: *Let G be a connected graph of order $n \in \mathbb{N}$. Then $\widetilde{\gamma}_{cCv}(G) = \widetilde{\gamma}_{cCvR}(G)$ if and only if $\widetilde{\gamma}_{cCv}(G) = n$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Assume that $\widetilde{\gamma}_{cCv}(G) = \widetilde{\gamma}_{cCvR}(G)$. Then we get $|V_1| + |V_2| = |V_1| + 2|V_2| = \widetilde{\omega}_G^{cCvR}(\alpha)$. Consequently, we have $|V_2| = 0$. By Theorem 2, it implies that $|V_0| = 0$.Thus, $\widetilde{\gamma}_{cCv}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$.

Conversely, assume that $\widetilde{\gamma}_{cCv}(G) = n$. By Theorem 4, $n = \widetilde{\gamma}_{cCv}(G) \leq \widetilde{\gamma}_{cCvR}(G) \leq n$. Accordingly, we conclude that $\widetilde{\gamma}_{cCv}(G) = \widetilde{\gamma}_{cCvR}(G)$. □

Definition 1: *Let G be a connected graph. Then G is called an outer-connected convex Roman graph provided that $\widetilde{\gamma}_{cCvR}(G) = 2\widetilde{\gamma}_{cCv}(G)$.*

In view of Definition 1, the characterization below is satisfied.

Theorem 18: *Let G be a connected graph and let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Then the following statements are mathematically equivalent:*

(i) *G is an outer-connected convex Roman graph;*

(ii) *G has a $\widetilde{\gamma}_{cCvR}$-function α such that $|V_1| = 0$; and*

(ii) *G has a $\widetilde{\gamma}_{cCvR}$-function α such that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Suppose that G is an outer-connected convex Roman graph. Then $\widetilde{\gamma}_{cCvR}(G) = 2\widetilde{\gamma}_{cCv}(G)$. So, we get $2\widetilde{\gamma}_{cCv}(G) = 2(|V_1| + |V_2|) = 2|V_1| + 2|V_2| = \widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2|$. Consequently, we have that $|V_1| = 0$. Thus $(i) \implies (ii)$. Next, assume that G has a $\widetilde{\gamma}_{cCvR}$-function α such that $|V_1| = 0$. By Theorem 6, it follows that V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. In that case, we conclude that $(ii) \implies (iii)$. Furthermore, suppose that graph G has a $\widetilde{\gamma}_{cCvR}$-function α for which V_2 is a $\widetilde{\gamma}_{cCv}$-set on G. This means that $|V_2| = \widetilde{\gamma}_{cCv}(G)$. By Theorem 6, it follows that $V_1 = \emptyset$. Therefore, we get $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = 2\widetilde{\gamma}_{cCv}(G)$ and hence, $(iii) \implies (i)$. This proves Theorem 18. □

The next result is a quick consequence of Theorem 18.

Corollary 6: *Let G be a connected graph. If $\widetilde{\gamma}_c(G) = 1$, then G is an outer-connected convex Roman graph with $\widetilde{\gamma}_{cCvR}(G) = 2$.*

Proof: Let $\alpha = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{cCvR}$-function on G. Assume $\widetilde{\gamma}_c(G) = 1$. Then it implies that $V_1 = \emptyset$. Consequently, we get $\widetilde{\gamma}_c(G) = |V_2| = 1$. Since α is a $\widetilde{\gamma}_{cCvR}$-function on G, by Theorem 6, we have that V_2

is a $\widetilde{\gamma}_{cCv}$-set on G, that is, $\widetilde{\gamma}_{cCv}(G) = |V_2|$. Invoking Theorem 18, G is an outer-connected convex Roman graph and $\widetilde{\gamma}_{cCvR}(G) = \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = 2(1) = 2$. □

Theorem 19: *Let G be a connected graph and $\alpha = (V_0, V_1 = \emptyset, V_2)$ be an OCCvRDF on G. Then, V_2 is a minimal outer-connected convex dominating set on G if and only if for all $u \in V_2$, there exists $v \in V_0$ such that $N_G(v) \cap V_2 = \{u\}$ or $d_G(u, w) \neq 2$ for any $w \in V_2 \setminus \{u\}$.*

Proof: Let $\alpha = (V_0, V_1 = \emptyset, V_2)$ be a OCCvRDF on G. By Theorem 3, V_2 is an outer-connected convex dominating set on G. Suppose that V_2 is a minimal outer-connected convex dominating set on G. Then it follows that for all $u \in V_2$, $V_2 \setminus \{u\}$ is not an outer-connected convex dominating set on G. Thus, it implies that there exists $v \in V(G) \setminus (V_2 \setminus \{u\})$ such that $d_G(v, w) \neq z$ for any $w \in V_2 \setminus \{u\}$. If $v \neq u$, then since V_2 is an outer-connected convex dominating set, v is dominated by V_2. Accordingly, we have that $d_G(u, v) = 1$. This indicates that $N_G(v) \cap V_2 = \{u\}$. If $v = u$, then it follows that $d_G(u, w) \neq 1$ for any $w \in V_2 \setminus \{u\}$.

Conversely, we suppose that for all $u \in V_2$, there exists $v \in V_0$ such that $N_G(v) \cap V_2 = \{u\}$. Then it means that $v \in V_0$ is not dominated by $V_2 \setminus \{u\}$. Consequently, it follows that V_2 is a minimal outer-connected convex dominating set on G. In addition, we suppose that for all $u \in V_2$, $d_G(u, w) \neq 1$ for any $w \in V_2 \setminus \{u\}$. Then this follows that u is not dominated for any $w \in V_2 \setminus \{u\}$. In that case, $V_2 \setminus \{u\}$ is not an outer-connected convex dominating set on G. Therefore, we conclude that V_2 is a minimal outer-connected convex dominating set on G. □

Theorem 20: *Let G and H be connected noncomplete graphs. Then*

$$\widetilde{\gamma}_{cCvR}(G + H) = \begin{cases} 2, & \text{if } \widetilde{\gamma}_c(G) = 1 \text{ or } \widetilde{\gamma}_c(H) = 1, \\ 4, & \text{if } \widetilde{\gamma}_c(G) \neq 1 \text{ and } \widetilde{\gamma}_c(H) \neq 1. \end{cases}$$

Proof: It is worth noting that $G + H \neq K_1$. Then in view of Theorem 10(ii), it implies that $\widetilde{\gamma}_{cCvR}(G + H) \geq 2$. Assume that $\widetilde{\gamma}_c(G) = 1$ or $\widetilde{\gamma}_c(H) = 1$. Then we have that $\widetilde{\gamma}_{cCvR}(G + H) \leq 2$. Accordingly, we get $\widetilde{\gamma}_{cCvR}(G + H) = 2$. Suppose that $\widetilde{\gamma}_c(G) \neq 1$ and $\widetilde{\gamma}_c(H) \neq 1$. Then $\widetilde{\gamma}_{cCvR}(G) \geq 2$ and $\widetilde{\gamma}_{cCvR}(H) \geq 2$. Then it means that $\widetilde{\gamma}_{cCvR}(G + H) \geq 4$. Let $a \in V(G)$ and $b \in V(H)$. Then we have that $ab \in E(G + H)$. In that case, we have $\{a, b\}$ is a convex set on $G + H$. Observe that $V(G) \subseteq N_G(a)$ and $V(H) \subseteq N_H(b)$. Hence, it implies that $V(G + H) = N_{G+H}[\{a, b\}]$. Also, it is worth noting that $\langle V(G + H) \setminus \{a, b\} \rangle$ is a connected subgraph of $G + H$. Hence, we set $V_0 = V(G + H) \setminus \{a, b\}$, $V_1 = \emptyset$ and $V_2 = \{a, b\}$. Then we have that $\alpha = (V_0, V_1, V_2)$ is an OCCvRDF on $G + H$. Thus, we get $\widetilde{\gamma}_{cCvR}(G + H) \leq \widetilde{\omega}_G^{cCvR}(\alpha) = |V_1| + 2|V_2| = 2|V_2| = 2|\{a, b\}| = 4$. Therefore, we conclude that $\widetilde{\gamma}_{cCvR}(G + H) = 4$. □

Corollary 7: *Let G and H be complete graphs. Then $\widetilde{\gamma}_{cCvR}(G + H) = 2$.*

4 Conclusions

This study has investigated a new variation of the convex Roman dominating function called the outer-connected convex Roman dominating function. Some important combinatorial properties of outer-connected convex Roman dominating functions in a connected graph were given, including some equalities and inequalities, bounds, and characterizations of the outer-connected convex Roman domination number for small values, and exact values derived from some special graphs. The paper suggested that the newly introduced parameter should be characterized under binary product operations of two graphs as future research to supplement the current results. Furthermore, the study can be extended to computer science and informatics, which deal with computational complexity or polynomial-time algorithms.

Acknowledgement: The author is grateful to VSU-Main and MSU-IIT for the support and mathematical learning, respectively. Moreover, the author would like to thank the referees for giving significant comments to improve this paper.

Funding Statement: The author(s) received no specific funding for this study.

Data Availability Statement: Not applicable.

Ethics Approval: Not applicable.

Use of Generative-AI tools declaration: The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflicts of Interest: The author has no competing interests to disclose.

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