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Boolean-Centred Negation and Involutive Envelopes in B-Algebras

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Abstract: This paper studies B-algebras through the Boolean-centre-valued operators $x^{\nabla} = 1 \Rightarrow x$ and $x^{\Delta} = 0 \Leftarrow x$. The operator $\cdot^{\Delta}$ defines a centre-valued negation satisfying $(x \vee y)^{\Delta} = x^{\Delta} \wedge y^{\Delta}$ for all $x, y \in A$, together with the inequality $x^{\Delta} \vee y^{\Delta} \leq (x \wedge y)^{\Delta}$. Equality in the latter holds on the Boolean centre $B(A)$; an explicit example confirms that restricting to $B(A)$ is necessary. The Boolean centre, equipped with the restricted operations, satisfies the equations of one standard axiomatisation of Nelson algebras [14]. An involutive distributive lattice $\mathcal{N}(A) = A \times A$ is constructed together with an injective map $\iota(x) = (x, x^{\Delta})$. The embedding preserves joins globally and preserves meets on $B(A)$. Prime-spectrum and prime-filter representation results are obtained for B-algebras. Under an injectivity condition on the pseudo-supplement map, an ultrafilter representation follows. The bi-implication $x \leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$ characterises equality and satisfies a transitivity inequality.

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Keywords: B-algebras; centre-valued negation; distributive involutive lattice; bi-implication; prime-filter representation; prime-spectrum representation.

1 Introduction

Algebraic structures arising from logic furnish a medium for non-classical reasoning. Lattice-theoretic methods and Boolean algebras, as developed in the classical work of Birkhoff [1], have shaped this programme alongside Heyting algebras [9,14], which provide algebraic semantics for intuitionistic logic. Nelson algebras [12,13,15,16] enrich this setting with a strong negation sharing formal features with constructive falsity.

Among distributive structures with Boolean-valued implication, B-algebras—introduced by Epstein and Horn [5] as abstractions of Post algebras [2]—occupy a natural intermediary position. A B-algebra is a bounded distributive lattice whose implication takes values in tAlgebraic structures arising from logic furnish a medium for non-classical reasoning. Lattice-theoretic methods and Boolean algebras, as developed in the classical work of Birkhoff [1], have shaped this programme alongside Heyting algebras [9,14], which provide algebraic semantics for intuitionistic logic. Nelson algebras [12,13,15,16] enrich this setting with a strong negation sharing formal features with constructive falsity.

Among distributive structures with Boolean-valued implication, B-algebras—introduced by Epstein and Horn [5] as abstractions of Post algebras [2]—occupy a natural intermediary position. A B-algebra is a bounded distributive lattice whose implication takes values in the Boolean centre $B(A)$, enabling controlled interaction between classical and non-classical features [11]. Not every bounded distributive lattice admits such an implication; the Boolean-valued maximality condition of Definition 2 places a restriction on the underlying lattice. By maximality, $x \Rightarrow y$ is uniquely determined whenever it exists. The operators x^{∇} and x^{Δ} generate a Boolean-centred negation operator within this structure.

Remark 1: *The terms* pseudo-supplement *and* dual pseudo-supplement *are internal labels for the operators of Definition 3. They bear no relation to pseudo-complements, supplements, or dual pseudo-complements in the classical lattice-theoretic sense [1,7], and they do not generalise any of those notions.*

Three restrictions hold throughout and are not restated at individual results. The De Morgan equality $(x \wedge y)^{\Delta} = x^{\Delta} \vee y^{\Delta}$ holds only for $x, y \in B(A)$. The embedding ι preserves meets only on $B(A)$. The compatibility inequality for $\tilde{h}$ is one-sided. All three share the same source: Boolean complements are unavailable outside $B(A)$.

Motivation

Three existing frameworks come close to the present setting, and each leaves a gap that motivates the constructions below. Heyting algebras furnish a negation $\neg x = x \to 0$, but this negation is Boolean-valued only in special cases; a Heyting algebra therefore does not, by itself, isolate a distinguished classical fragment against which a controlled negation can be measured. Nelson algebras supply a strong negation with constructive-falsity content, yet the standard route to a Nelson algebra is the twist-product $L \times L$ of Vakarelov [16] and Sendlewski [15], which imposes explicit external compatibility conditions between the two coordinates. Residuated lattices generalise implication further, but their residuation need not land in any distinguished subalgebra, so no analogue of a Boolean centre is available for comparison. B-algebras occupy the gap between these three settings: the implication of Definition 2 is Boolean-valued by definition, so every B-algebra carries its own distinguished classical fragment $B(A)$ without extraneous hypotheses. This paper works out the negation theory that the Boolean-valued implication makes possible—the operators $\cdot^{\nabla}$ and $\cdot^{\Delta}$, the involutive envelope $\mathcal{N}(A)$, and the associated spectral and representation theorems—and in doing so builds the twist-product-free involutive construction $\mathcal{N}(A) = A \times A$ directly from B-algebra operations, with no compatibility conditions imposed from outside the algebra. The basic identities of Proposition 1 originate with Epstein and Horn [5], but the negation-theoretic, Nelson-theoretic, and representation-theoretic consequences of those identities have not previously been developed; this gap is what the present paper addresses.he Boolean centre $B(A)$, enabling controlled interaction between classical and non-classical features [11]. Not every bounded distributive lattice admits such an implication; the Boolean-valued maximality condition of Definition 2 places a restriction on the underlying lattice.

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Main Contributions

The operators $\cdot^{\nabla}$ and $\cdot^{\Delta}$ induce several structures naturally associated with a B-algebra. Specifically, $\cdot^{\Delta}$ acts as a Boolean-centred negation operator; the product $\mathcal{N}(A) = A \times A$ yields an involutive distributive lattice; and the prime spectrum carries a canonical order-reversing map determined by $\cdot^{\Delta}$. These constructions produce representation results and clarify the role of Boolean-centre elements in the associated negation theory.

(1) De Morgan-type identities for $\cdot^{\Delta}$ are established, including a global inequality and an equality valid on $B(A)$.
(2) The Boolean centre satisfies the identities of one standard equational presentation of Nelson algebras.
(3) An involutive distributive lattice $\mathcal{N}(A) = A \times A$ is constructed with an embedding $\iota : A \to \mathcal{N}(A)$ and a functor $\mathcal{N} : \mathbf{Balg} \to \mathbf{InvLat}$.
(4) A partial spectral representation framework based on prime filters is developed for B-algebras.
(5) Prime-filter and ultrafilter representation results are obtained under suitable hypotheses.

Section 4 analyses the De Morgan-type properties of $\cdot^{\Delta}$ and constructs the involutive lattice $\mathcal{N}(A) = A \times A$ with embedding ι. The Boolean centre $B(A)$ satisfies the Nelson identities of Rasiowa [14] with Boolean lattice reduct; the global algebra A does not become a Nelson algebra. Core elements (Definition 5) determine a Boolean substructure of A on which $\cdot^{\nabla}$ and $\cdot^{\Delta}$ coincide; the core embedding (Theorem 7) represents A inside a product of Boolean intervals.

Twist-product constructions [15,16] rely on external compatibility conditions absent within B-algebras; the present framework requires none. The implication $\Rightarrow$ is not assumed to satisfy full residuation outside $B(A)$; only the maximality property of Definition 2 is used. Morphisms of **InvLat** preserve the De Morgan involution, reflecting the intended Nelson-type reading of $\mathcal{N}(A)$.

2 Preliminaries

Standard references for lattice theory are Birkhoff [1] and Grätzer [7]; for Boolean algebras, Halmos [8] and Givant–Halmos [6]. The concept of B-algebras originates with Epstein and Horn [5].

Definition 1: *Let $A = (A, \vee, \wedge, 0, 1)$ be a bounded distributive lattice. The* Boolean centre *of A is*

$$B(A) = \{x \in A : \exists\, x' \in A,\ x \vee x' = 1 \ \text{ and }\ x \wedge x' = 0\}.$$

Since 0 and 1 are complemented in every bounded distributive lattice, $\{0, 1\} \subseteq B(A)$ always holds. Hence the maximality condition of Definition 2 below is meaningful.

Under the operations inherited from A, the set $B(A)$ forms a Boolean algebra, and the complement x' of each $x \in B(A)$ is unique.

Definition 2: *A B-*algebra *is a bounded distributive lattice A together with a binary operation $\Rightarrow : A \times A \to B(A)$ such that for all $x, y \in A$, the element $x \Rightarrow y$ is the greatest $b \in B(A)$ satisfying $x \wedge b \leq y$. The* dual implication *is $x \Leftarrow y := y \Rightarrow x$. A* homomorphism of B-algebras *is a bounded lattice homomorphism $f : A \to B$ satisfying $f(x \Rightarrow y) = f(x) \Rightarrow f(y)$ for all $x, y \in A$.*

The implication $x \Rightarrow y$ is uniquely determined for each pair (x, y): if $b_1, b_2 \in B(A)$ both satisfy $x \wedge b_i \leq y$, then

$$x \wedge (b_1 \wedge b_2) = (x \wedge b_1) \wedge b_2 \leq y,$$

so the set $\{b \in B(A) : x \wedge b \leq y\}$ is closed under finite meets and has a unique greatest element. Since $x \Rightarrow y \in B(A)$ for all $x, y \in A$, the implication is Boolean-valued by definition.

Remark 3: *When A carries a Heyting implication $\to$, the B-algebra implication satisfies $x \Rightarrow y = (x \to y)^{\nabla}$, the greatest Boolean-centre element below $x \to y$.*

Definition 3: *Let A be a B-algebra. For each $x \in A$, set*

$$x^{\Delta} := 0 \Leftarrow x \qquad \text{and} \qquad x^{\nabla} := 1 \Rightarrow x. \tag{1}$$

The element x^{Δ} is the dual pseudo-supplement *of x: the greatest $b \in B(A)$ with $x \wedge b = 0$. The element x^{∇} is the* pseudo-supplement *of x: the greatest $b \in B(A)$ with $b \leq x$. Both lie in $B(A)$ and bound x from below and above, respectively, within the Boolean centre. On $B(A)$, $\cdot^{\Delta}$ coincides with Boolean complementation (Proposition 4). The scope of these terms is fixed by Remark 2.*

Definition 4: *A* centre-valued negation *on a bounded distributive lattice A is a unary operation whose image lies in $B(A)$. The operator $\cdot^{\Delta}$ provides such a negation on every B-algebra.*

Since $x^{\Delta} = 0 \Leftarrow x$ and every B-algebra homomorphism preserves $\Rightarrow$ and 0, every such homomorphism automatically preserves $\cdot^{\Delta}$:

$$f(x^{\Delta}) = f(0 \Leftarrow x) = 0 \Leftarrow f(x) = f(x)^{\Delta}.$$

The following identities were established in [5].

Proposition 1 (Basic identities [5])**:** *For all $x, y, z \in A$ and $b \in B(A)$ with complement b':*

(i) $x \Rightarrow x = 1$; $0 \Rightarrow x = 1$; $x \Rightarrow 1 = 1$; $1 \Rightarrow x = x^{\triangledown}$.

(ii) $x \Rightarrow (x \Rightarrow y) = x \Rightarrow y$.

(iii) $z \Rightarrow (x \wedge y) = (z \Rightarrow x) \wedge (z \Rightarrow y)$.

(iv) $(x \vee y) \Rightarrow z = (x \Rightarrow z) \wedge (y \Rightarrow z)$.

(v) $(x \wedge b) \Rightarrow y = b' \vee (x \Rightarrow y)$.

(vi) $x \Rightarrow (b \vee y) = b \vee (x \Rightarrow y)$.

(vii) *If $x \leq y$ then $z \Rightarrow x \leq z \Rightarrow y$ and $y \Rightarrow z \leq x \Rightarrow z$.*

Anti-monotonicity of $\cdot^{\Delta}$ follows from item (*(vii)*): $x \leq y$ implies $y^{\Delta} \leq x^{\Delta}$.

Proposition 2: $x^{\triangledown} \leq x \leq (x^{\Delta})'$ *for every $x \in A$.*

Proof: Since $x^{\triangledown}$ is the greatest $b \in B(A)$ with $b \leq x$, the inequality $x^{\triangledown} \leq x$ is immediate. Since $x^{\Delta} \in B(A)$, its Boolean complement $(x^{\Delta})'$ exists in $B(A)$. The equality $x \wedge x^{\Delta} = 0$ then gives $x \leq (x^{\Delta})'$. □

Remark 4: *The inequality $x^{\Delta} \leq x$ does not follow from the axioms: on $B(A)$, $x^{\Delta} = x'$, and $x' \leq x$ forces $x = 1$.*

Corollary 1: *For every $x \in A$, $(x^{\Delta})^{\Delta} = x$ if and only if $x \in B(A)$.*

Proof: By Proposition 4(*(1)*), $(x^{\Delta})^{\Delta} = (x^{\Delta})'$. For $x \in B(A)$: $x^{\Delta} = x'$, so $(x^{\Delta})' = x$. Conversely, if $(x^{\Delta})' = x$ and $x^{\Delta} \in B(A)$, then $x \in B(A)$. □

Definition 5 (Core element [10])**:** *An element $x \in A$ is a* core element *if $x^{\Delta} = x^{\triangledown}$. The set of core elements is $C(A) = \{x \in A : x^{\Delta} = x^{\triangledown}\}$.*

Core elements determine a Boolean substructure of A on which the operators $\cdot^{\triangledown}$ and $\cdot^{\Delta}$ coincide. Theorem 1 gives $C(A) \subseteq B(A)$, so every core element belongs to the Boolean centre. The core embedding (Theorem 7) represents A inside a product of Boolean intervals $[0, c]$, one for each $c \in C(A)$.

Theorem 1 (Core characterisation [10])**:** *For $x \in A$, the following are equivalent:*

(1) $x^{\Delta} = x^{\triangledown}$;

(2) *there exists $b \in B(A)$ with $x^{\Delta} = x \wedge b$ and $x^{\triangledown} = x \vee b$;*

(3) $(x^{\Delta})^{\Delta} = x^{\Delta}$ *and* $(x^{\triangledown})^{\triangledown} = x^{\triangledown}$;

(4) *there exist $b \leq c$ in $B(A)$ with $x \in [b, c]$, $x^{\Delta} = b$, and $x^{\triangledown} = c$.*

In particular, $C(A) \subseteq B(A)$.

Remark 5: *Theorem 1 and Proposition 3 are proved in [10] and are invoked here as stated.*

Proposition 3 ([10])**:** *$C(A)$ is a Boolean subalgebra of $B(A)$.*

Definition 6: *The* bi-implication *of $x, y \in A$ is $x \leftrightarrow y := (x \Rightarrow y) \wedge (y \Rightarrow x)$.*

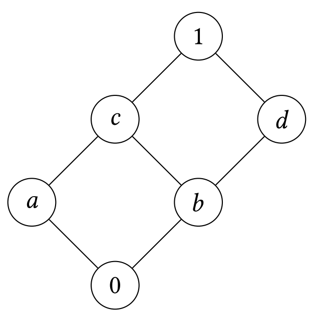
Throughout, A denotes a B-algebra with Boolean centre $B(A)$.

3 Running Examples

Example 1 (Four-element chain)**:** *Let $A = \{0 < a < b < 1\}$ be the four-element chain. The only complemented elements are 0 and 1, so $B(A) = \{0, 1\}$. Setting $x \Rightarrow y = 1$ when $x \leq y$ and $x \Rightarrow y = 0$ otherwise gives a valid B-algebra by the maximality property of Definition 2. Since 0 is the greatest element of $\{0, 1\}$ below either a or b, one has $a^{\nabla} = b^{\nabla} = 0$ with $a \neq b$. The pseudo-supplement map is therefore not injective, and the ultrafilter representation of Section 6.3 does not apply. The prime-filter lattice embedding of Section 6.1 applies without restriction.*

Example 2 (Three-element chain)**:** *Let $H = \{0 < a < 1\}$ with $B(H) = \{0, 1\}$ and $x \Rightarrow y$ the Boolean projection of the Heyting implication. The maximality property of Definition 2 holds. Then $a^{\Delta} = a^{\nabla} = 0$, and the embedding of Section 4 gives $\iota(a) = (a, 0)$ with $\neg\iota(a) = (0, a)$.*

Example 3 (Six-element algebra)**:** *Let $A = \{0, a, b, c, d, 1\}$ with cover relations $0 \prec a$, $0 \prec b$, $a \prec c$, $b \prec c$, $b \prec d$, $c \prec 1$, $d \prec 1$.*



Since $b \vee d = 1$ and $b \wedge d = 0$, one has $B(A) = \{0, b, d, 1\} \cong \mathbf{2}^2$. The pseudo-supplements are $0^{\nabla} = 0$, $a^{\nabla} = 0$, $b^{\nabla} = b$, $c^{\nabla} = b$, $d^{\nabla} = d$, $1^{\nabla} = 1$. A routine verification gives $C(A) = B(A)$, and $a \leftrightarrow d = b \neq 1$, confirming $a \neq d$.

Strict De Morgan inequality outside **$B(A)$.** *With $b \in B(A)$ and $a \notin B(A)$: $a^{\Delta} = 0$, $b^{\Delta} = d$, and $a \wedge b = 0$ gives $(a \wedge b)^{\Delta} = 0^{\Delta} = 1$. Thus $a^{\Delta} \vee b^{\Delta} = d < 1 = (a \wedge b)^{\Delta}$. This shows the restriction of equality (4) to $B(A)$ is genuinely necessary.*

4 Involutive Lattice Structures Associated with B-Algebras

4.1 De Morgan Inequalities and Double Negation

Lemma 1 (De Morgan inequalities for $\cdot^{\Delta}$)**:** *For all $x, y \in A$:*

$$x^{\Delta} \vee y^{\Delta} \leq (x \wedge y)^{\Delta}, \tag{2}$$
$$(x \vee y)^{\Delta} = x^{\Delta} \wedge y^{\Delta}. \tag{3}$$

For $x, y \in B(A)$, equality holds in (2)*:*

$$(x \wedge y)^{\Delta} = x^{\Delta} \vee y^{\Delta} \qquad (x, y \in B(A)). \tag{4}$$

Proof: Inequality (2)**.** Since $x^{\Delta}, y^{\Delta} \in B(A)$ and $B(A)$ is closed under $\vee$, the element $x^{\Delta} \vee y^{\Delta}$ lies in $B(A)$. Distributivity gives

$$(x \wedge y) \wedge (x^{\Delta} \vee y^{\Delta}) = (x \wedge y \wedge x^{\Delta}) \vee (x \wedge y \wedge y^{\Delta}) = 0,$$

using $x \wedge x^{\Delta} = 0$ and $y \wedge y^{\Delta} = 0$. By maximality of $(x \wedge y)^{\Delta}$, the inequality follows.

Equality (3)**.** Anti-monotonicity and $x \leq x \vee y$ give $(x \vee y)^{\Delta} \leq x^{\Delta}$; the argument for y^{Δ} is the same. For the reverse, distributivity gives $(x \vee y) \wedge (x^{\Delta} \wedge y^{\Delta}) = 0$, so maximality gives $x^{\Delta} \wedge y^{\Delta} \leq (x \vee y)^{\Delta}$.

Equality (4) **for** $x, y \in B(A)$**.** By (2) it suffices to verify the reverse inequality. Let $b \in B(A)$ satisfy $(x \wedge y) \wedge b = 0$, equivalently $x \wedge (y \wedge b) = 0$. Since $y, b \in B(A)$, maximality of x^{Δ} gives $y \wedge b \leq x^{\Delta}$. Decomposing $b = (b \wedge y) \vee (b \wedge y')$ and using $y^{\Delta} = y'$ (valid because $y \in B(A)$), one obtains $b \leq x^{\Delta} \vee y^{\Delta}$. Setting $b = (x \wedge y)^{\Delta}$ gives the reverse inequality. □

Remark 6: *The argument uses y', which exists only when $y \in B(A)$. For $y \in A \setminus B(A)$, equality* (4) *need not hold, as Example 3 demonstrates. Problem **(P1)** asks for global conditions.*

Proposition 4: *For all $x \in A$:*

(1) $(x^{\Delta})^{\Delta} = (x^{\Delta})'$*, the Boolean complement of x^{Δ} in $B(A)$.*
(2) *If $x \in B(A)$, then $(x^{\Delta})^{\Delta} = x$.*

Proof: *(1)*: Since $x^{\Delta} \in B(A)$, the greatest $b \in B(A)$ with $x^{\Delta} \wedge b = 0$ is $(x^{\Delta})'$.
(2): For $x \in B(A)$, $x^{\Delta} = x'$, so $(x^{\Delta})^{\Delta} = (x')' = x$. □

Remark 7: *Corollary 1 records that $\cdot^{\Delta}$ is not an involution on all of A. On $B(A)$ it restricts to Boolean complementation; on the full algebra it acts as a centre-valued negation in the sense of Definition 4.*

4.2 Nelson-type Structure on the Boolean Centre

Lemma 2: *$B(A)$ is closed under $\vee$, $\wedge$, and $\Rightarrow$.*

Proof: Closure under lattice operations is standard. Since $x \Rightarrow y \in B(A)$ for all $x, y \in A$, closure under $\Rightarrow$ is immediate. $\square$

Theorem 2: *The Boolean centre $B(A)$, equipped with the restricted operations, satisfies the five equations of one standard axiomatisation of Nelson algebras [14], with Boolean lattice reduct.*

Remark 8: *The formulation follows Rasiowa [14]. Since $B(A)$ is Boolean, $x^{\Delta} = x'$ and $x \Rightarrow y = x' \vee y$ on $B(A)$; the five identities reduce to standard Boolean identities on that domain.*

*The involutive construction $\mathcal{N}(A)$ of Definition 9 is not a Nelson algebra in general: closure of $N_0(A)$ under implication is open (Problem **(P3)**), a first-coordinate analogue of axiom (N4) holds for $\iota(a)$ with $a \in B(A)$ only, and residuation is absent. The global algebra A does not become a Nelson algebra.*

Proof: For $x, y \in B(A)$, set $x^{\Delta} = x'$ and $x \Rightarrow y = x' \vee y$. The five identities of Rasiowa [14] are verified directly.

(N1) $(x^{\Delta})^{\Delta} = (x')' = x.$
(N2) $(x \vee y)^{\Delta} = (x \vee y)' = x' \wedge y' = x^{\Delta} \wedge y^{\Delta}.$
(N3) $x \wedge x^{\Delta} = x \wedge x' = 0.$
(N4) $(x \Rightarrow y) \wedge (x \Rightarrow y^{\Delta}) = (x' \vee y) \wedge (x' \vee y') = x' = x^{\Delta}.$
(N5) $x \wedge (x \Rightarrow y) = x \wedge (x' \vee y) = x \wedge y \leq y.$

$\square$

4.3 The Involutive Construction and Its Restricted Substructure

Definition 7: *The category* **InvLat** *has as objects bounded distributive involutive lattices: bounded distributive lattices L equipped with an involution $\neg : L \to L$ satisfying $\neg\neg l = l$ and the De Morgan laws. Morphisms are bounded lattice homomorphisms preserving the involution. Since the De Morgan laws hold on a distributive lattice, the involution reverses the lattice order: $l_1 \leq l_2$ implies $\neg l_2 \leq \neg l_1$.*

Definition 8: *The category* **Balg** *has as objects all B-algebras and as morphisms the homomorphisms of Definition 2.*

Definition 9: *Let A be a B-algebra. The* involutive construction *$\mathcal{N}(A) = A \times A$ carries the operations*

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2,\ y_1 \vee y_2), \tag{5}$$

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2,\ y_1 \wedge y_2), \tag{6}$$

$$\neg(x, y) = (y, x), \tag{7}$$

with $\mathbf{0} = (0, 1)$ and $\mathbf{1} = (1, 0)$. The involution is the coordinate-swap $\neg$, not $\cdot^{\Delta}$. The embedding is $\iota(x) = (x, x^{\Delta})$, and the restricted substructure *is $N_0(A) = \{(x, y) \in A \times A : x \wedge y = 0\}$.*

The operations in (5)–(6) *are twisted: the meet increases the second coordinate while the join decreases it. The order on* $\mathcal{N}(A)$ *is therefore not coordinatewise; it satisfies*

$$(x_1, y_1) \le (x_2, y_2) \iff x_1 \le x_2 \text{ in } A \text{ and } y_2 \le y_1 \text{ in } A.$$

Proposition 5: *$N_0(A)$ is closed under $\wedge$, $\vee$, and $\neg$ in $\mathcal{N}(A)$, and $\iota(A) \subseteq N_0(A)$.*

Proof: Let $(x_1, y_1), (x_2, y_2) \in N_0(A)$, so $x_i \wedge y_i = 0$.

Meet: $(x_1 \wedge x_2) \wedge (y_1 \vee y_2) \le (x_1 \wedge y_1) \vee (x_2 \wedge y_2) = 0$ by distributivity.

Join: $(x_1 \vee x_2) \wedge (y_1 \wedge y_2) \le (x_1 \wedge y_1) \vee (x_2 \wedge y_2) = 0$.

Negation: $\neg(x, y) = (y, x)$ and $y \wedge x = 0$.

Embedding: $a \wedge a^{\Delta} = 0$ for all $a \in A$. □

Remark 9: *Closure of $N_0(A)$ under a natural implication is Problem **(P3)**. For $(x, y) \in N_0(A)$, axiom (N3) holds: since $x \wedge y = 0$,*

$$(x, y) \wedge \neg(x, y) = (x \wedge y,\ x \vee y) = (0,\ x \vee y) \le (0, 1) = \mathbf{0},$$

where the inequality uses $x \vee y \le 1$ in A and the twisted order on $\mathcal{N}(A)$.

Theorem 3 (Involutive construction)**:** *Let A be a B-algebra. Then:*

(1) *$\mathcal{N}(A)$ is a distributive involutive lattice satisfying the De Morgan laws for $\neg$. Axiom* (N3) *holds globally on $N_0(A)$. A first-coordinate analogue of axiom* (N4) *holds for $\iota(a)$ with $a \in B(A)$; the general case is Problem **(P3)**.*

(2) *The map $\iota : A \hookrightarrow N_0(A) \subseteq \mathcal{N}(A)$ is injective. Join preservation holds for all $x, y \in A$. The inequality $\iota(x) \wedge \iota(y) \le \iota(x \wedge y)$ holds for all $x, y \in A$, with equality when $x, y \in B(A)$.*

(3) *For every distributive involutive lattice N and every bounded lattice homomorphism $h : A \to U(N)$ satisfying $h(x^{\Delta}) = \neg_N h(x)$ for all $x \in A$, the map $\tilde{h} : N_0(A) \to N$ defined by $\tilde{h}(x, y) = h(x) \wedge \neg_N h(y)$ is an order-compatible factorisation: it is monotone with respect to the twisted order, satisfies $\tilde{h} \circ \iota = h$, and $\tilde{h}(\neg(x, y)) \le \neg_N \tilde{h}(x, y)$ for every $(x, y) \in N_0(A)$.*

Proof: ***(1).*** Distributivity holds in $\mathcal{N}(A)$ because it holds coordinatewise in A and the second coordinate is reversed. *Involution:* $\neg\neg(x, y) = (x, y)$. *De Morgan law:* $\neg\big((x_1, y_1) \vee (x_2, y_2)\big) = (y_1 \wedge y_2,\ x_1 \vee x_2) = \neg(x_1, y_1) \wedge \neg(x_2, y_2)$; the law for $\wedge$ is analogous. *Axiom (N3):* For $(x, y) \in N_0(A)$, since $x \wedge y = 0$, $(x, y) \wedge \neg(x, y) = (0,\ x \vee y) \le (0, 1) = \mathbf{0}$ in the twisted order. *First-coordinate analogue of (N4) for $\iota(a)$, $a \in B(A)$:* Since $a \in B(A)$, $a \Rightarrow b = a' \vee b$, so $(a' \vee b) \wedge (a' \vee b') = a' = a^{\Delta}$. The first coordinate therefore satisfies the analogue of (N4).

(2). *Join preservation:* By Lemma 1 (3), $\iota(x \vee y) = (x \vee y,\ x^{\Delta} \wedge y^{\Delta}) = \iota(x) \vee \iota(y)$. *Inequality:* Lemma 1 (2) gives $x^{\Delta} \vee y^{\Delta} \le (x \wedge y)^{\Delta}$, so $\iota(x) \wedge \iota(y) \le \iota(x \wedge y)$ in the twisted order. *Equality on $B(A)$:* Lemma 1 (4) gives equality when $x, y \in B(A)$. *Injectivity:* $\iota(x) = \iota(y)$ implies $x = y$.

(3). Define $\tilde{h}(x, y) = h(x) \wedge \neg_N h(y)$.

Monotonicity. For $(x_1, y_1) \leq (x_2, y_2)$, one has $x_1 \leq x_2$ and $y_2 \leq y_1$. Order-preservation of h gives $h(x_1) \leq h(x_2)$. Since $\neg_N$ reverses order (Definition 7), $y_2 \leq y_1$ gives $\neg_N h(y_1) \leq \neg_N h(y_2)$. Hence $\tilde{h}(x_1, y_1) \leq \tilde{h}(x_2, y_2)$.

Factorisation. From $h(x^\Delta) = \neg_N h(x)$: $\tilde{h}(\iota(x)) = h(x) \wedge \neg_N(\neg_N h(x)) = h(x)$.

Compatibility inequality. $\tilde{h}(\neg(x, y)) = h(y) \wedge \neg_N h(x)$ and $\neg_N \tilde{h}(x, y) = \neg_N h(x) \vee h(y)$; since $a \wedge b \leq a \vee b$, the inequality follows.

Remark 10: *Full equality $\tilde{h}(\neg(x, y)) = \neg_N \tilde{h}(x, y)$ is Problem **(P4)**.*

□

4.4 Functorial Assignment

Proposition 6 (Functoriality)**:** *Define $\mathcal{N} : \mathbf{Balg} \to \mathbf{InvLat}$ on objects by $A \mapsto A \times A$ and on morphisms by $f \mapsto [(x, y) \mapsto (f(x), f(y))]$. Then $\mathcal{N}$ is a functor.*

Proof: Since f preserves the bounded lattice operations, the induced map preserves the twisted meet and twisted join coordinatewise. For the twisted meet: $\mathcal{N}(f)(x_1 \wedge x_2,\ y_1 \vee y_2) = (f(x_1) \wedge f(x_2),\ f(y_1) \vee f(y_2))$; the argument for the twisted join is analogous. Preservation of $\neg$: $\mathcal{N}(f)(\neg(x, y)) = (f(y), f(x)) = \neg \mathcal{N}(f)(x, y)$. Composition and identity are immediate. □

Corollary 2: *Every bounded lattice homomorphism $h : A \to U(N)$ satisfying $h(x^\Delta) = \neg_N h(x)$ for all $x \in A$ admits a monotone factorisation $\tilde{h} : N_0(A) \to N$ with $\tilde{h} \circ \iota = h$ and $\tilde{h}(\neg(x, y)) \leq \neg_N \tilde{h}(x, y)$ for all $(x, y) \in N_0(A)$.*

$$\begin{tikzcd} \mathbf{Bool} \arrow[r, hook, "\text{inclusion}"] & \mathbf{Balg} \arrow[r, "\mathcal{N}"] \arrow[dr, "h"'] & \mathbf{InvLat} \\ & & N \arrow[u, "U"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "h"'] & N_0(A) \arrow[d, "\tilde{h}"] \\ & N \end{tikzcd}$$

The dashed arrow below indicates that a dual equivalence has not been established.

$$\begin{tikzcd} \mathbf{Balg} \arrow[r, dashed, "\mathrm{Spec}"] & \mathbf{SpNeg}^{\mathrm{op}} \end{tikzcd}$$

5 Prime-Spectrum Representation

5.1 Prime Spectrum and the Map σ

Definition 10: *A proper filter $P \subseteq A$ is* prime *if $x \vee y \in P$ implies $x \in P$ or $y \in P$. The* prime spectrum *is* $\mathrm{Spec}(A) = \{P \subseteq A : P$ *is a prime filter*$\}$. *For each $x \in A$, set $\widehat{x} = \{P \in \mathrm{Spec}(A) : x \in P\}$.*

The collection $\{\widehat{x} : x \in A\}$ satisfies $\widehat{x} \cap \widehat{y} = \widehat{x \wedge y}$ and forms a basis of compact open sets for the *spectral topology* on $\mathrm{Spec}(A)$, making $\mathrm{Spec}(A)$ a compact T_0 space; see [7] and [4, Chapter 8]. The *specialisation order* is $P \leq Q$ iff $P \subseteq Q$.

Definition 11: *For each $P \in \mathrm{Spec}(A)$, define $\sigma(P) = \{x \in A : x^{\Delta} \notin P\}$. Whenever $\sigma(P)$ is a prime filter, it is regarded as an element of* $\mathrm{Spec}(A)$. *All continuity arguments use inverse images under σ; openness of direct images is not assumed.*

Lemma 3: *For every $P \in \mathrm{Spec}(A)$ and all $x, y \in A$: if $x \Rightarrow y \in P$ then either $x \notin P$ or $y \in P$.*

Proof: If $x \Rightarrow y \in P$ and $x \in P$, then $x \wedge (x \Rightarrow y) \in P$; since $x \wedge (x \Rightarrow y) \leq y$, upper-closure gives $y \in P$. $\square$

Proposition 7: *If the global De Morgan equality holds for all $x, y \in A$, then $\sigma(P)$ is a prime filter for every $P \in \mathrm{Spec}(A)$. Without that assumption, $\sigma(P)$ satisfies upper-closure, primeness, and closure under $\wedge$ on $B(A)$. Moreover:*

(1) $\sigma^{-1}(\widehat{x}) = \mathrm{Spec}(A) \setminus \widehat{x^{\Delta}}$ *for every $x \in A$; in particular σ is a continuous order-reversing self-map of* $\mathrm{Spec}(A)$.

(2) *If $P \subseteq Q$ then $\sigma(Q) \subseteq \sigma(P)$.*

(3) *For every $x \in B(A)$ and $P \in \mathrm{Spec}(A)$: $x \in \sigma(\sigma(P))$ iff $x \in P$.*

Remark 11: *Since $1 \in P$ and $1^{\Delta} = 0 \notin P$, one has $1 \in \sigma(P)$. If $0 \in \sigma(P)$ then $0^{\Delta} = 1 \notin P$, contradicting $1 \in P$. Hence $\sigma(P)$ is proper whenever it is a filter.*

Proof: *Upper-closure:* If $x \in \sigma(P)$ and $x \leq z$, then $z^{\Delta} \leq x^{\Delta}$ by anti-monotonicity, so $z^{\Delta} \notin P$ and $z \in \sigma(P)$.

Closure under $\wedge$ *on* $B(A)$*:* For $x, y \in \sigma(P) \cap B(A)$, Lemma 1 (4) gives $(x \wedge y)^{\Delta} = x^{\Delta} \vee y^{\Delta}$. Were $(x \wedge y)^{\Delta} \in P$, primeness would force $x^{\Delta} \in P$ or $y^{\Delta} \in P$, a contradiction. Under the global De Morgan equality the argument extends to all $x, y \in \sigma(P)$; see Problem **(P1)**.

Primeness: If $x \vee y \in \sigma(P)$, then by (3), $x^{\Delta} \wedge y^{\Delta} \notin P$. Primeness forces $x^{\Delta} \notin P$ or $y^{\Delta} \notin P$.

(1): $P \in \sigma^{-1}(\widehat{x})$ iff $P \notin \widehat{x^{\Delta}}$, so $\sigma^{-1}(\widehat{x}) = \mathrm{Spec}(A) \setminus \widehat{x^{\Delta}}$ is clopen.

(2): If $P \subseteq Q$ and $x^{\Delta} \notin Q$, then $x^{\Delta} \notin P$.

(3): $x \in \sigma(\sigma(P))$ iff $(x^{\Delta})^{\Delta} \in P$. By Proposition 4(*(2)*), $(x^{\Delta})^{\Delta} = x$ for $x \in B(A)$. $\square$

Remark 12: *Part (3) shows σ is involutive on Boolean elements. Full bijectivity of σ on* $\mathrm{Spec}(A)$ *is open (Problem* ***(P2)****).*

5.2 Spectral Negation Structures

The category defined below isolates the spectral and topological behaviour induced by $\cdot^{\Delta}$. It serves as an intermediate structure between Priestley spaces and involutive spectral systems, adapted to the Boolean-valued implication setting.

Definition 12: *A* spectral negation structure *is a pair (X, σ) where X is a compact T_0 space with a clopen basis $\mathcal{B}$ and $\sigma : X \to X$ is continuous and reverses the specialisation order, $\mathcal{B}$ is closed under* $\mathbf{n}\, U := \sigma^{-1}(X \setminus U)$*, and* $\mathbf{n}(U \cup V) = \mathbf{n}\, U \cap \mathbf{n}\, V$ *for all $U, V \in \mathcal{B}$. A* morphism *is a continuous specialisation-order-preserving map $f : X \to X'$ with $f \circ \sigma = \sigma' \circ f$. The category* **SpNeg** *is defined for the purposes of the present partial representation framework.*

Remark 13: *The forward image $\sigma(U)$ need not be open, so* $\mathbf{n}\, U = \sigma^{-1}(X \setminus U)$ *is the correct formulation; it matches Proposition 7(1) directly. The two formulations coincide when σ is bijective (Problem* ***(P2)****).*

Proposition 8: *For any spectral negation structure (X, σ), the structure* $(\mathcal{B}, \cup, \cap, \mathbf{n}, \emptyset, X)$ *is a bounded distributive lattice satisfying* $U \cap (\mathbf{n}\, U \cup V) \subseteq V$ *for all $U, V \in \mathcal{B}$.*

Proof: $U \cap (\mathbf{n}\, U \cup V) = (U \cap \mathbf{n}\, U) \cup (U \cap V) = U \cap V \subseteq V$. $\square$

Theorem 4 (Prime-spectrum lattice representation)**:** *For every B-algebra A:*

(1) $(\mathrm{Spec}(A), \sigma)$ *is a spectral negation structure.*

(2) *The map* $\eta_A : A \to \mathrm{Clopen}(\mathrm{Spec}(A))$*,* $x \mapsto \widehat{x}$*, is an injective lattice homomorphism; for* $x \in B(A)$*,* $\eta_A(x^{\Delta}) = \mathbf{n}\, \widehat{x}$*.*

(3) *Every B-algebra homomorphism $h : A \to B$ induces a morphism* $\mathrm{Spec}(h) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$*,* $P \mapsto h^{-1}(P)$*.*

Proof: *(1)* follows from Proposition 7 and Remark 13.

(2): Lattice homomorphism properties follow from filter and primality; injectivity follows from Theorem 5.

(3): Every B-algebra homomorphism preserves $\cdot^{\Delta}$. For any $P \in \mathrm{Spec}(B)$ and $x \in A$:

$$x \in \sigma(h^{-1}(P)) \iff x^{\Delta} \notin h^{-1}(P) \iff h(x^{\Delta}) \notin P \iff h(x)^{\Delta} \notin P \iff h(x) \in \sigma(P),$$

so $\sigma(h^{-1}(P)) = h^{-1}(\sigma(P))$. □

A dual equivalence $\mathbf{Balg} \simeq \mathbf{SpNeg}^{\mathrm{op}}$ has not been established; completing it requires surjectivity of η_A and bijectivity of σ (Problem **(P2)**). Extending Priestley duality [4] to B-algebras remains open.

6 Representation Theory

6.1 Prime-Filter Lattice Embedding

Theorem 5 (Prime filter separation)**:** *If $x \nleq y$ in A, there exists $P \in \mathrm{Spec}(A)$ with $x \in P$ and $y \notin P$.*

Proof: The principal filter $F = \{a \in A : x \leq a\}$ contains x but not y. A filter P maximal among those containing F and excluding y exists by Zorn's lemma and is prime by standard arguments [7]. □

Theorem 6 (Prime-filter lattice embedding)**:** *Every B-algebra A embeds injectively as a lattice into $\prod_{P \in \mathrm{Spec}(A)} \mathbf{2}$, where $\mathbf{2} = \{0, 1\}$. Implication preservation is not claimed.*

Proof: Define $\phi : A \to \prod_P \mathbf{2}$ by $\phi(x)(P) = 1$ iff $x \in P$. Preservation of $\wedge$ and $\vee$ follows from filter and primality; injectivity follows from Theorem 5. Lemma 3 yields only a one-sided implication condition, which falls short of full implication preservation. □

Corollary 3: *Every B-algebra embeds injectively as a lattice into a direct product of copies of* $\mathbf{2}$.

6.2 Core-Based Embedding

Theorem 7 (Core embedding [10])**:** *The map $\Phi : A \to \prod_{c \in C(A)} [0, c]$ given by $\Phi(x) = (x \wedge c)_{c \in C(A)}$ is an injective lattice homomorphism.*

Proof: Distributivity gives $(x \vee y) \wedge c = (x \wedge c) \vee (y \wedge c)$ and $(x \wedge y) \wedge c = (x \wedge c) \wedge (y \wedge c)$ for each $c \in C(A)$. Since $1 \in C(A)$, the coordinate at $c = 1$ recovers x, giving injectivity. □

6.3 Ultrafilter Representation

The following embedding is conditional on an injectivity hypothesis that need not hold in general (Example 1).

Lemma 4: $(x \wedge y)^{\nabla} = x^{\nabla} \wedge y^{\nabla}$ *for all $x, y \in A$.*

Proof: Since $x^{\nabla} \wedge y^{\nabla} \in B(A)$ and $x^{\nabla} \wedge y^{\nabla} \leq x \wedge y$, maximality gives $x^{\nabla} \wedge y^{\nabla} \leq (x \wedge y)^{\nabla}$. For the reverse: $(x \wedge y)^{\nabla} \leq x \wedge y \leq x$ with $(x \wedge y)^{\nabla} \in B(A)$ gives $(x \wedge y)^{\nabla} \leq x^{\nabla}$; the bound y^{∇} follows by symmetry. □

Theorem 8: *Let A be a B-algebra in which $x \mapsto x^{\nabla}$ is injective. Then A embeds injectively as a lattice into $\prod_{u \in \mathrm{Ult}(B(A))} \mathbf{2}$. Lattice operations are preserved; implication is not.*

Proof: The Stone representation of $B(A)$ gives an injective Boolean embedding $\psi : B(A) \hookrightarrow \prod_u \mathbf{2}$. For each ultrafilter u, set $\varphi_u(x) = [x^{\nabla} \in u]$. By Lemma 4, each φ_u is a lattice homomorphism. Injectivity of the composite follows from injectivity of $x \mapsto x^{\nabla}$ and of ψ. $\square$

7 Bi-Implication and Equality

Theorem 9: *For $x, y \in A$, $x = y$ if and only if $x \leftrightarrow y = 1$.*

Proof: If $x = y$ then $x \leftrightarrow y = 1 \wedge 1 = 1$. Conversely, $x \leftrightarrow y = 1$ gives $x \Rightarrow y = 1$ and $y \Rightarrow x = 1$; by the maximality property of Definition 2, $x \leq y$ and $y \leq x$. $\square$

Theorem 10: $(x \leftrightarrow y) \wedge (y \leftrightarrow z) \leq x \leftrightarrow z$ *for all $x, y, z \in A$.*

Proof: Set $b = x \Rightarrow y$ and $c = y \Rightarrow z$. Then $x \wedge (b \wedge c) \leq y \wedge c \leq z$, so $b \wedge c \leq x \Rightarrow z$. Symmetrically $(z \Rightarrow y) \wedge (y \Rightarrow x) \leq z \Rightarrow x$. Taking meets gives the result. $\square$

Corollary 4: *The relation $\{(x, y) \in A^2 : x \leftrightarrow y = 1\}$ is the diagonal of A.*

8 Comparison with Related Constructions

Twist-products. The constructions of Vakarelov [16] and Sendlewski [15] represent Nelson algebras as subalgebras of $L \times L$ subject to explicit compatibility conditions. The involutive construction $\mathcal{N}(A) = A \times A$ requires no external constraints; $N_0(A)$ operates within the B-algebra framework without added assumptions.

Twist-product	**Involutive construction (this paper)**
Subalgebra of $L \times L$ with constraints	Full product $A \times A$; restricted to $N_0(A)$
External compatibility conditions required	Defined via B-algebra operations alone
Representation-based	Functorial; monotone factorisation (Corollary 2)

Pseudo-complement versus dual pseudo-supplement. In a distributive lattice, the pseudo-complement $x^* = \max\{y : x \wedge y = 0\}$ may fail to exist. In a B-algebra, x^{Δ} always exists, lies in $B(A)$, and the pair (x^{∇}, x^{Δ}) approximates x from below and above within the Boolean centre. The one-sided De Morgan inequality holds globally; equality fails outside the Boolean centre (Lemma 1), and Example 3 confirms this.

Residuated lattices and related structures. In a residuated lattice, the residuation $x\backslash y$ need not produce output in any distinguished Boolean fragment. The B-algebra implication is always Boolean-valued, making $B(A)$ essential to the structure. The present framework differs from standard residuated lattice settings precisely because the implication is Boolean-valued by definition. Bilattices [3] build involutive structure externally rather than from a Boolean-valued implication. Heyting–Nelson systems [12,15] impose compatibility conditions on the Heyting structure; the B-algebra setting avoids such conditions because the implication is already Boolean-valued. Those frameworks do not assume Boolean-valued implication, so the spectral map σ, the centre-valued De Morgan structure, and the functorial involutive assignment do not arise there.

9 Illustrative Examples

Example 4 (Boolean algebras)**:** *Every Boolean algebra B is a B-algebra with $x \Rightarrow y = x' \vee y$, $B(B) = B$, $x^{\nabla} = x$, $x^{\Delta} = x'$, and $C(B) = B$. The bi-implication equals* 1 *precisely when $x = y$.*

Example 5 (Three-element chain, continued)**:** *In Example 2, $a \leftrightarrow 0 = 0 \wedge 1 = 0 \neq 1$, confirming $a \neq 0$.*

Example 6 (Six-element algebra, continued)**:** *In Example 3: $a \leftrightarrow d = b \neq 1$; $b \leftrightarrow d = 0 \neq 1$; $b \leftrightarrow b = 1$. The transitivity inequality gives $(b \leftrightarrow a) \wedge (a \leftrightarrow d) \leq b \leftrightarrow d$, which reduces to $0 \wedge b = 0 \leq 0$.*

10 Applications to Constructive Logic

For $x \in B(A)$, x^{Δ} is the Boolean complement of x, so excluded middle holds on $B(A)$. Outside $B(A)$, $\cdot^{\Delta}$ acts as a Boolean-centred negation operator without forcing excluded middle, sharing formal features with the strong negation of Nelson's construction [13]. Join preservation of ι places every B-algebra inside a distributive involutive structure via the functor $\mathcal{N}$. The chain of inclusions at the map level reads:

$$\text{Boolean algebra} \xrightarrow{\text{id}} B\text{-algebra} \xrightarrow{\iota} N_0(A).$$

Every Heyting algebra participates via $x \Rightarrow y = (x \to y)^{\nabla}$ (Remark 3). The bi-implication models logical equivalence: $x \leftrightarrow y = 1$ iff $x = y$ (Theorem 9), and Theorem 10 supplies an algebraic transitivity property.

11 Open Problems

(P1) **Global De Morgan equality.** Identify conditions under which $(x \wedge y)^{\Delta} = x^{\Delta} \vee y^{\Delta}$ holds for all $x, y \in A$, and determine whether this characterises a subvariety of **Balg**.

(P2) **Full spectral correspondence.** Extend Theorem 4 to a full categorical equivalence $\mathbf{Balg} \simeq \mathbf{SpNeg}^{\mathrm{op}}$ by establishing surjectivity of η_A, naturality of the reconstruction map, and bijectivity of σ.

(P3) **Closure of $N_0(A)$ under implication.** For which B-algebras A does $N_0(A)$ satisfy all five Nelson identities of [14], including closure under a natural implication, and for which does axiom (N4) hold without restricting to Boolean first coordinates?

(P4) **Injective pseudo-supplement characterisation.** Characterise B-algebras for which $x \mapsto x^{\nabla}$ is injective, and determine for which pairs (A, N) the monotone factorisation $\tilde{h}$ of Corollary 2 satisfies $\tilde{h}(\neg(x, y)) = \neg_N \tilde{h}(x, y)$.

(P5) **Subdirectly irreducible algebras.** Classify the subdirectly irreducible B-algebras, and determine which subvarieties of **Balg** are finitely axiomatisable.

12 Conclusions

The operators $\cdot^{\nabla}$ and $\cdot^{\Delta}$ admit a rich structural theory within B-algebras. The De Morgan identities of Lemma 1 separate the global behaviour of $\cdot^{\Delta}$ from its restriction to the Boolean centre. A global inequality and a full join identity hold for all $x, y \in A$; the meet identity requires $x, y \in B(A)$ because the proof uses y', unavailable outside $B(A)$. Example 3 provides a strict instance, and this gap propagates into the meet-preservation property of ι and the prime-filter argument for σ.

The involutive lattice construction $\mathcal{N}(A)$ yields a functor from **Balg** to **InvLat**, while the prime-spectrum construction gives a partial spectral representation framework associated with $\cdot^{\Delta}$. The Boolean centre $B(A)$ satisfies the five equational Nelson identities of Rasiowa (Theorem 2), with Boolean lattice reduct; the global algebra A does not. In $N_0(A)$, axiom (N3) holds globally; a first-coordinate analogue of axiom (N4) holds for $\iota(a)$ with $a \in B(A)$, and the general case remains open (Problem **(P3)**).

A continuous order-reversing self-map σ of $\mathrm{Spec}(A)$ arises from Proposition 7, making $(\mathrm{Spec}(A), \sigma)$ a spectral negation structure (Theorem 4). The prime-filter lattice embedding (Theorem 6) is unconditional; the ultrafilter embedding (Theorem 8) requires injectivity of $\cdot^{\nabla}$, with Example 1 showing that condition is not automatic. The bi-implication characterises equality (Theorem 9) and satisfies a transitivity inequality (Theorem 10). These results point toward further work on global De Morgan identities, stronger spectral dualities, and the full structure of $N_0(A)$.

Beyond the five problems of Section 11, several broader directions remain open. A finite-model search, in the style of the verification used for the running examples, could locate small B-algebras separating the open problems from their nearest known partial answers, and could test candidate axiomatisations before a full proof is attempted. The relationship between B-algebras and substructural or paraconsistent logics is not addressed here and may clarify how the centre-valued negation $\cdot^{\Delta}$ compares with negations studied in that literature. On the categorical side, the variety generated by **Balg** and its free algebras have not been described, and the functor $\mathcal{N} : \mathbf{Balg} \to \mathbf{InvLat}$ of Proposition 6 raises the question of which involutive lattices arise, up to isomorphism, as $\mathcal{N}(A)$ for some B-algebra A. Finally, the restriction to finite or atomic B-algebras used in several results above (for example Theorem 7) is convenient but not obviously necessary, and identifying the correct infinite-algebra analogues is left for future work.

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N. Rafi: Contributed to the formulation of the research framework, validated the findings, provided supervision and research resources, participated in the interpretation of results, and critically reviewed and edited the manuscript. All authors have carefully reviewed and approved the final manuscript and accept responsibility for the integrity and accuracy of the work presented. All authors have read and approved the final version of the manuscript for publication.

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References

1. G. Birkhoff, *Lattice Theory*, *American Mathematical Society*, Providence, 3rd ed., **1967**. https://doi.org/10.1090/coll/025
2. C. C. Chang and A. Horn, Prime ideal characterisations of generalised Post algebras, In *Proceedings of Symposia in Pure Mathematics*, *2* **(1961)**, 43–48. https://doi.org/10.1090/pspum/002
3. R. Cignoli, I. M. L. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, *Kluwer*, Dordrecht, **2000**.
4. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, *Cambridge University Press*, Cambridge, 2nd ed., **2012**. https://doi.org/10.1017/CBO9780511809088
5. G. Epstein and A. Horn, P-algebras, an abstraction from Post algebras, *Algebra Universalis 4(1)* **(1974)**, 195–206. https://doi.org/10.1007/BF02485725
6. S. Givant and P. Halmos, *Introduction to Boolean Algebras*, *Springer*, New York, **2009**. https://doi.org/10.1007/978-0-387-68436-9
7. G. Grätzer, *General Lattice Theory*, *Birkhäuser*, Basel, **1978**. https://doi.org/10.1007/978-3-0348-7633-9
8. P. R. Halmos, *Lectures on Boolean Algebras*, *Van Nostrand*, Princeton, **1963**. https://doi.org/10.1007/978-1-4612-9855-7
9. A. Heyting, Die formalen Regeln der intuitionistischen Logik, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* **(1930)**, 42–56.
10. N. K. Kakumanu, B. Tharuni and S. S. Raju, Structural theory of B-algebras: core elements, bi-implication, and Stone-type representations, **(2024)**. https://doi.org/10.13140/RG.2.2.17232.26887
11. N. K. Kakumanu, S. S. Raju and N. Rafi, Semi-B-algebras, *Bulletin of the International Mathematical Virtual Institute 14(1)* **(2024)**, 13–22. https://doi.org/10.7251/BIMVI2401013N
12. A. Monteiro, Sur les algèbres de Heyting symétriques, *Portugaliæ Mathematica 39(1–4)* **(1980)**, 1–237.
13. D. Nelson, Constructible falsity, *The Journal of Symbolic Logic 14(1)* **(1949)**, 16–26. https://doi.org/10.2307/2268973
14. H. Rasiowa, *An Algebraic Approach to Non-Classical Logics*, *North-Holland*, Amsterdam, **1974**.

15. A. Sendlewski, Nelson algebras through Heyting ones, *Studia Logica 49(1)* **(1990)**, 105–126. https://doi.org/10.1007/BF00401557
16. D. Vakarelov, Notes on $\mathcal{N}$-lattices and constructive logic with strong negation, *Studia Logica 36(1–2)* **(1977)**, 109–125.