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Research article

Ideal Structures in Leras Topological Spaces

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Abstract: This paper presents the notion of ideal leras topological spaces as an extension of leras topological spaces. The connections between these two classes of spaces are examined in detail. Different forms of closed sets arising in ideal leras topological spaces are introduced and their interrelations are discussed. Several characterizations and fundamental properties associated with these sets are obtained. Examples are included to clarify the introduced concepts and the corresponding results.

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1 Introduction

In 1964, Levine introduced the concept of simple extension topological spaces and investigated several weak forms of open sets in such spaces. This work led to the development of many generalized structures in topology and motivated researchers to study weaker variants of open and closed sets.

The study of weakly open sets has become an important area in general topology because these sets provide a wider framework than ordinary open sets. Different kinds of weak open sets have been introduced by various authors to generalize classical topological notions and to analyze relationships among different classes of generalized sets.

Recently, in 2026, I. Rajasekaran introduced the notion of leras topological spaces. In these spaces, the members of the topology are called leras open sets. Along with this concept, related notions such as leras closed sets, leras interior, and leras closure operators were also introduced and studied. Moreover, several weaker forms of leras open sets and their basic properties were investigated.

The theory of ideals has also received considerable attention in topology due to its effectiveness in generating new topological structures and studying generalized forms of openness and closedness. Hamlett and Jankovic [1] studied local functions in ideal topological spaces and obtained a new topology associated with ideals. Their work inspired further investigations in ideal topological spaces.

Although the framework of leras topological spaces has proved useful for unifying several generalized open-set structures obtained through simple extensions, it does not, in its original form, account for the additional flexibility that ideals bring to a topological space. In classical topology, the introduction of an ideal alongside a topology has repeatedly shown its value: it permits the construction of a finer topology through the associated local function, offers a systematic way to weaken or strengthen closedness conditions, and unifies several previously unrelated generalized continuity and covering properties under a single framework. Given how effectively ideals have enriched standard and other generalized topological settings, it is natural to ask whether a comparable enrichment is possible for leras topological spaces, and what new structures such a combination would reveal.

This question is the central motivation for the present study. Since leras open sets already arise from a two-part construction involving two ordinary topologies and a fixed pair of complementary non-open sets, superimposing an ideal on this structure is not a routine generalization; the interaction between the ideal and the leras topology must be examined carefully to ensure that the resulting local function and closure operator behave consistently with the established leras theory. Existing ideal-topological results, such as those of Hamlett and Jankovic [1], cannot be applied directly to leras spaces because the open sets in this setting are not generated by a single topology in the usual sense. A dedicated treatment is therefore required, and this is precisely the gap the present paper aims to fill.

The importance of undertaking this study, rather than remaining within the existing leras or classical ideal frameworks separately, is twofold. First, it extends the applicability of leras topological spaces by embedding them into the broader and well-tested machinery of ideal topological spaces, thereby making available tools such as local functions and $\star$-closure operators that have proved effective elsewhere in generating new topologies and new classes of sets. Second, it enables the identification of set classes, namely $l^\star$-closed, $l^\star$-dense in itself, and $l^\star$-perfect sets, that have no counterpart in the ideal-free leras setting, and whose interrelationships, as illustrated by the examples in this paper, are non-trivial and do not collapse into one another. Such classes may, in turn, serve as tools for future investigations into generalized continuity, separation properties, and covering properties within leras-type spaces.

Motivated by these ideas, in this paper we introduce a new class of spaces called ideal leras topological spaces, which extends the concept of leras topological spaces. The relationship between leras topological spaces and ideal leras topological spaces is examined in detail. Several new classes of closed sets are introduced and their interrelationships are studied. Fundamental properties and characterizations of these sets are established. Suitable examples are included to illustrate the introduced concepts and the obtained results.

Before proceeding further, we recall some preliminary definitions and basic concepts that will be used throughout the paper.

2 Preliminaries

Definition 2.1: *An ideal [2] I on a nonempty set X is a nonempty collection of subsets of X which satisfies*

1. *$A \in I$ and $B \subset A$ implies $B \in I$,*
2. *$A \in I$ and $B \in I$ implies $A \cup B \in I$.*

Given a topological space (X, τ) with an ideal I on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator

$$(.)^\star : \mathcal{P}(X) \to \mathcal{P}(X),$$

called a local function [6] of A with respect to τ and I is defined as follows:
For $A \subset X$,

$$A^\star(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau : x \in U\}$.
A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(I, \tau)$, called the $\star$-topology, finer than τ, is defined by

$$cl^\star(A) = A \cup A^\star(I, \tau).$$

When there is no chance for confusion, we simply write $A^\star$ for $A^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$.
If I is an ideal on X, then the space (X, τ, I) is called an ideal space.

Definition 2.2: *[3] Let X be a non-empty set equipped with a topology τ. Levine introduced the collection*

$$\tau(B) = \{O \cup (\widetilde{O} \cap B) : O, \widetilde{O} \in \tau, \text{ fixed } B \notin \tau\},$$

where $B \notin \tau$, and referred to it as the simple extension of τ by B.

1. *$\emptyset, X \in \tau(B)$,*
2. *the union of any collection of members of $\tau(B)$ belongs to $\tau(B)$,*
3. *the intersection of finitely many members of $\tau(B)$ belongs to $\tau(B)$.*

The pair $(X, \tau(B))$ is called a simply extended topological space (briefly, SETS). The members of $\tau(B)$ are known as B-open sets, while their complements are termed B-closed sets.

For any subset S of X, the B-closure of S, denoted by $B_C(S)$, is defined as the intersection of all B-closed sets containing S.

Similarly, the B-interior of S, denoted by $B_I(S)$, is defined as the union of all B-open sets contained in S.

Definition 2.3: *[5] Let $(X, \tau(B))$ be a simply extended topological space. Define*

$$LR_B(X) = \left\{\mathbb{R} \cup \mathbb{S} : \mathbb{R} = (O \cap B),\ \mathbb{S} = (\widetilde{O} \cap \widetilde{B})\right\},$$

where:

1. $O, \widetilde{O} \in \tau$ *and* $B, \widetilde{B} \notin \tau$,
2. $B \cup \widetilde{B} = X$.

The family $LR_B(X)$ *is called a leras topology on* X *if it satisfies the following axioms:*

1. $\emptyset, X \in LR_B(X)$,
2. *the union of any collection of members of* $LR_B(X)$ *belongs to* $LR_B(X)$,
3. *the intersection of finitely many members of* $LR_B(X)$ *belongs to* $LR_B(X)$.

Then the triple $(X, \tau(B), LR_B(X))$ *is called a leras topological space. The elements of* $LR_B(X)$ *are called leras-open (briefly,* lr*-open) sets. The complements of* lr*-open sets are called leras-closed (briefly,* lr*-closed) sets. The collection of all* lr*-closed sets is denoted by* $LR_{B^c}(X)$.

Let $(X, \tau(B), LR_B(X))$ *be a leras topological space, where* $\mathbb{L}^X = LR_B(X)$, *and let* $(Y, \tau(B), LR_B(Y))$ *be a leras topological space, where* $\mathbb{L}^Y = LR_B(Y)$.

Definition 2.4: *[5] The leras closure and the leras interior of a set* A *are denoted by* $l_C(A)$ *and* $l_I(A)$, *respectively, and defined by*

1. $l_C(A) = \bigcap\{N : A \subseteq N,\ N \text{ is } lr\text{-closed}\}$,
2. $l_I(A) = \bigcup\{N : N \subseteq A,\ N \text{ is } lr\text{-open}\}$.

3 Ideal leras topological spaces

To begin our study, we first recall some basic notions and preliminary definitions that will be used throughout this section.

Let $\mathbb{L}^X$ be a leras topological space. A leras topological space $\mathbb{L}^X$ with an ideal I on X is called an *ideal leras topological space* and is denoted by $(X, I, \mathbb{L}^X)$ or $\mathbb{L}^X_I$.

Definition 3.1: *Let* $\mathbb{L}^X_I$ *be an ideal leras topological space. For a subset* $K_l \subset X$,

$$K_l^\star(I, \mathbb{L}^X) = \{x \in X : K_l \cap H \notin I \text{ for every } H \in \mathbb{L}^X(x)\},$$

where $\mathbb{L}^X(x) = \{H \in \mathbb{L}^X : x \in H\}$ *is called the* leras local function *(l-local function) of* K_l *with respect to* I *and* $\mathbb{L}^X$.

Example 3.2: *Let* $(X, \mathbb{L}^X)$ *be a leras topological space with an ideal* I *on* X.

1. *If* $I = \{\emptyset\}$, *then* $H_l^\star = l_C(H)$.
2. *If* $I = \mathcal{P}(X)$, *then* $H_l^\star = \emptyset$.

Theorem 3.3: *Let* $(X, \mathbb{L}^X)$ *be a leras topological space with ideals* I, I' *on* X *and let* H, M *be subsets of* X. *Then:*

1. $H \subseteq M \Rightarrow H_l^\star \subseteq M_l^\star$.
2. $I \subseteq I' \Rightarrow H_l^\star(I') \subseteq H_l^\star(I)$.
3. $H_l^\star = l_C(H_l^\star) \subseteq l_C(H)$.

4. $(H_l^\star)_l^\star \subseteq H_l^\star$.
5. $H_l^\star \cup M_l^\star = (H \cup M)_l^\star$.
6. $H_l^\star - M_l^\star \subseteq (H - M)_l^\star - M_l^\star \subseteq (H - M)_l^\star$.
7. *If* $S \in \mathbb{L}^X$, *then* $S \cap H_l^\star = S \cap (S \cap H)_l^\star \subseteq (S \cap H)_l^\star$.
8. *If* $R \in I$, *then* $(H \cup R)_l^\star = H_l^\star = (H - R)_l^\star$.

Proof:

1. $H \subseteq M \Rightarrow H_l^\star \subseteq M_l^\star$.
   Let $x \in H_l^\star$. Then for every $K_l \in \mathcal{K}_l(x)$, we have $K_l \cap H \notin I$. Since $H \subseteq M$, we have $K_l \cap H \subseteq K_l \cap M$. By the hereditary property of I, if $K_l \cap M \in I$ then $K_l \cap H \in I$, a contradiction. Therefore $K_l \cap M \notin I$ for all $K_l \in \mathcal{K}_l(x)$, so $x \in M_l^\star$. Hence $H_l^\star \subseteq M_l^\star$.
2. $I \subseteq I' \Rightarrow H_l^\star(I') \subseteq H_l^\star(I)$.
   Let $x \in H_l^\star(I')$. Then for every $K_l \in \mathcal{K}_l(x)$, $K_l \cap H \notin I'$. Since $I \subseteq I'$, if $K_l \cap H \in I$ then $K_l \cap H \in I'$, contradicting $K_l \cap H \notin I'$. Thus $K_l \cap H \notin I$ for all $K_l \in \mathcal{K}_l(x)$, giving $x \in H_l^\star(I)$. Therefore $H_l^\star(I') \subseteq H_l^\star(I)$.
3. $H_l^\star = l_C(H_l^\star) \subseteq l_C(H)$.
   Step 1: We show $H \cup H_l^\star$ is lr-closed, which implies $l_C(H) \subseteq H \cup H_l^\star$ and hence $H_l^\star \subseteq l_C(H)$.
   Let $x \notin H \cup H_l^\star$. Then $x \notin H$ and $x \notin H_l^\star$, so there exists $K_l \in \mathcal{K}_l(x)$ with $K_l \cap H \in I$. We claim $K_l \cap H_l^\star = \emptyset$. Suppose $y \in K_l \cap H_l^\star$. Since K_l is lr-open and $y \in K_l$, we have $K_l \in \mathcal{K}_l(y)$, and $K_l \cap H \in I$, contradicting $y \in H_l^\star$. Therefore $K_l \cap H_l^\star = \emptyset$, so

   $$K_l \cap (H \cup H_l^\star) = (K_l \cap H) \cup (K_l \cap H_l^\star) = K_l \cap H \in I.$$

   This means $x \notin (H \cup H_l^\star)_l^\star$, so $(H \cup H_l^\star)_l^\star \subseteq H \cup H_l^\star$, which shows $H \cup H_l^\star$ is lr-closed. Since $H \subseteq H \cup H_l^\star$, we have $l_C(H) \subseteq H \cup H_l^\star$. In particular, $H_l^\star \subseteq H \cup H_l^\star$ and any lr-closed set containing H also contains $H_l^\star$, giving $H_l^\star \subseteq l_C(H)$.
   Step 2: We show $l_C(H_l^\star) \subseteq H_l^\star$.
   Applying Step 1 to $H_l^\star$ in place of H gives $H_l^\star \cup (H_l^\star)_l^\star$ is lr-closed. By part (4) (proved independently below), $(H_l^\star)_l^\star \subseteq H_l^\star$, so $H_l^\star \cup (H_l^\star)_l^\star = H_l^\star$ is lr-closed. Therefore $l_C(H_l^\star) \subseteq H_l^\star$. Combined with $H_l^\star \subseteq l_C(H_l^\star)$, we obtain $H_l^\star = l_C(H_l^\star)$.
4. $(H_l^\star)_l^\star \subseteq H_l^\star$.
   Let $x \in (H_l^\star)_l^\star$. Then for every $K_l \in \mathcal{K}_l(x)$, $K_l \cap H_l^\star \notin I$, so in particular $K_l \cap H_l^\star \neq \emptyset$. Pick $y \in K_l \cap H_l^\star$. Since K_l is lr-open and $y \in K_l$, we have $K_l \in \mathcal{K}_l(y)$. Since $y \in H_l^\star$, it follows that $K_l \cap H \notin I$. Since $K_l \in \mathcal{K}_l(x)$ was arbitrary and $K_l \cap H \notin I$, we conclude $x \in H_l^\star$. Hence $(H_l^\star)_l^\star \subseteq H_l^\star$.
5. $H_l^\star \cup M_l^\star = (H \cup M)_l^\star$.
   ($\supseteq$): Since $H \subseteq H \cup M$ and $M \subseteq H \cup M$, part (1) gives $H_l^\star \subseteq (H \cup M)_l^\star$ and $M_l^\star \subseteq (H \cup M)_l^\star$. Hence $H_l^\star \cup M_l^\star \subseteq (H \cup M)_l^\star$.
   ($\subseteq$): Suppose $x \in (H \cup M)_l^\star$ but $x \notin H_l^\star \cup M_l^\star$. Then:
   - $x \notin H_l^\star$: there exists $K_l^1 \in \mathcal{K}_l(x)$ with $K_l^1 \cap H \in I$.
   - $x \notin M_l^\star$: there exists $K_l^2 \in \mathcal{K}_l(x)$ with $K_l^2 \cap M \in I$.

Let $K_l = K_l^1 \cap K_l^2 \in \mathcal{K}_l(x)$ (since $\mathbb{L}^X$ is closed under finite intersections). Then

$$K_l \cap (H \cup M) = (K_l \cap H) \cup (K_l \cap M) \subseteq (K_l^1 \cap H) \cup (K_l^2 \cap M) \in I,$$

since I is closed under finite unions and subsets. This contradicts $x \in (H \cup M)_l^\star$. Therefore $(H \cup M)_l^\star \subseteq H_l^\star \cup M_l^\star$, and so $(H \cup M)_l^\star = H_l^\star \cup M_l^\star$.

6. $H_l^\star - M_l^\star \subseteq (H - M)_l^\star - M_l^\star \subseteq (H - M)_l^\star$.
Let $x \in H_l^\star - M_l^\star$. Then $x \in H_l^\star$ and $x \notin M_l^\star$. Since $x \notin M_l^\star$, there exists $K_l^0 \in \mathcal{K}_l(x)$ with $K_l^0 \cap M \in I$. For any $K_l \in \mathcal{K}_l(x)$, set $K = K_l \cap K_l^0 \in \mathcal{K}_l(x)$. Then $K \cap M \subseteq K_l^0 \cap M \in I$, so $K \cap M \in I$. Since $x \in H_l^\star$, $K \cap H \notin I$. Now

$$K \cap H = \big(K \cap (H - M)\big) \cup (K \cap H \cap M).$$

Since $K \cap H \cap M \subseteq K \cap M \in I$, if $K \cap (H - M) \in I$ then $K \cap H \in I$, a contradiction. Therefore $K \cap (H - M) \notin I$. Since this holds for every $K_l \in \mathcal{K}_l(x)$, we obtain $x \in (H - M)_l^\star$. Also $x \notin M_l^\star$, so $x \in (H - M)_l^\star - M_l^\star$. Hence $H_l^\star - M_l^\star \subseteq (H - M)_l^\star - M_l^\star \subseteq (H - M)_l^\star$.

7. *If $S \in \mathbb{L}^X$, then $S \cap H_l^\star = S \cap (S \cap H)_l^\star \subseteq (S \cap H)_l^\star$.*
Let $x \in S \cap H_l^\star$. Then $x \in S$ and $x \in H_l^\star$. For any $K_l \in \mathcal{K}_l(x)$, the intersection $K_l \cap S$ is lr-open (as $S \in \mathbb{L}^X$) and contains x, so $K_l \cap S \in \mathcal{K}_l(x)$. Since $x \in H_l^\star$:

$$(K_l \cap S) \cap H = K_l \cap (S \cap H) \notin I.$$

Since $K_l \in \mathcal{K}_l(x)$ was arbitrary, this shows $x \in (S \cap H)_l^\star$. Hence $S \cap H_l^\star \subseteq (S \cap H)_l^\star$, and intersecting both sides with S gives

$$S \cap H_l^\star = S \cap (S \cap H_l^\star) \subseteq S \cap (S \cap H)_l^\star \subseteq (S \cap H)_l^\star.$$

8. *If $R \in I$, then $(H \cup R)_l^\star = H_l^\star = (H - R)_l^\star$.*
Part A: $(H \cup R)_l^\star = H_l^\star$.
($\subseteq$): Let $x \in (H \cup R)_l^\star$. For every $K_l \in \mathcal{K}_l(x)$, $K_l \cap (H \cup R) \notin I$. Since $K_l \cap R \subseteq R \in I$, we have $K_l \cap R \in I$. If $K_l \cap H \in I$, then $K_l \cap (H \cup R) = (K_l \cap H) \cup (K_l \cap R) \in I$, a contradiction. So $K_l \cap H \notin I$, giving $x \in H_l^\star$.
($\supseteq$): Since $H \subseteq H \cup R$, part (1) gives $H_l^\star \subseteq (H \cup R)_l^\star$. Hence $(H \cup R)_l^\star = H_l^\star$.
Part B: $(H - R)_l^\star = H_l^\star$.
($\subseteq$): Since $H - R \subseteq H$, part (1) gives $(H - R)_l^\star \subseteq H_l^\star$.
($\supseteq$): Let $x \in H_l^\star$. For every $K_l \in \mathcal{K}_l(x)$, $K_l \cap H \notin I$. Since $K_l \cap H \cap R \subseteq R \in I$, we have $K_l \cap H \cap R \in I$. Now

$$K_l \cap H = \big(K_l \cap (H - R)\big) \cup (K_l \cap H \cap R).$$

If $K_l \cap (H - R) \in I$, then $K_l \cap H \in I$, a contradiction. So $K_l \cap (H - R) \notin I$ for all $K_l \in \mathcal{K}_l(x)$, giving $x \in (H - R)_l^\star$. Hence $H_l^\star \subseteq (H - R)_l^\star$, and thus $(H - R)_l^\star = H_l^\star$.

□

Theorem 3.4: *If $\mathbb{L}_I^X$ is an ideal leras topological space with an ideal I and $H \subseteq H_l^\star$, then $H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H)$.*

Proof:

Suppose $H \subseteq H_l^\star$.

Step 1: $H_l^\star = l_C(H_l^\star) = l_C(H)$.

By Theorem 3.3(3), $H_l^\star = l_C(H_l^\star) \subseteq l_C(H)$. Since $H \subseteq H_l^\star$, monotonicity of l_C gives $l_C(H) \subseteq l_C(H_l^\star)$. Therefore $H_l^\star = l_C(H_l^\star) = l_C(H)$.

Step 2: $l_C(H) = l_C^\star(H)$.

Since $H \subseteq H_l^\star$ by hypothesis, $H \cup H_l^\star = H_l^\star$. Thus, using Definition 3.5,
$l_C^\star(H) = H \cup H_l^\star = H_l^\star = l_C(H)$.

Combining Steps 1 and 2:

$$H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H). \ \square$$

Definition 3.5: *The set operator $l_C^\star$ is called the* leras $\star$-closure *and is defined as*

$$l_C^\star(H) = H \cup H_l^\star.$$

Example 3.6: *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$, and fix $B = \{a, b\} \notin \tau$ and $\widetilde{B} = \{c\} \notin \tau$, so that $B \cup \widetilde{B} = X$.*

By Definition 2.3, the leras topology is

$$\mathbb{L}^X = \big\{\emptyset,\ \{a, b\},\ \{c\},\ X\big\},$$

which is closed under arbitrary unions and finite intersections. The lr-closed sets are

$$(\mathbb{L}^X)^c = \big\{\emptyset,\ \{a, b\},\ \{c\},\ X\big\}.$$

The leras neighbourhoods of each point are:

$$\mathcal{K}_l(a) = \{\{a, b\},\ X\}, \quad \mathcal{K}_l(b) = \{\{a, b\},\ X\}, \quad \mathcal{K}_l(c) = \{\{c\},\ X\}.$$

Take the ideal $I = \{\emptyset, \{a\}\}$ on X, and let $H = \{a, b\}$.

$H_l^\star$ and $l_C^\star(H)$ using $H_l^\star = \big\{x \in X \ :\ K_l \cap H \notin I \text{ for all } K_l \in \mathcal{K}_l(x)\big\}$.

***Check** $x = a$**:** $\{a, b\} \cap \{a, b\} = \{a, b\} \notin I$ and $X \cap \{a, b\} = \{a, b\} \notin I$. So $a \in H_l^\star$.*

***Check** $x = b$**:** $\{a, b\} \cap \{a, b\} = \{a, b\} \notin I$ and $X \cap \{a, b\} = \{a, b\} \notin I$. So $b \in H_l^\star$.*

***Check** $x = c$**:** $\{c\} \cap \{a, b\} = \emptyset \in I$. Since there exists $K_l \in \mathcal{K}_l(c)$ with $K_l \cap H \in I$, we have $c \notin H_l^\star$.*

Therefore $H_l^\star = \{a, b\}$, and the leras $\star$-closure is

$l_C^\star(H) = H \cup H_l^\star = \{a, b\} \cup \{a, b\} = \{a, b\}$.

Theorem 3.7: *The operator $l_C^\star$ satisfies:*

1. $H \subseteq l_C^\star(H)$.
2. $l_C^\star(\emptyset) = \emptyset$ *and* $l_C^\star(X) = X$.
3. $H \subseteq M \Rightarrow l_C^\star(H) \subseteq l_C^\star(M)$.
4. $l_C^\star(H) \cup l_C^\star(M) = l_C^\star(H \cup M)$.
5. $l_C^\star(l_C^\star(H)) = l_C^\star(H)$.

Proof:

1. By Definition, $l_C^\star(H) = H \cup H_l^\star \supseteq H$.
2. $l_C^\star(\emptyset) = \emptyset$ *and* $l_C^\star(X) = X$.
   For any $x \in X$ and $K_l \in \mathcal{K}_l(x)$, $K_l \cap \emptyset = \emptyset \in I$, so no point belongs to $\emptyset_l^\star$, giving $\emptyset_l^\star = \emptyset$. Therefore $l_C^\star(\emptyset) = \emptyset \cup \emptyset = \emptyset$.
   By part (1), $X \subseteq l_C^\star(X) = X \cup X_l^\star \subseteq X$, so $l_C^\star(X) = X$.
3. $H \subseteq M \Rightarrow l_C^\star(H) \subseteq l_C^\star(M)$.
   By Theorem 3.3(1), $H_l^\star \subseteq M_l^\star$. Therefore $l_C^\star(H) = H \cup H_l^\star \subseteq M \cup M_l^\star = l_C^\star(M)$.
4. $l_C^\star(H) \cup l_C^\star(M) = l_C^\star(H \cup M)$.
   Using Definition 3.5 and Theorem 3.3(5):

$$\begin{aligned} l_C^\star(H) \cup l_C^\star(M) &= (H \cup H_l^\star) \cup (M \cup M_l^\star) \\ &= (H \cup M) \cup (H_l^\star \cup M_l^\star) \\ &= (H \cup M) \cup (H \cup M)_l^\star \\ &= l_C^\star(H \cup M). \end{aligned}$$

5. $l_C^\star(l_C^\star(H)) = l_C^\star(H)$.
   Let $A = l_C^\star(H) = H \cup H_l^\star$. By Theorem 3.3(5):

$$(H \cup H_l^\star)_l^\star = H_l^\star \cup (H_l^\star)_l^\star.$$

   By Theorem 3.3(4), $(H_l^\star)_l^\star \subseteq H_l^\star$, so $H_l^\star \cup (H_l^\star)_l^\star = H_l^\star$. Therefore:

$$l_C^\star(A) = A \cup A_l^\star = (H \cup H_l^\star) \cup H_l^\star = H \cup H_l^\star = A = l_C^\star(H). \ \square$$

Definition 3.8: *A basis $\beta(I, \mathbb{L}^X)$ for $(\mathbb{L}^X)_I^\star$ is given by*

$\beta(I, \mathbb{L}^X) = \{H - M \, : \, H \in \mathbb{L}^X, \ M \in I\}$.

Theorem 3.9: *Let $\mathbb{L}_I^X$ be an ideal leras topological space with an ideal I on X and $H \subseteq X$. If $H \subseteq H_l^\star$, then $H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H)$.*

Proof:

Suppose $H \subseteq H_l^\star$.

Step 1: $H_l^\star = l_C(H_l^\star) = l_C(H)$.

By Theorem 3.3(3), $H_l^\star = l_C(H_l^\star) \subseteq l_C(H)$. Since $H \subseteq H_l^\star$, monotonicity of l_C gives $l_C(H) \subseteq l_C(H_l^\star)$. Therefore $H_l^\star = l_C(H_l^\star) = l_C(H)$.

Step 2: $l_C(H) = l_C^\star(H)$.

Since $H \subseteq H_l^\star$ by hypothesis, $H \cup H_l^\star = H_l^\star$. Thus $l_C^\star(H) = H \cup H_l^\star = H_l^\star = l_C(H)$.

Combining Steps 1 and 2: $H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H)$. □

Definition 3.10: *A subset H of an ideal leras topological space $\mathbb{L}_I^X$ is called:*

1. $l^\star$-closed *if $H_l^\star \subseteq H$,*
2. $l^\star$-dense in itself *if $H \subseteq H_l^\star$,*
3. $l^\star$-perfect *if $H = H_l^\star$.*

Remark 3.11: *For the relationships among the several sets defined in this paper, we have the following diagram:*

$l^\star$-perfect $\longrightarrow$ $l^\star$-dense in itself $\longrightarrow$ not necessarily $l^\star$-closed

and

$l^\star$-perfect $\longrightarrow$ $l^\star$-closed.

The following examples show that the converse implications do not hold in general.

Example 3.12: *We use the setup of Example 3.6: $X = \{a, b, c\}$, $\mathbb{L}^X = \{\emptyset, \{a, b\}, \{c\}, X\}$, ideal $I = \{\emptyset, \{a\}\}$, with leras neighbourhoods $\mathcal{K}_l(a) = \mathcal{K}_l(b) = \{\{a, b\}, X\}$ and $\mathcal{K}_l(c) = \{\{c\}, X\}$.*

Case 1:** $H = \{a, b\}$ **is $l^\star$-perfect.

- *$x = a$: $\{a, b\} \cap \{a, b\} = \{a, b\} \notin I$, $X \cap \{a, b\} = \{a, b\} \notin I$. So $a \in H_l^\star$.*
- *$x = b$: same as $x = a$, so $b \in H_l^\star$.*
- *$x = c$: $\{c\} \cap \{a, b\} = \emptyset \in I$. So $c \notin H_l^\star$.*

*Thus $H_l^\star = \{a, b\} = H$. Since $H = H_l^\star$, the set $H = \{a, b\}$ is $l^\star$-**perfect**, hence also $l^\star$-dense in itself and $l^\star$-closed.*

Case 2:** $H = \{b, c\}$ **is $l^\star$-dense in itself but not $l^\star$-perfect.

- *$x = a$: $\{a, b\} \cap \{b, c\} = \{b\} \notin I$, $X \cap \{b, c\} = \{b, c\} \notin I$. So $a \in H_l^\star$.*
- *$x = b$: $\{a, b\} \cap \{b, c\} = \{b\} \notin I$, $X \cap \{b, c\} = \{b, c\} \notin I$. So $b \in H_l^\star$.*
- *$x = c$: $\{c\} \cap \{b, c\} = \{c\} \notin I$, $X \cap \{b, c\} = \{b, c\} \notin I$. So $c \in H_l^\star$.*

*Thus $H_l^\star = X = \{a, b, c\}$. Since $H = \{b, c\} \subseteq X = H_l^\star$, the set H is **$l^\star$-dense in itself**. However, $H \neq H_l^\star$ (since $a \in H_l^\star \setminus H$), so H is **not $l^\star$-perfect**. This shows that $l^\star$-dense in itself does not imply $l^\star$-perfect.*

Case 3:** $H = \{a\}$ **is $l^\star$-closed but neither $l^\star$-dense in itself nor $l^\star$-perfect.

- *$x = a$: $\{a, b\} \cap \{a\} = \{a\} \in I$. So $a \notin H_l^\star$.*
- *$x = b$: $\{a, b\} \cap \{a\} = \{a\} \in I$. So $b \notin H_l^\star$.*
- *$x = c$: $\{c\} \cap \{a\} = \emptyset \in I$. So $c \notin H_l^\star$.*

*Thus $H_l^\star = \emptyset \subseteq \{a\} = H$, so $H = \{a\}$ is **$l^\star$-closed**. Since $H = \{a\} \nsubseteq \emptyset = H_l^\star$, it is **not $l^\star$-dense in itself** and **not $l^\star$-perfect**. This shows that $l^\star$-closed does not imply $l^\star$-dense in itself.*

Lemma 3.13: *Let $\mathbb{L}_I^X$ be an ideal leras topological space and $H \subseteq X$. If H is $l^\star$-dense in itself, then $H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H)$.*

Proof:

Let H be $l^\star$-dense in itself, so $H \subseteq H_l^\star$. By Theorem 3.9, $H_l^\star = l_C(H_l^\star) = l_C(H)$. Since $H \subseteq H_l^\star$, we also have $l_C^\star(H) = H \cup H_l^\star = H_l^\star$. Therefore $H_l^\star = l_C(H_l^\star) = l_C(H) = l_C^\star(H)$. □

Definition 3.14: *An ideal I in a space $\mathbb{L}_I^X$ is called $\mathbb{L}^X$-codense if*

$\mathbb{L}^X \cap I = \{\emptyset\}$.

Theorem 3.15: *Let $\mathbb{L}_I^X$ be a leras ideal topological space and let I be $\mathbb{L}^X$-codense. Then $X = X_l^\star$.*

Proof:

Since $X_l^\star \subseteq X$ always, it suffices to show $X \subseteq X_l^\star$. Let $x \in X$ and let $K_l \in \mathcal{K}_l(x)$ be any lr-open set containing x. Since I is $\mathbb{L}^X$-codense, $\mathbb{L}^X \cap I = \{\emptyset\}$. Since $x \in K_l$, we have $K_l \neq \emptyset$, so $K_l \notin I$. Therefore $K_l \cap X = K_l \notin I$. Since $K_l \in \mathcal{K}_l(x)$ was arbitrary, $x \in X_l^\star$. Hence $X = X_l^\star$. □

Theorem 3.16: *Let $\mathbb{L}_I^X$ be a leras ideal topological space. Then the following are equivalent:*

1. *$X = X_l^\star$.*
2. *$\mathbb{L}^X \cap I = \{\emptyset\}$.*
3. *If $R \in I$, then $l_I(R) = \emptyset$.*
4. *For every $H \in \mathbb{L}^X$, $H \subseteq H_l^\star$.*

Proof:

We prove the equivalences in the cycle (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (2)**:** Suppose $X = X_l^\star$. Let $K_l \in \mathbb{L}^X \cap I$. Suppose for contradiction that $K_l \neq \emptyset$, so there exists $x \in K_l$. Since $K_l \in \mathbb{L}^X$ and $x \in K_l$, we have $K_l \in \mathcal{K}_l(x)$. By assumption $x \in X = X_l^\star$, so for every $K \in \mathcal{K}_l(x)$, $K \cap X \notin I$. In particular, $K_l \cap X = K_l \notin I$, contradicting $K_l \in I$. Hence $K_l = \emptyset$, and so $\mathbb{L}^X \cap I = \{\emptyset\}$.

(2) ⇒ (3)**:** Suppose $\mathbb{L}^X \cap I = \{\emptyset\}$. Let $R \in I$. Recall that $l_I(R) = \bigcup\{K_l \in \mathbb{L}^X : K_l \subseteq R\}$. Let $K_l \in \mathbb{L}^X$ with $K_l \subseteq R$. Since $R \in I$ and I is hereditary, $K_l \in I$. But $\mathbb{L}^X \cap I = \{\emptyset\}$, so $K_l = \emptyset$. Hence every lr-open set contained in R is empty, giving $l_I(R) = \emptyset$.

(3) ⇒ (4)**:** Suppose that for every $R \in I$, $l_I(R) = \emptyset$. Let $H \in \mathbb{L}^X$ and let $x \in H$. We show $x \in H_l^\star$. Let $K_l \in \mathcal{K}_l(x)$. Then $x \in K_l \cap H$, so $K_l \cap H \neq \emptyset$. Suppose for contradiction that $K_l \cap H \in I$. Since $H \in \mathbb{L}^X$ and $K_l \in \mathbb{L}^X$, their intersection $K_l \cap H \in \mathbb{L}^X$. By hypothesis, $l_I(K_l \cap H) = \emptyset$. But $K_l \cap H \in \mathbb{L}^X$ and $K_l \cap H \subseteq K_l \cap H$, so $K_l \cap H \subseteq l_I(K_l \cap H) = \emptyset$, giving $K_l \cap H = \emptyset$, which contradicts $x \in K_l \cap H$. Therefore $K_l \cap H \notin I$, and since $K_l \in \mathcal{K}_l(x)$ was arbitrary, $x \in H_l^\star$. Hence $H \subseteq H_l^\star$.

(4) ⇒ (1)**:** Suppose that for every $H \in \mathbb{L}^X$, $H \subseteq H_l^\star$. We show $X \subseteq X_l^\star$. Let $x \in X$ and $K_l \in \mathcal{K}_l(x)$. Then $K_l \in \mathbb{L}^X$, so by hypothesis $K_l \subseteq (K_l)_l^\star$, meaning $x \in (K_l)_l^\star$. By the definition of the local function, for every $U \in \mathcal{K}_l(x)$, $U \cap K_l \notin I$. Note that for any $V \in \mathcal{K}_l(x)$, $V \in \mathbb{L}^X$ and $V \neq \emptyset$ (since $x \in V$), so by condition (2) (which follows from condition (4) as shown above) $V \notin I$. Hence $V \cap X = V \notin I$ for all $V \in \mathcal{K}_l(x)$, giving $x \in X_l^\star$. Therefore $X = X_l^\star$. □

4 Conclusions

In this paper, the concept of ideal leras topological spaces was introduced as a natural extension of leras topological spaces. The leras local function and leras $\star$-closure operator were defined and their fundamental properties were established. New classes of sets, namely $l^\star$-closed, $l^\star$-dense in itself, and $l^\star$-perfect sets, were investigated and their relationships were studied. Several characterizations and illustrative examples were provided. These results contribute to the development of ideal structures in leras topology and offer a basis for further research in generalized topological spaces.

The present work opens several directions for future investigation. First, the notion of $\mathbb{L}^X$-codense ideals was introduced only briefly in this paper; a systematic study of codense and completely codense ideals in leras topological spaces, along with their effect on the leras $\star$-topology, would be a natural continuation of this work. In particular, it would be of interest to characterize when the leras $\star$-topology $(\mathbb{L}^X)^\star$ coincides with $\mathbb{L}^X$ itself, extending known results from classical ideal topological spaces to this setting.

Second, the basis $\beta(I, \mathbb{L}^X)$ defined in this paper for the leras $\star$-topology deserves further exploration. Establishing that $(\mathbb{L}^X)^\star$ is indeed generated by this basis, and studying the resulting topology's separation axioms, compactness-type properties, and connectedness in relation to the original leras topology, would strengthen the structural understanding of ideal leras spaces.

Third, building on the notions of $l^\star$-closed, $l^\star$-dense in itself, and $l^\star$-perfect sets introduced here, one could define and study further weak and strong forms of open and closed sets in ideal leras topological spaces, such as leras-I-open sets, leras-I-continuous functions, and leras-I-compact spaces, in analogy with corresponding notions already well developed in ideal topological spaces. Investigating decomposition theorems for leras continuity using ideals, in the spirit of Noiri and Popa [4], is another promising avenue.

Fourth, the interplay between the ideal I and the two ordinary topologies underlying the simple extension construction (that is, the topology τ and the sets $B, \widetilde{B}$ used to define $LR_B(X)$) has not been fully explored in this paper. A more refined study of how choices of I, B, and $\widetilde{B}$ jointly influence the leras local function and the resulting $\star$-topology may yield sharper characterizations and new classes of ideal leras spaces.

Finally, extending the present framework to function spaces, product spaces, and subspaces of ideal leras topological spaces, as well as examining possible applications to digital topology and rough set theory, where ideal structures have already found practical use, represent worthwhile directions for future research.

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