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Research article

On Ideal Leras Generalized Closed Sets in Ideal Leras Topological Spaces

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Abstract: The purpose of this paper is to extend the study of generalized closed sets in leras topological spaces to the ideal setting. Using the leras local function and leras $\star$-closure operator associated with an ideal, we introduce and investigate two new classes of sets, called *ideal leras generalized closed sets* (I_{lrg}-closed sets) and *ideal leras generalized open sets* (I_{lrg}-open sets), in ideal leras topological spaces. We show that every lrg-closed set is I_{lrg}-closed, and that every $l^\star$-closed set is I_{lrg}-closed, while the converses fail in general, thereby placing the new classes correctly within the existing hierarchy of sets in leras and ideal leras topological spaces. Several characterizations of I_{lrg}-closed sets are obtained, including formulations in terms of the leras $\star$-closure operator, the leras closures of singletons, and the non-existence of nonempty lr-closed sets in certain associated difference sets. Additional properties, such as finite unions of I_{lrg}-closed sets being I_{lrg}-closed and sufficient conditions under which I_{lrg}-closed sets coincide with lrg-closed sets, are also established. Appropriate examples are constructed throughout to illustrate the developed concepts and to demonstrate that the implications obtained cannot, in general, be reversed.

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Keywords: leras-open set, leras-closed set, leras generalized-closed set, ideal leras topological space, $l^\star$-closed

1 Introduction

Generalized open and closed sets play a significant role in the development of modern topology. In 1964, [4] introduced generalized closed sets and initiated the study of weaker forms of openness and closedness. Later, several researchers extended these ideas to different generalized topological structures. I. Rajasekaran et al. [10] introduced leras topological spaces and studied concepts such as leras closed sets, leras interior, and leras closure. Further investigations led to the study of generalized closed sets in leras topological spaces [9].

The notion of an ideal on a topological space, together with the associated local function and $\star$-topology introduced by Kuratowski [2] and Vaidyanathaswamy [11], and later developed systematically by Janković and Hamlett [1], has proved to be a powerful device for refining topological structures. Equipping a space with an ideal makes it possible to describe smallness or negligibility of sets within a purely topological framework, and this in turn yields finer closure operators, sharper separation properties, and new generalized classes of sets that are not visible at the level of the original topology alone. This ideal-topological approach has been applied fruitfully to numerous generalized topological structures obtained by weakening the open-set axioms, such as semi-open sets [3], pre-open sets [5], and α-open sets [6], among others, and continues to generate new decompositions and characterizations in the literature [7].

Since leras topological spaces [10] constitute a genuinely new generalized topological framework arising from the interaction of simply extended topologies and their associated leras topology the natural next step is to examine how the ideal-theoretic machinery behaves once it is transplanted onto this setting. This motivates the introduction of ideal leras topological spaces and, within them, the leras local function and leras $\star$-closure operator [8]. Building on this foundation, the present paper takes the further step of combining the generalized closed sets already studied in leras topological spaces (namely, lrg-closed sets [9]) with the ideal structure, producing a strictly larger and more flexible class of sets.

The importance of this study over the existing ones is threefold. First, from a structural point of view, the class of lrg-closed sets in a leras topological space is rigid in the sense that it depends only on the leras topology; incorporating an ideal allows the notion of closedness to be relaxed further, in a controlled way, by neglecting sets that are declared small by the ideal. As shown in this paper, every lrg-closed set is I_{lrg}-closed, but not conversely, so the newly introduced class properly enlarges the existing one and captures examples that the earlier theory could not accommodate. Second, the ideal leras local function and $\star$-closure operators developed in [8] had not previously been used to construct a generalized-closedness notion analogous to Levine's original construction; doing so here provides a natural bridge between the ideal theory on leras spaces and the generalized-closed-set theory on leras spaces, unifying two lines of investigation that had so far developed independently. Third, on the practical side, the characterizations obtained here (in terms of the leras $\star$-closure, leras closures of singletons, and the absence of nonempty lr-closed sets in certain difference sets) give workable criteria for verifying I_{lrg}-closedness without appealing directly to the definition, which is expected to be useful in further decomposition-type results and in comparing leras-type generalized closed sets with their classical or ideal-topological counterparts.

Motivated by these considerations, this paper introduces I_{lrg}-closed sets and I_{lrg}-open sets in ideal leras topological spaces. Their relationships with existing classes of sets lr-closed, lrg-closed, $l^\star$-closed, $l^\star$-dense in itself, and $l^\star$-perfect sets are investigated, and several basic properties and characterizations are established, supported throughout by illustrative examples showing the sharpness of the implications obtained.

2 Preliminaries

Definition 2.1: *An ideal [2] I on a nonempty set X is a nonempty collection of subsets of X which satisfies*

1. $A \in I$ *and* $B \subset A$ *implies* $B \in I$,
2. $A \in I$ *and* $B \in I$ *implies* $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X, and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator

$$(\cdot)^{\star} : \mathcal{P}(X) \to \mathcal{P}(X),$$

called a *local function* [11] of A with respect to τ and I, is defined as follows. For $A \subset X$,

$$A^{\star}(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau : x \in U\}$.

A Kuratowski closure operator $cl^{\star}(\cdot)$ for a topology $\tau^{\star}(I, \tau)$, called the $\star$-*topology* (finer than τ), is defined by

$$cl^{\star}(A) = A \cup A^{\star}(I, \tau).$$

When there is no chance for confusion, we write $A^{\star}$ for $A^{\star}(I, \tau)$ and $\tau^{\star}$ for $\tau^{\star}(I, \tau)$. If I is an ideal on X, the space (X, τ, I) is called an *ideal space.*

Definition 2.2 ([4])**:** *Let X be a nonempty set equipped with a topology τ. Levine introduced the collection*

$$\tau(B) = \{O \cup (\widetilde{O} \cap B) : O, \widetilde{O} \in \tau,\ B \notin \tau \textit{ fixed}\},$$

and referred to it as the simple extension *of τ by B. One verifies:*

1. $\emptyset, X \in \tau(B)$,
2. *the union of any collection of members of $\tau(B)$ belongs to $\tau(B)$,*
3. *the intersection of finitely many members of $\tau(B)$ belongs to $\tau(B)$.*

The pair $(X, \tau(B))$ is called a simply extended topological space *(briefly, SETS). The members of $\tau(B)$ are known as B-open sets, while their complements are termed B-closed sets.*

*For any subset S of X, the B-*closure *of S, denoted $B_C(S)$, is the intersection of all B-closed sets containing S, and the B-*interior *of S, denoted $B_I(S)$, is the union of all B-open sets contained in S.*

Definition 2.3 ([10])**:** *Let $(X, \tau(B))$ be a simply extended topological space. Define*

$$LR_B(X) = \left\{\mathbb{R} \cup \mathbb{S} : \mathbb{R} = O \cap B,\ \mathbb{S} = \widetilde{O} \cap \widetilde{B}\right\},$$

where

1. $O, \widetilde{O} \in \tau$ *and* $B, \widetilde{B} \notin \tau$,
2. $B \cup \widetilde{B} = X$.

The family $LR_B(X)$ is called a leras topology *on X if it satisfies the following axioms:*

1. $\emptyset, X \in LR_B(X)$,
2. *the union of any collection of members of $LR_B(X)$ belongs to $LR_B(X)$,*

3. *the intersection of finitely many members of $LR_B(X)$ belongs to $LR_B(X)$.*

The triple $(X, \tau(B), LR_B(X))$ is called a leras topological space. *The elements of $LR_B(X)$ are called* leras-open *(briefly, lr-open) sets; their complements are called* leras-closed *(briefly, lr-closed) sets. The collection of all lr-closed sets is denoted $LR_{B^c}(X)$.*

We write $\mathbb{L}^X = LR_B(X)$ for the leras topology on X, and similarly $\mathbb{L}^Y = LR_B(Y)$ for a leras topology on Y.

Definition 2.4 ([10])**:** *The* leras closure *and* leras interior *of a set A are denoted $l_C(A)$ and $l_I(A)$, respectively, and defined by*

1. $l_C(A) = \bigcap\{N : A \subseteq N,\ N \text{ is } lr\text{-closed}\}$,
2. $l_I(A) = \bigcup\{N : N \subseteq A,\ N \text{ is } lr\text{-open}\}$.

Definition 2.5: *[9] A subset A of a leras topological space $\mathbb{L}^X$ is said to be a*

1. leras generalized closed set *(lrg-closed set) if $l_C(A) \subseteq M$ whenever $A \subseteq M$ and M is lr-open;*
2. leras generalized open set *(lrg-open set) if its complement $X - A$ is lrg-closed in X.*

A leras topological space $\mathbb{L}^X$ together with an ideal I on X is called an *ideal leras topological space*, denoted $(X, I, \mathbb{L}^X)$ or $\mathbb{L}_I^X$.

Definition 2.6: *[8] Let $\mathbb{L}_I^X$ be an ideal leras topological space. For a subset $K_l \subset X$,*

$$K_l^\star(I, \mathbb{L}^X) = \{x \in X : K_l \cap H \notin I \text{ for every } H \in \mathbb{L}^X(x)\},$$

where $\mathbb{L}^X(x) = \{H \in \mathbb{L}^X : x \in H\}$, is called the leras local function *(l-local function) of K_l with respect to I and $\mathbb{L}^X$.*

Theorem 2.7: *[8] Let $(X, \mathbb{L}^X)$ be a leras topological space with ideals I, I' on X, and let $H, M \subseteq X$. Then:*

1. $H \subseteq M \Rightarrow H_l^\star \subseteq M_l^\star$.
2. $I \subseteq I' \Rightarrow H_l^\star(I') \subseteq H_l^\star(I)$.
3. $H_l^\star = l_C(H_l^\star) \subseteq l_C(H)$.
4. $(H_l^\star)_l^\star \subseteq H_l^\star$.
5. $H_l^\star \cup M_l^\star = (H \cup M)_l^\star$.
6. $H_l^\star - M_l^\star = (H - M)_l^\star - M_l^\star \subseteq (H - M)_l^\star$.
7. *If $S \in \mathbb{L}^X$, then $S \cap H_l^\star = S \cap (S \cap H)_l^\star \subseteq (S \cap H)_l^\star$.*
8. *If $R \in I$, then $(H \cup R)_l^\star = H_l^\star = (H - R)_l^\star$.*

Theorem 2.8: *[8] If $\mathbb{L}_I^X$ is an ideal leras topological space and $H \subseteq H_l^\star$, then $H_l^\star = l_C(H_l^\star) = l_C(H)$.*

Definition 2.9: *[8] The set operator $l_C^\star$, called the* leras $\star$-closure*, is defined by*

$$l_C^\star(H) = H \cup H_l^\star.$$

Theorem 2.10: *[8] The operator $l_C^\star$ satisfies:*

1. $H \subseteq l_C^\star(H)$.
2. $l_C^\star(\emptyset) = \emptyset, \quad l_C^\star(X) = X$.
3. $H \subseteq M \Rightarrow l_C^\star(H) \subseteq l_C^\star(M)$.
4. $l_C^\star(H) \cup l_C^\star(M) = l_C^\star(H \cup M)$.
5. $l_C^\star(l_C^\star(H)) = l_C^\star(H)$.

Definition 2.11: *[8] A subset H of an ideal leras topological space $\mathbb{L}_I^X$ is called*

1. $l^\star$-closed *if* $H_l^\star \subseteq H$,
2. $l^\star$-dense in itself *if* $H \subseteq H_l^\star$,
3. $l^\star$-perfect *if* $H = H_l^\star$,
4. $\mathbb{L}^X$-codense *if* $\mathbb{L}^X \cap I = \{\emptyset\}$.

Lemma 2.12: *[8] Let $\mathbb{L}_I^X$ be an ideal leras topological space and $H \subseteq X$. If H is $l^\star$-dense in itself, then*

$$H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H).$$

3 On I_{lrg}-Closed Sets and I_{lrg}-Open Sets

Definition 3.1: *Let K be a subset of an ideal leras topological space $\mathbb{L}_I^X$. Then:*

1. *K is said to be* ideal leras-generalized closed *(briefly, I_{lrg}-closed) if $K_l^\star \subseteq S$ whenever $K \subseteq S$ and S is lr-open.*
2. *K is said to be* ideal leras-generalized open *(briefly, I_{lrg}-open) if $X - K$ is I_{lrg}-closed.*

Example 3.2: *Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $B = \{a, b\} \notin \tau$.*

Simple extension. *The simple extension of τ by B is*

$$\tau(B) = \Big\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\Big\}.$$

Thus $(X, \tau(B))$ is a SETS. The B-closed sets are $\Big\{X, \{b, c\}, \{a, c\}, \{c\}, \{a\}, \emptyset\Big\}$.

Leras topology. *Set $\widetilde{B} = X - B = \{c\}$, so $B \cup \widetilde{B} = X$. Computing all sets of the form $(O \cap B) \cup (O' \cap \widetilde{B})$ for $O, O' \in \tau$ yields*

$$\mathbb{L}^X = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\Big\},$$

which is the discrete topology on X. Every subset of X is therefore both lr-open and lr-closed, and $(\mathbb{L}^X)^c = \mathbb{L}^X$.

Ideal. *Let $I = \{\emptyset, \{c\}\}$. One verifies the hereditary and finite-union properties, so $(X, \tau(B), LR_B(X), I)$ is an ideal leras topological space $\mathbb{L}_I^X$.*

Leras local function. *Since every singleton is lr-open and $I = \{\emptyset, \{c\}\}$:*

$$\{a\}_l^\star = \{a\}, \quad \{b\}_l^\star = \{b\}, \quad \{c\}_l^\star = \emptyset, \quad \{a, b\}_l^\star = \{a, b\}, \quad \{a, c\}_l^\star = \{a\}, \quad \{b, c\}_l^\star = \{b\}.$$

I_{lrg}-closed sets. *A set K is I_{lrg}-closed if $K_l^\star \subseteq S$ whenever $K \subseteq S$ and S is lr-open.*

- *$K = \{c\}$: $K_l^\star = \emptyset \subseteq S$ for every lr-open S containing $\{c\}$. Hence $\{c\}$ is I_{lrg}-closed. Since $l_C(\{c\}) = \{c\} \subseteq \{c\}$ and $\{c\}$ is lr-open, it is also lrg-closed.*
- *$K = \{a, c\}$: $K_l^\star = \{a\}$. The lr-open sets containing $\{a, c\}$ are $\{a, c\}$ and X. We have $\{a\} \subseteq \{a, c\}$ and $\{a\} \subseteq X$, so $\{a, c\}$ is I_{lrg}-closed. Since $l_C(\{a, c\}) = \{a, c\}$, it is lrg-closed too.*
- *$K = \{b, c\}$: $K_l^\star = \{b\} \subseteq S$ for every lr-open $S \supseteq \{b, c\}$. Hence $\{b, c\}$ is I_{lrg}-closed.*

Theorem 3.3: *In an ideal leras topological space $\mathbb{L}_I^X$, every lrg-closed set is I_{lrg}-closed.*

Proof: Let K be an lrg-closed set in $\mathbb{L}_I^X$. By Definition 2.5(1),

$$l_C(K) \subseteq S \quad \text{whenever } K \subseteq S \text{ and } S \text{ is } lr\text{-open}.$$

By Theorem 2.7(3), $K_l^\star \subseteq l_C(K_l^\star) \subseteq l_C(K)$. Let S be any lr-open set with $K \subseteq S$. Then $l_C(K) \subseteq S$, and therefore

$$K_l^\star \subseteq l_C(K) \subseteq S.$$

Since S was arbitrary, K is I_{lrg}-closed by Definition 3.1(1). □

Remark 3.4: *The converse of Theorem 3.3 need not hold in general; an I_{lrg}-closed set need not be lrg-closed, as illustrated in Example 3.5 below.*

Example 3.5: *Let $X = \{a, b, c\}$ with leras topology $\mathbb{L}^X = \{\emptyset, \{a, b\}, X\}$ and ideal $I = \{\emptyset, \{a\}\}$.*
Basic data. *The lr-closed sets are $\{\emptyset, \{c\}, X\}$. One checks that I satisfies both ideal axioms.*
Leras local functions. *Using Definition 2.6 with $I = \{\emptyset, \{a\}\}$:*

$$\{a\}_l^\star = \emptyset, \qquad \{b\}_l^\star = \{b, c\}, \qquad \{c\}_l^\star = \{c\},$$
$$\{a, b\}_l^\star = \{b, c\}, \qquad \{a, c\}_l^\star = \{c\}, \qquad \{b, c\}_l^\star = \{b, c\}.$$

We verify the representative cases in detail.

Case $K = \{a\}$. *For any $x \in X$ and any lr-open set $H \ni x$, the intersection $H \cap \{a\}$ belongs to $\{\emptyset, \{a\}\} = I$. Hence no point satisfies the local-function condition, and $\{a\}_l^\star = \emptyset$.*

Case $K = \{b\}$. *The lr-open sets are $\emptyset$, $\{a, b\}$, and X. For $x = a$: the lr-open sets containing a are $\{a, b\}$ and X; both meet $\{b\}$ in $\{b\} \notin I$, so $a \in \{b\}_l^\star$. For $x = b$: $\{a, b\} \cap \{b\} = \{b\} \notin I$, so $b \in \{b\}_l^\star$. For $x = c$: the only lr-open set containing c is X, and $X \cap \{b\} = \{b\} \notin I$, so $c \in \{b\}_l^\star$. Hence $\{b\}_l^\star = X$.*

The complete table of leras local functions is:

$$\{a\}_l^\star = \emptyset, \qquad \{b\}_l^\star = X, \qquad \{c\}_l^\star = \{c\},$$
$$\{a, b\}_l^\star = X, \qquad \{a, c\}_l^\star = \{c\}, \qquad \{b, c\}_l^\star = X.$$

I_{lrg}-closedness versus lrg-closedness. *Consider $K = \{a\}$. Since $\{a\}_l^\star = \emptyset$, the condition $K_l^\star \subseteq S$ holds trivially for every lr-open set S containing K. Hence $\{a\}$ is I_{lrg}-closed.*

However, the leras closure of $\{a\}$ is

$$l_C(\{a\}) = \bigcap \Big\{ F \,:\, F \text{ is } lr\text{-closed},\ \{a\} \subseteq F \Big\} = X,$$

since the only lr-closed sets are $\emptyset$, $\{c\}$, and X, and $\{a\}$ is not contained in $\{c\}$. The lr-open set $S = \{a, b\}$ satisfies $\{a\} \subseteq S$, but

$$l_C(\{a\}) = X \nsubseteq \{a, b\} = S.$$

Hence $\{a\}$ is not *lrg-closed.*

Therefore $\{a\}$ is I_{lrg}-closed but not lrg-closed, showing that the converse of Theorem 3.3 fails in general.

Theorem 3.6: *Let $\mathbb{L}_I^X$ be an ideal leras topological space and $K \subseteq X$. Then K is I_{lrg}-closed if and only if $l_C^\star(K) \subseteq S$ whenever $K \subseteq S$ and S is lr-open in X.*

Proof:

($\Rightarrow$) Suppose K is I_{lrg}-closed and let S be lr-open with $K \subseteq S$. By Definition 3.1(1), $K_l^\star \subseteq S$. Since also $K \subseteq S$, Definition 2.9 gives

$$l_C^\star(K) = K \cup K_l^\star \subseteq S.$$

($\Leftarrow$) Suppose $l_C^\star(K) \subseteq S$ whenever $K \subseteq S$ and S is lr-open. For any such S,

$$K_l^\star \subseteq K \cup K_l^\star = l_C^\star(K) \subseteq S.$$

Hence K is I_{lrg}-closed by Definition 3.1(1). □

Theorem 3.7: *Let $\mathbb{L}_I^X$ be an ideal leras topological space and $K \subseteq X$. The following are equivalent:*

1. *K is I_{lrg}-closed.*
2. *$l_C^\star(K) \subseteq S$ whenever $K \subseteq S$ and S is lr-open in X.*
3. *For every $x \in l_C^\star(K)$, $l_C(\{x\}) \cap K \neq \emptyset$.*
4. *$l_C^\star(K) - K$ contains no nonempty lr-closed set.*
5. *$K_l^\star - K$ contains no nonempty lr-closed set.*

Proof:

We prove the implications in the cyclic order (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) ⇒ (2). Suppose K is I_{lrg}-closed and let S be lr-open with $K \subseteq S$. Then $K_l^\star \subseteq S$ by Definition 3.1(1), and since $K \subseteq S$,

$$l_C^\star(K) = K \cup K_l^\star \subseteq S.$$

(2) ⇒ (3). Suppose (2) holds. Assume for contradiction that there exists $x \in l_C^\star(K)$ with $l_C(\{x\}) \cap K = \emptyset$, i.e., $K \subseteq X - l_C(\{x\})$. Since $X - l_C(\{x\})$ is lr-open, condition (2) gives $l_C^\star(K) \subseteq X - l_C(\{x\})$, so $x \notin l_C^\star(K)$, a contradiction.

(3) ⇒ (4). Suppose (3) holds. Let F be an lr-closed set with $F \subseteq l_C^\star(K) - K$. For any $x \in F$, we have $x \in l_C^\star(K)$, so by (3), $l_C(\{x\}) \cap K \neq \emptyset$. But $x \in F \subseteq X - K$, and since F is lr-closed, $l_C(\{x\}) \subseteq l_C(F) = F \subseteq X - K$, giving $l_C(\{x\}) \cap K = \emptyset$, a contradiction unless $F = \emptyset$.

(4) ⇒ (5). By Definition 2.9, $l_C^\star(K) = K \cup K_l^\star$, so

$$K_l^\star - K \subseteq (K \cup K_l^\star) - K = l_C^\star(K) - K.$$

Any lr-closed set contained in $K_l^\star - K$ is also contained in $l_C^\star(K) - K$, which by (4) is empty of lr-closed sets.

(5) ⇒ (1). Suppose (5) holds. We first justify that $K_l^\star$ is lr-closed. By Theorem 2.7(3), $K_l^\star = l_C(K_l^\star) \subseteq l_C(K)$. In particular, $K_l^\star = l_C(K_l^\star)$, and since $l_C(K_l^\star)$ is by definition the intersection of all lr-closed sets containing $K_l^\star$ (hence itself lr-closed, being such an intersection), it follows that $K_l^\star$ coincides with an lr-closed set. Thus $K_l^\star$ is lr-closed.

Now let S be lr-open with $K \subseteq S$. Since $K \subseteq S$, $K_l^\star - S \subseteq K_l^\star - K$. Because $K_l^\star$ is lr-closed (as shown above) and S is lr-open, $X - S$ is lr-closed, so $K_l^\star - S = K_l^\star \cap (X - S)$ is an intersection of two lr-closed sets, and is therefore itself lr-closed. This lr-closed set is contained in $K_l^\star - K$, which by hypothesis (5) contains no nonempty lr-closed set. Hence $K_l^\star - S = \emptyset$, i.e., $K_l^\star \subseteq S$. Since S was an arbitrary lr-open set containing K, this shows $K_l^\star \subseteq S$ whenever $K \subseteq S$ and S is lr-open. Hence K is I_{lrg}-closed by Definition 3.1(1). □

Corollary 3.8: *Let $\mathbb{L}_I^X$ be an ideal leras topological space and let $K \subseteq X$ be an I_{lrg}-closed set. Then the following are equivalent:*

(1) *K is $l^\star$-closed.*
(2) *$l_C^\star(K) - K$ is lr-closed.*
(3) *$K_l^\star - K$ is lr-closed.*

Proof:

(1) ⇒ (2). If K is $l^\star$-closed, then $K_l^\star \subseteq K$ by Definition 2.11(1), so $l_C^\star(K) = K$, giving $l_C^\star(K) - K = \emptyset$, which is lr-closed.

(2) ⇒ (3). Suppose $l^\star_C(K) - K$ is lr-closed. Since $K^\star_l - K \subseteq l^\star_C(K) - K$ and K is I_{lrg}-closed, Theorem 3.7(5) says $K^\star_l - K$ contains no nonempty lr-closed set. But $K^\star_l - K$ is itself lr-closed (being a subset of the lr-closed set $l^\star_C(K) - K$), so $K^\star_l - K = \emptyset$, which is lr-closed.

(3) ⇒ (1). Suppose $K^\star_l - K$ is lr-closed. Since K is I_{lrg}-closed, Theorem 3.7(5) gives that $K^\star_l - K$ contains no nonempty lr-closed set. Since $K^\star_l - K$ is itself lr-closed, we must have $K^\star_l - K = \emptyset$, i.e., $K^\star_l \subseteq K$. Hence K is $l^\star$-closed by Definition 2.11(1). □

Theorem 3.9: *Let $\mathbb{L}^X_I$ be an ideal leras topological space. If a subset $K \subseteq X$ is both $l^\star$-dense in itself and I_{lrg}-closed, then K is lrg-closed.*

Proof:

Since K is $l^\star$-dense in itself, $K \subseteq K^\star_l$, and by Lemma 2.12:

$$K^\star_l = l_C(K^\star_l) = l_C(K) = l^\star_C(K).$$

Let S be any lr-open set with $K \subseteq S$. Since K is I_{lrg}-closed, $K^\star_l \subseteq S$, and therefore

$$l_C(K) = K^\star_l \subseteq S.$$

Hence K is lrg-closed by Definition 2.5(1). □

Corollary 3.10: *Let $\mathbb{L}^X_I$ be an ideal leras topological space with $I = \{\emptyset\}$. Then a subset $K \subseteq X$ is I_{lrg}-closed if and only if it is lrg-closed.*

Proof:

By Theorem 3.3, every lrg-closed set is I_{lrg}-closed, regardless of the choice of ideal. It remains to prove the converse when $I = \{\emptyset\}$.

Step 1: $K^\star_l = l_C(K)$ when $I = \{\emptyset\}$. Recall from Definition 2.6 that, for $K \subseteq X$,

$$K^\star_l = \{x \in X \,:\, K \cap H \notin I \text{ for every } H \in \mathbb{L}^X(x)\},$$

where $\mathbb{L}^X(x)$ denotes the collection of all lr-open sets containing x. When $I = \{\emptyset\}$, the condition $K \cap H \notin I$ is equivalent to $K \cap H \neq \emptyset$. Hence

$$K^\star_l = \{x \in X \,:\, K \cap H \neq \emptyset \text{ for every } H \in \mathbb{L}^X(x)\}.$$

This is precisely the set of points x every lr-open neighbourhood of which meets K nontrivially – that is, $K^\star_l$ is exactly the leras closure $l_C(K)$ of K with respect to the leras topology $\mathbb{L}^X$. Indeed:

- If $x \in K^\star_l$, then every lr-open $H \ni x$ satisfies $H \cap K \neq \emptyset$, so x cannot lie in the complement of any lr-closed set containing K (since such a complement is lr-open and disjoint from K). Hence x belongs to every lr-closed set containing K, i.e., $x \in l_C(K)$.

- Conversely, if $x \in l_C(K)$, then x belongs to every lr-closed set containing K. If some lr-open $H \in \mathbb{L}^X(x)$ satisfied $H \cap K = \emptyset$, then $X - H$ would be an lr-closed set containing K but not containing x, contradicting $x \in l_C(K)$. Hence every lr-open $H \ni x$ meets K, so $x \in K_l^\star$.

Therefore $K_l^\star = l_C(K)$ whenever $I = \{\emptyset\}$; that is, under the trivial ideal the leras local function of K collapses exactly to the ordinary leras closure of K, and no proper subset of $l_C(K)$ can arise as $K_l^\star$.

Step 2: I_{lrg}-closed ⇒ lrg-closed.

Suppose K is I_{lrg}-closed and $I = \{\emptyset\}$. Let S be any lr-open set with $K \subseteq S$. By Definition 3.1(1), $K_l^\star \subseteq S$. By Step 1, $K_l^\star = l_C(K)$, so this becomes $l_C(K) \subseteq S$.

Since S was an arbitrary lr-open set containing K, this shows that $l_C(K) \subseteq S$ whenever $K \subseteq S$ and S is lr-open. By Definition 2.5(1), K is lrg-closed.

Combining Steps 1 and 2 with Theorem 3.3, we conclude that, when $I = \{\emptyset\}$, K is I_{lrg}-closed if and only if K is lrg-closed. □

Theorem 3.11: *Let $\mathbb{L}_I^X$ be an ideal leras topological space. The following statements are equivalent:*

1. *Every subset of X is I_{lrg}-closed.*
2. *Every lr-open set is $l^\star$-closed.*

Proof:

(1) ⇒ (2). Let S be any lr-open set. By hypothesis, S is I_{lrg}-closed. Since $S \subseteq S$ and S is lr-open, Definition 3.1(1) gives $S_l^\star \subseteq S$, so S is $l^\star$-closed.

(2) ⇒ (1). Let $K \subseteq X$ and let S be lr-open with $K \subseteq S$. By hypothesis, S is $l^\star$-closed, so $S_l^\star \subseteq S$. By Theorem 2.7(1), $K_l^\star \subseteq S_l^\star \subseteq S$. Hence K is I_{lrg}-closed. □

Theorem 3.12: *In an ideal leras topological space $\mathbb{L}_I^X$, every $l^\star$-closed set is I_{lrg}-closed.*

Proof:

Let K be $l^\star$-closed, so $K_l^\star \subseteq K$ by Definition 2.11(1). For any lr-open S with $K \subseteq S$, $K_l^\star \subseteq K \subseteq S$. Hence K is I_{lrg}-closed by Definition 3.1(1). □

Example 3.13: *In the space of Example 3.5 ($\mathbb{L}^X = \{\emptyset, \{a,b\}, X\}$, $I = \{\emptyset, \{a\}\}$), take $K = \{b,c\}$. We have $\{b,c\}_l^\star = X \nsubseteq \{b,c\}$, so $\{b,c\}$ is* not *$l^\star$-closed. However, the only lr-open superset of $\{b,c\}$ is X, and $X \subseteq X$, so $\{b,c\}$* is *I_{lrg}-closed. Hence the converse of the preceding theorem fails in general.*

Remark 3.14: *The relationships among the classes of sets studied in this paper are summarised in the following diagram, where a single arrow denotes implication and the absence of a reverse arrow indicates that the converse fails in general.*

$$\begin{array}{ccccc} l^\star\text{-dense in itself} & \Leftarrow & l^\star\text{-perfect} & \Rightarrow & l^\star\text{-closed} \\ & & & & \Downarrow \\ lr\text{-closed} & \Rightarrow & lrg\text{-closed} & \Rightarrow & I_{lrg}\text{-closed} \end{array}$$

None of the above implications reverses in general.

Example 3.15: *We collect here the key non-implications.*

1. I_{lrg}***-closed*** $\nRightarrow$ lrg***-closed.*** *In Example 3.5, the set* $\{a\}$ *satisfies* $\{a\}_l^\star = \emptyset \subseteq S$ *for every* lr*-open* S *containing* $\{a\}$*, hence it is* I_{lrg}*-closed. However,* $l_C(\{a\}) = X \nsubseteq \{a, b\}$*, so* $\{a\}$ is not lrg*-closed.*
2. I_{lrg}***-closed*** $\nRightarrow$ $l^\star$***-closed.*** *In Example 3.5, the set* $K = \{b, c\}$ *is* I_{lrg}*-closed (the only* lr*-open superset is* X*, and* $\{b, c\}_l^\star = X \subseteq X$*), yet* $\{b, c\}_l^\star = X \nsubseteq \{b, c\}$*, so* $\{b, c\}$ is not $l^\star$*-closed.*
3. lrg***-closed*** $\nRightarrow$ lr***-closed.***
   Let $X = \{a, b, c\}$*, with leras topology* $\mathbb{L}^X = \{\emptyset, \{a\}, X\}$*. The* lr*-closed sets are* $\{X, \{b, c\}, \emptyset\}$
   Consider the set $K = \{b\}$*. The only* lr*-open set containing* K *is* X*. Also,* $l_C(K) = \{b, c\} \subseteq X$*. Hence* K *is* lrg*-closed.*
   However, $K = \{b\} \notin \{X, \{b, c\}, \emptyset\}$*, so* K *is not* lr*-closed.*
   Therefore, lrg*-closed* $\nRightarrow$ lr*-closed.*
4. $l^\star$***-dense in itself*** $\nRightarrow$ $l^\star$***-perfect.***
   Let $X = \{a, b, c\}$*, with leras topology* $\mathbb{L}^X = \{\emptyset, \{a\}, X\}$*, and ideal* $I = \{\emptyset, \{b\}\}$*. Take* $H = \{a\}$*.*
   Compute $H_l^\star$*. For* $x = a$*, every* lr*-open set containing* a *is either* $\{a\}$ *or* X*, and each intersects* H *in* $\{a\} \notin I$*. Hence* $a \in H_l^\star$*.*
   For $x = b$*, the only* lr*-open set containing* b *is* X*, and* $X \cap H = \{a\} \notin I$*. Thus* $b \in H_l^\star$*.*
   Similarly, for $x = c$*,* $X \cap H = \{a\} \notin I$*, so* $c \in H_l^\star$*.*
   Therefore, $H_l^\star = X$*. Since* $H = \{a\} \subseteq X = H_l^\star$*, the set* H *is* $l^\star$*-dense in itself. But* $H \neq H_l^\star$*, so* H *is not* $l^\star$*-perfect.*
   Hence $l^\star$*-dense in itself* $\nRightarrow$ $l^\star$*-perfect.*

Theorem 3.16: *Let* $\mathbb{L}_I^X$ *be an ideal leras topological space and* $K \subseteq X$*. Whenever* K *is both* I_{lrg}*-closed and* lr*-open, it follows that* K *is* $l^\star$*-closed.*

Proof:

Since $K \subseteq K$ and K is lr-open, Definition 3.1(1) gives $K_l^\star \subseteq K$. Hence K is $l^\star$-closed by Definition 2.11(1). □

Theorem 3.17: *For any two* I_{lrg}*-closed sets* K *and* S *in* $\mathbb{L}_I^X$*, their union* $K \cup S$ *is also* I_{lrg}*-closed.*

Proof:

Let U be lr-open with $K \cup S \subseteq U$. Then $K \subseteq U$ and $S \subseteq U$. Since K and S are I_{lrg}-closed, $K_l^\star \subseteq U$ and $S_l^\star \subseteq U$. By Theorem 2.7(5),

$$(K \cup S)_l^\star = K_l^\star \cup S_l^\star \subseteq U.$$

Hence $K \cup S$ is I_{lrg}-closed. □

Remark 3.18: *The class of I_{lrg}-closed sets is not generally closed under finite intersections, as shown in the following example.*

Example 3.19: *Let $X = \{a, b, c\}$ with leras topology $\mathbb{L}^X = \{\emptyset, \{c\}, X\}$ and ideal $I = \{\emptyset, \{a\}\}$.*

The lr-closed sets are $\{\emptyset, \{a, b\}, X\}$.

Leras local functions. *Since $I = \{\emptyset, \{a\}\}$, the condition $H \cap K \notin I$ requires $H \cap K \neq \emptyset$ and $H \cap K \neq \{a\}$.*

For $K = \{c\}$: the only lr-open sets containing a or b are X, and $X \cap \{c\} = \{c\} \notin I$, so $a, b \in \{c\}^\star_l$. Clearly $c \in \{c\}^\star_l$. Thus $\{c\}^\star_l = X$.

Similarly, $\{a, c\}^\star_l = X$ and $\{b, c\}^\star_l = X$ (since $\{c\} \subseteq \{a, c\}$ forces every lr-open neighbourhood to intersect $\{a, c\}$ outside I).

*I_{lrg}**-closedness of** $K_1 = \{a, c\}$ **and** $K_2 = \{b, c\}$**.** The lr-open sets containing $\{a, c\}$ are: $\{c\}$ does not contain a, so the only lr-open superset of $\{a, c\}$ is X. Then $\{a, c\}^\star_l = X \subseteq X$, so $\{a, c\}$ is I_{lrg}-closed. By an identical argument, $\{b, c\}$ is I_{lrg}-closed.*

***Intersection** $K_1 \cap K_2 = \{c\}$ **is not** I_{lrg}**-closed.** We have $\{c\}^\star_l = X$. The lr-open set $S = \{c\}$ satisfies $\{c\} \subseteq S$, but*

$$\{c\}^\star_l = X \nsubseteq \{c\} = S.$$

Hence $\{c\}$ is not *I_{lrg}-closed.*

This shows that the intersection of two I_{lrg}-closed sets need not be I_{lrg}-closed.

Theorem 3.20: *Let K be an I_{lrg}-closed subset of $\mathbb{L}^X_I$. If $K \subseteq S \subseteq K^\star_l$, then S is also I_{lrg}-closed.*

Proof:

Let U be lr-open with $S \subseteq U$. Since $K \subseteq S \subseteq U$ and K is I_{lrg}-closed, $K^\star_l \subseteq U$. Now $S \subseteq K^\star_l$, so by Theorem 2.7(1) and Theorem 2.7(4),

$$S^\star_l \subseteq (K^\star_l)^\star_l \subseteq K^\star_l \subseteq U.$$

Hence S is I_{lrg}-closed by Definition 3.1(1). □

Definition 3.21: *Let $\mathbb{L}^X_I$ be an ideal leras topological space and $K \subseteq X$. The $l^\star$-*interior *of K, denoted $l^\star_I(K)$, is defined by*

$$l^\star_I(K) = X - l^\star_C(X - K) = X - \big((X - K) \cup (X - K)^\star_l\big).$$

Theorem 3.22: *Let $\mathbb{L}^X_I$ be an ideal leras topological space and $K \subseteq X$. Then K is I_{lrg}-open if and only if every lr-closed subset D of K satisfies $D \subseteq l^\star_I(K)$.*

Proof:

($\Rightarrow$) Suppose K is I_{lrg}-open, so $X-K$ is I_{lrg}-closed. Let D be lr-closed with $D \subseteq K$. Then $X-K \subseteq X-D$, and since $X-D$ is lr-open and $X-K$ is I_{lrg}-closed, Definition 3.1(1) gives $(X-K)^\star_l \subseteq X-D$. Taking complements,

$$D \subseteq X-(X-K)^\star_l = l^\star_I(K).$$

($\Leftarrow$) Suppose every lr-closed $D \subseteq K$ satisfies $D \subseteq l^\star_I(K) = X-(X-K)^\star_l$. Let U be lr-open with $X-K \subseteq U$, so $X-U \subseteq K$. Since $X-U$ is lr-closed and contained in K, by hypothesis

$$X-U \subseteq X-(X-K)^\star_l,$$

i.e., $(X-K)^\star_l \subseteq U$. Hence $X-K$ is I_{lrg}-closed, and K is I_{lrg}-open. □

Theorem 3.23: *Let $\mathbb{L}^X_I$ be an ideal leras topological space and $K \subseteq X$. The following conditions are equivalent:*

1. *K is I_{lrg}-closed.*
2. *$K \cup (X-K^\star_l)$ is I_{lrg}-closed.*
3. *$K^\star_l - K$ is I_{lrg}-open.*

Proof:

(1) ⇒ (2). Suppose K is I_{lrg}-closed. By Theorem 3.7(5), $K^\star_l - K$ contains no nonempty lr-closed set. Let U be lr-open with $K \cup (X-K^\star_l) \subseteq U$. Then

$$X-U \subseteq X-\left(K \cup (X-K^\star_l)\right) = K^\star_l - K.$$

Since $X-U$ is lr-closed and $K^\star_l - K$ contains no nonempty lr-closed set, $X-U=\emptyset$, i.e., $U=X$. Hence $(K \cup (X-K^\star_l))^\star_l \subseteq X$ trivially, so $K \cup (X-K^\star_l)$ is I_{lrg}-closed.

(2) ⇒ (3). Note that $X-(K^\star_l - K) = K \cup (X-K^\star_l)$, which is I_{lrg}-closed by hypothesis. Hence $K^\star_l - K$ is I_{lrg}-open by Definition 3.1(2).

(3) ⇒ (1). Suppose $K^\star_l - K$ is I_{lrg}-open, so $X-(K^\star_l - K) = K \cup (X-K^\star_l)$ is I_{lrg}-closed. Let F be lr-closed with $F \subseteq K^\star_l - K$. Then $K \cup (X-K^\star_l) \subseteq X-F$, and since $X-F$ is lr-open and $K \cup (X-K^\star_l)$ is I_{lrg}-closed,

$$\left(K \cup (X-K^\star_l)\right)^\star_l \subseteq X-F.$$

In particular, no nonempty lr-closed set is contained in $K^\star_l - K$, so $F=\emptyset$. By Theorem 3.7(5), K is I_{lrg}-closed. □

4 Discussion

The results obtained in this paper indicate that the class of I_{lrg}-closed sets provides a genuinely finer tool for describing closedness in ideal leras topological spaces than either lr-closed or lrg-closed sets alone, since it accounts for the additional flexibility furnished by the ideal I. This suggests a few directions in which the present framework could be developed or applied further.

From a theoretical standpoint, the characterizations established in Theorem 3.7 – particularly the reformulation in terms of the absence of nonempty lr-closed sets in $K_l^\star - K$ – parallel analogous characterizations of generalized closed sets in classical ideal topological spaces. This parallel suggests that other ideal-topological notions, such as $\star$-normality, $\star$-compactness, or codensity-based decomposition theorems, may admit natural analogues in the ideal leras setting, using the leras local function and leras $\star$-closure operator introduced in [8] in place of their classical counterparts.

From an applied standpoint, since leras topological spaces are built from the simple extension of a topology by a fixed set B, they offer a flexible combinatorial mechanism for modelling situations in which a base structure (τ) is refined by a distinguished, non-open reference set (B). Equipping such spaces with an ideal, as done here, further allows one to designate certain subsets as "negligible" with respect to that refinement. This dual flexibility may be of interest in contexts such as rough set theory and information systems, where approximation spaces are frequently built from a base relation together with a notion of negligible or boundary elements; the I_{lrg}-closed sets introduced here could potentially serve as a finer approximation tool in such settings, though this connection would need to be made precise in future work.

Finally, the examples constructed in this paper, all on finite sets, suggest that a systematic computational study of lr-open sets and leras local functions on finite spaces – for instance, via enumeration or software-assisted search – could facilitate the discovery of further examples and counterexamples, and thereby support a more complete classification of the relationships among the various classes of sets in ideal leras topological spaces.

5 Conclusions

In this paper, we introduced and investigated the notions of I_{lrg}-closed sets and I_{lrg}-open sets in ideal leras topological spaces. Several characterizations and fundamental properties of these sets were established, and their relationships with lrg-closed, lr-closed, and $l^\star$-closed sets were studied. Various examples were provided to show that the converse implications do not hold in general. The obtained results extend existing concepts in leras topology and provide a basis for further study of generalized closed sets in ideal leras topological spaces.

As future work, the classes of I_{lrg}-closed and I_{lrg}-open sets introduced here may be used to define and study new forms of generalized continuity (such as I_{lrg}-continuous and I_{lrg}-irresolute mappings), separation axioms, and compactness/connectedness notions in ideal leras topological spaces. Further generalized classes, such as $I_{lrg}^\star$-closed or I_{lrg}-semi-closed sets, along with related decomposition theorems, may also be explored in this setting.

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