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Research article

Analysis of a host-vector disease model with fixed moments pulse interventions

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Abstract: In our study, we propose a vector-host epidemic model with fixed-time pulse interventions guided by population size of susceptible humans and susceptible vectors. This approach aims to reflect the situation where public health interventions, namely, vaccination campaigns, mass treatments, insecticide spraying, or sterile mosquito releases, are generally implemented periodically rather than continuously. A saturated incidence function is used to fit the situation where the number of infective cases is getting larger with the limited medical facilities. The model incorporates a general rule of implementation of control measures such us vaccination, treatment and mosquitoes elimination. We theoretically investigate the dynamic behavior with the appearance of $(k+m)T$- periodic solutions. Firstly, we derive the existence and stability of the order-1 $(k+1)T$- disease-free periodic solution with nonnegative integer k and the order-m $(1+m)T$-disease-free periodic solution with m being any positive integer. Furthermore, we study the existence and stability of the disease-free $(1+2)T$-periodic solution. In addition, numerical simulations are performed to support the theoretical results. Our results indicate that there exists a critical monitoring period or vaccination rate such that the intervention strategy can successfully control the disease.

Mathematics Subject Classification: [2020]: 34D20, 37C75, 92B05
Keywords: Vector-borne disease; Impulsive dynamical system; Periodic solution; Stability.

1 Introduction

Vector-borne diseases remains a major health challenge for global public health (see [1]). These diseases are mainly caused by a vector and the most common is mosquito. This vector is responsible to the transmission of malaria, dengue fever, chikungunya, zika, yellow fever and West Nile virus (see [3], [4], [14]). Among the prevention methods of these disease, vaccination and treatment remains effective method used to control the disease propagation. Recently, several studies have been carried out to investigate different strategies for vaccination and treatment implementation method. Wang et al. [13], formulated a mathematical model for vector-borne disease by incorporating age of vaccination and infection to analyze their effects for the spread of vector-borne. In [6], formulated a stochastic vector-borne epidemic model with to discuss the effects of stochastic perturbation a vaccination age in the spread of vector-borne. Further, Tang et al. [12] developed a fractional order model to analyse the dynamics of a vector-borne infection with effect of imperfect vaccination. Moreover, Ndii [5] studied the effects of vaccination, vector controls and media on dengue transmission dynamics with a seasonally varying mosquito population. Many studies have taken into account prevention and control strategies pulsed at fixed times, assuming that interventions were administered at fixed times ([2], [9], [10], [11], [15]). However, these models do not take into account the influence of the prevalence of infectious diseases on the administration of interventions, which can be a waste of medical resources. In a context of insufficient resources, particularly in regions with a high health challenge, a good implementation of treatment policies is important for the control of vector-borne epidemics. Furthermore, contrary to continuous models, impulsive dynamical systems offer a natural mathematical framework by coupling continuous dynamics between intervention times with discrete jumps representing the instantaneous effect of control measures. In this work, we formulated a mathematical model for a vector-borne epidemic with a fixed-time impulsive interventions with a threshold policy guided by the populations of susceptible humans and vectors. At the end of each monitoring period T, we checked whether or not it is possible to administer a treatment. This formulation thus allows for a more faithful and operational modeling of the strategies actually implemented in the field. The aim here is to assess the theoretical effects of this type of implementation policy in controlling or eradicating an epidemic involving a vector. We structure the remainder part of this paper as follows: in section 2, we formulate the model with some basic properties. Section 3 is devoted to the study of the existence and uniqueness of disease-free periodic solution (DFPS). Numerical simulations are carried out and presented in section 4. In section 5 the paper ends with a conclusion.

2 Model formulation and preliminaries

In this part, we give some preliminaries results and the model definition.

2.1 Preliminaries results

We consider the following impulsive system

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx_1(t)}{dt} &= F_1(x_1(t), x_2(t), x_3(t), x_4(t)),\\
\frac{dx_2(t)}{dt} &= F_2(x_1(t), x_2(t), x_3(t), x_4(t)),\\
\frac{dx_3(t)}{dt} &= F_3(x_1(t), x_2(t), x_3(t), x_4(t)),\\
\frac{dx_4(t)}{dt} &= F_4(x_1(t), x_2(t), x_3(t), x_4(t)),
\end{aligned}\right\} t \neq nT, n \in \mathbb{N},\\
\\
x_1(nT^+) = \psi_1\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),\\
x_2(nT^+) = \psi_2\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),\\
x_3(nT^+) = \psi_3\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),\\
x_4(nT^+) = \psi_4\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),
\end{cases}
\tag{1}
$$

where x_1, x_2, x_3, $x_4 \in \mathbb{R}$ and ψ_1, ψ_2, ψ_3 and ψ_4 are positive and suitable smooth functions. A solution X(t)=$(x_1(t), x_2(t), x_3(t), x_4(t))$ of (1) is a function defined in $\mathbb{R}^+$, with non-negative components, continuously differentiable in $\mathbb{R}^+ \setminus \{nT, n \in \mathbb{N}\}$. Let φ be the flow associated to the first four equations of system (1), then

$$X(t) = \varphi(t, X_0),\ \ 0 < t \leq T,$$

where $X_0 = X(0)$ and $X(T^+) = \psi(X(T)) = \psi(\varphi(T, X_0))$ with $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$.

We now state the following definition about a periodic solution (PS).

Definition 1: *Suppose that $X(t)$ is a solution of system* (1). *$X(t)$ is called a hT-periodic solution (hT-PS) of system* (1), *if $X((i+h)T) = X(iT)$ while $X((i+l)T) \neq X(iT)$, $l = 1, ..., h-1$ hold for all $i \in \mathbb{N}$.*

Now, let

$$\Omega = \{(t, x_1, x_2, x_3, x_4)/x_1(t) \geq E_T, t = nT, n \in \mathbb{N}\},$$

where E_T is the population size threshold of individuals x_1 that determines whether we apply or not intervention. Further, we consider the following impulsive model with threshold policy.

$$
\begin{cases}
\left.\begin{aligned}
\frac{dx_1(t)}{dt} &= F_1(x_1(t), x_2(t), x_3(t), x_4(t)),\\
\frac{dx_2(t)}{dt} &= F_2(x_1(t), x_2(t), x_3(t), x_4(t)),\\
\frac{dx_3(t)}{dt} &= F_3(x_1(t), x_2(t), x_3(t), x_4(t)),\\
\frac{dx_4(t)}{dt} &= F_4(x_1(t), x_2(t), x_3(t), x_4(t)),
\end{aligned}\right\}(t, x_1, x_2, x_3, x_4) \notin \Omega,\\
\\
\left.\begin{aligned}
x_1(nT^+) &= \psi_1\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),\\
x_2(nT^+) &= \psi_2\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),\\
x_3(nT^+) &= \psi_3\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),\\
x_4(nT^+) &= \psi_4\big(x_1(nT), x_2(nT), x_3(nT), x_4(nT)\big),
\end{aligned}\right\}(t, x_1, x_2, x_3, x_4) \in \Omega.
\end{cases}
\tag{2}
$$

For system (2), there may exist new types of periodic solution (PS) with the period being integral multiples of T, which are generated after k monitoring moments without implementing prevention and control strategy and m monitoring moments with carrying out interventions. Then, the following definition is given.

Definition 2: *Let $\tilde{X}(t) = (\tilde{x_1}(t), \tilde{x_2}(t), \tilde{x_3}(t), \tilde{x_4}(t))$ be a solution of system (2). If there are k monitoring timings $t_i \in \{nT, n = 1, ..., k+m\}$, $i = 1, ..., k$, such that $\tilde{x}_1(t_i) < E_T$, and m monitoring timings $t_j \in \{nT, n = 1, ..., k+m\}$, $j = 1, ..., m$, such that $\tilde{x}_1(t_j) \geq E_T$, then $\tilde{X}(t)$ is called a $(k+m)T$-PS, with k being any non-negative integer and m being any positive integer. Particularly, if $k = 0$ and m is any positive integer, then $(0+m)T$-PS $(\tilde{x_1}(t), \tilde{x_2}(t), \tilde{x_3}(t), \tilde{x_4}(t))$ of system (2) corresponds to the order-m mT-PS of model (1). All $(k+m)T$-PS for any non-negative integer k and positive integer m are referred as the order-m PS.*

2.2 The model

We formulate a mathematical model with saturated incidence functions and fixed-time pulse interventions guided by population size of susceptible humans and vectors. We divide the human population denoted by N_h into three distinct subclasses which are susceptible S_h, infectious I_h and recovered R_h, so that at any instant t $N_h = S_h + I_h + R_h$. The total population of vectors N_v is categorized into two subclasses, namely, susceptible S_v and infectious vectors I_v, so $N_v = S_v + I_v$. Susceptible humans are recruited at a constant rate Λ_h and acquired infection after having a contact with an infectious vector I_v at the rate $\frac{\beta S_h I_v}{1+c_1 I_v}$, where $c_1 \geq 0$ is the saturation level of the force of infection and β is the disease transmission rate from infectious vectors to susceptible human. The natural death rate of human is μ_h, γ is the recovered rate and d is the disease-induced death rate. Susceptible vectors are recruited at a constant rate Λ_v and acquired infection after having a contact with an infectious human I_h at the rate $\frac{\xi S_v I_h}{1+c_2 I_h}$ where ξ is the disease transmission rate from human to vector and c_2 is the level of saturation. The natural death rate of vectors is μ_v. From the above description, we obtain the following system of differential equations :

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_h - \frac{\beta S_h I_v}{1+c_1 I_v} - \mu_h S_h, \\ \dfrac{dI_h(t)}{dt} = \frac{\beta S_h I_v}{1+c_1 I_v} - (\mu_h + \gamma + d) I_h, \\ \dfrac{dR_h(t)}{dt} = \gamma I_h - \mu_h R_h, \\ \dfrac{dS_v(t)}{dt} = \Lambda_v - \frac{\xi S_v I_h}{1+c_2 I_h} - \mu_v S_v, \\ \dfrac{dI_v(t)}{dt} = \frac{\xi S_v I_h}{1+c_2 I_h} - \mu_v I_v. \end{cases} \tag{3}$$

Now, we assume that the epidemic is under surveillance at periodic fixed times, when the size of the susceptible human and vectors populations are greater than or equal to S_{hT} and S_{vT} respectively, then a vaccination policy is applied at rate p_h, infected humans are treated at rate q_h and vectors are eliminated at rate p_v. Since the component R_h do not appear in the remaining equations of (3), we obtain the following impulsive model that describe the susceptibles-guided pulse interventions administered at monitoring timings :

$$\begin{cases} \left.\begin{array}{l} \dfrac{dS_h}{dt} = \Lambda_h - \frac{\beta S_h I_v}{1+c_1 I_v} - \mu_h S_h \\ \dfrac{dI_h}{dt} = \frac{\beta S_h I_v}{1+c_1 I_v} - (\mu_h + \gamma + d) I_h \\ \dfrac{dS_v}{dt} = \Lambda_v - \frac{\xi S_v I_h}{1+c_2 I_h} - \mu_v S_v \\ \dfrac{dI_v}{dt} = \frac{\xi S_v I_h}{1+c_2 I_h} - \mu_v I_v \end{array}\right\} (t, S_h, I_h, S_v, I_v) \notin \Gamma, \\ \\ \left.\begin{array}{l} S_h(t^+) = (1 - p_h) S_h(t) \\ I_h(t^+) = (1 - q_h) I_h(t) \\ S_v(t^+) = (1 - p_v) S_v(t) \\ I_v(t^+) = (1 - p_v) I_v(t) \end{array}\right\} (t, S_h, I_h, S_v, I_v) \in \Gamma, \end{cases} \tag{4}$$

where

$$\Gamma = \Big\{ (t, S_h, I_h, S_v, I_v) \in \mathbb{N} \times \mathbb{R}_4^+ / S_h(t) \geq S_{hT}, S_v(t) \geq S_{vT}, t = nT, n \in \mathbb{N} \Big\}.$$

3 Disease-free periodic solution (DFPS)

In this section, we investigate the dynamics of the disease-free periodic solution of system (4). In the absence of the disease, the infected compartment are empty, that is $I_h = 0$ and $I_v = 0$. Thus, the following system is obtained :

$$\begin{cases} \dfrac{dS_h(t)}{dt} = \Lambda_h - \mu_h S_h, \\ \dfrac{dS_v(t)}{dt} = \Lambda_v - \mu_v S_v, \\ S_h(t^+) = (1-p_h)S_h(t), \\ S_v(t^+) = (1-p_v)S_v(t). \end{cases} \tag{5}$$

3.1 The order-1 disease-free periodic solution (order-1 DFPS)

Here, we focus on the study of order-1 disease-free periodic solution, that is $(k+1)T$-DFPS. Solving system (5) with initial conditions

$$S_h(nT^+) = (1-p_h)S_h(nT) \ \text{ and } \ S_v(nT^+) = (1-p_v)S_v(nT),$$

with $n \in \mathbb{N}$, we obtain

$$S_h(t) = \frac{\Lambda_h}{\mu_h} + \left(S_h(nT^+) - \frac{\Lambda_h}{\mu_h}\right)e^{-\mu_h(t-nT)}$$

and

$$S_v(t) = \frac{\Lambda_v}{\mu_v} + \left(S_v(nT^+) - \frac{\Lambda_v}{\mu_v}\right)e^{-\mu_v(t-nT)}.$$

We now derive the existence of the $(0+1)T$-PPS (Positive Periodic Solution) of system (5). We assume that the interventions are carried out at each monitoring timing $nT = 1, 2, \cdots$, thus

$$S_h((n+1)T^+) = (1-p_h)\left[\frac{\Lambda_h}{\mu_h} + \left(S_h(nT^+) - \frac{\Lambda_h}{\mu_h}\right)e^{-\mu_h T}\right], \tag{6}$$

and

$$S_v((n+1)T^+) = (1-p_v)\left[\frac{\Lambda_v}{\mu_v} + \left(S_v(nT^+) - \frac{\Lambda_v}{\mu_v}\right)e^{-\mu_v T}\right]. \tag{7}$$

Let $S_h((n+1)T^+) = S_h(nT^+)$ and $S_v((n+1)T^+) = S_v(nT^+)$, then the unique fixed point is

$$\left(S_{1h}^0, S_{1v}^0\right) = \left(\frac{\Lambda_h(1-p_h)(1-e^{-\mu_h T})}{\mu_h(1-(1-p_h)e^{-\mu_h T})}, \frac{\Lambda_v(1-p_v)(1-e^{-\mu_v T})}{\mu_v(1-(1-p_v)e^{-\mu_v T})}\right),$$

where $S_h(nT^+) = S_{1h}^0$ and $S_v(nT^+) = S_{1v}^0$. Thus, there exist a $(0+1)T$-PPS of system (5) denoted by $\left(S_{hT}^{(0,1)}(t), S_{vT}^{(0,1)}(t)\right)$ with

$$S_{hT}^{(0,1)}(t) = \frac{\Lambda_h}{\mu_h}\left[1 - \frac{p_h e^{-\mu_h(t-nT)}}{1-(1-p_h)e^{-\mu_h T}}\right],\ nT < t \leq (n+1)T,\ n \in \mathbb{N},$$

and

$$S_{vT}^{(0,1)}(t) = \frac{\Lambda_v}{\mu_v}\left[1 - \frac{p_v e^{-\mu_v(t-nT)}}{1-(1-p_v)e^{-\mu_v T}}\right],\ nT < t \leq (n+1)T,\ n \in \mathbb{N}.$$

The lower and upper bounds of $S_{hT}^{(0,1)}(t)$ are given by

$$L_h^0 = \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h}{1-(1-p_h)e^{-\mu_h T}}\right) \text{ and } U_h^0 = \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h e^{-\mu_h T}}{1-(1-p_h)e^{-\mu_h T}}\right),$$

respectively. The lower and upper bounds of $S_{vT}^{(0,1)}(t)$ are respectively equal to

$$L_v^0 = \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v}{1-(1-p_v)e^{-\mu_v T}}\right) \text{ and } U_v^0 = \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v e^{-\mu_v T}}{1-(1-p_v)e^{-\mu_v T}}\right).$$

When $S_{hT} > U_h^0$ and $S_{vT} > U_v^0$, then no pulse intervention will be carried out and in this case $\left(S_{hT}^{(0,1)}(t), S_{vT}^{(0,1)}(t)\right)$ does not exist. On the other hand, if $S_{hT} \leq U_h^0$ and $S_{vT} \leq U_v^0$ holds, there is a positive $(0+1)$T-PPS $\left(S_{hT}^{(0,1)}(t), S_{vT}^{(0,1)}(t)\right)$ of system (5).

We now consider the general case, that is, the existence of the $(k+1)T$-PPS of system (5), where k is any positive integer. We assume that there is no intervention at time $(n+r)T$, $r = 1, \cdots, k$, while pulse interventions are administered at time $(n+k+1)T$. Then,

$$S_h((n+k+1)T^+) = (1-p_h)\left[\frac{\Lambda_h}{\mu_h} + \left(S_h(nT^+) - \frac{\Lambda_h}{\mu_h}\right)e^{-\mu_h(k+1)T}\right],$$

and

$$S_v((n+k+1)T^+) = (1-p_v)\left[\frac{\Lambda_v}{\mu_v} + \left(S_v(nT^+) - \frac{\Lambda_v}{\mu_v}\right)e^{-\mu_v(k+1)T}\right].$$

Let

$$S_h((n+k+1)T^+) = S_h(nT^+) \ \text{ and } \ S_v((n+k+1)T^+) = S_v(nT^+).$$

There exists a unique positive fixed point (S_{1h}^k, S_{1v}^k) given by

$$S_{1h}^k = \frac{\Lambda_h(1-p_h)(1-e^{-\mu_h(k+1)T})}{\mu_h(1-(1-p_h)e^{-\mu_h(k+1)T})} \text{ and } S_{1v}^k = \frac{\Lambda_v(1-p_v)(1-e^{-\mu_v(k+1)T})}{\mu_v(1-(1-p_v)e^{-\mu_v(k+1)T})}.$$

Hence, system (5) can have a $(k+1)T$-PS denoted by $(S_{hT}^{(k,1)}(t), S_{vT}^{(k,1)}(t))$, with

$$S_{hT}^{(k,1)}(t) = \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h e^{-\mu_h(t-nT)}}{1-(1-p_h)e^{-\mu_h(k+1)T}}\right),$$

$$S_{vT}^{(k,1)}(t) = \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v e^{-\mu_v(t-nT)}}{1-(1-p_v)e^{-\mu_v(k+1)T}}\right),$$

where $nT < t \leq (n+k+1)T$, $n \in \mathbb{N}$.

The existence of the $(k+1)T$-PPS holds when the pulse interventions are only carried out at time $(n+k+1)T$. Since, $S_{hT}^{(k,1)}(t)$ and $S_{vT}^{(k,1)}(t)$ are increasing for $t \in]nT, (n+k+1)T]$, thus we only need the relations

$$S_{hT}^{(k,1)}((n+k)T) < S_{hT} \leq S_{hT}^{(k,1)}((n+k+1)T)$$

and

$$S_{vT}^{(k,1)}((n+k)T) < S_{vT} \leq S_{vT}^{(k,1)}((n+k+1)T).$$

That is,

$$L_h^k < S_{hT} \leq U_h^k$$

and

$$L_v^k < S_{vT} \leq U_v^k,$$

where L_h^k and U_h^k are the lower and upper bounds of $S_{hT}^{(k,1)}(t)$ given by

$$L_h^k = \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h e^{-\mu_h kT}}{1-(1-p_h)e^{-\mu_h(k+1)T}}\right)$$

and

$$U_h^k = \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h e^{-\mu_h(k+1)T}}{1-(1-p_h)e^{-\mu_h(k+1)T}}\right),$$

L_v^k and U_v^k are the lower and upper bounds of $S_{vT}^{(k,1)}(t)$ given by

$$L_v^k = \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v e^{-\mu_v kT}}{1-(1-p_v)e^{-\mu_v(k+1)T}}\right)$$

and

$$U_v^k = \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v e^{-\mu_v(k+1)T}}{1-(1-p_v)e^{-\mu_v(k+1)T}}\right).$$

For any $(k_1, k_2) \in \mathbb{N}^2$, with $k_1 \neq k_2$, we now show that the $(k_1+1)T$-PPS and $(k_2+1)T$-PPS of system (5) can not co-exist. For that, we assume that $k_1 < k_2$ and thus $\forall k \in \mathbb{N}$, $L_h^k < U_h^k$ and $L_v^k < U_v^k$.

Compute that

$$\frac{L_h^{k+1}}{U_h^k} = \frac{1 - \dfrac{p_h e^{-\mu_h(k+1)T}}{1-(1-p_h)e^{-\mu_h(k+2)T}}}{1 - \dfrac{p_h e^{-\mu_h(k+1)T}}{1-(1-p_h)e^{-\mu_h(k+1)T}}} > 1, \tag{8}$$

and

$$\frac{L_v^{k+1}}{U_v^k} = \frac{1 - \dfrac{p_v e^{-\mu_v(k+1)T}}{1-(1-p_v)e^{-\mu_v(k+2)T}}}{1 - \dfrac{p_v e^{-\mu_v(k+1)T}}{1-(1-p_v)e^{-\mu_v(k+1)T}}} > 1. \tag{9}$$

Thus, from (8) and (9) we deduce that $\forall k \in \mathbb{N}$, $U_h^k < L_h^{k+1}$ and $U_v^k < L_v^{k+1}$.

Further,

$$\lim_{k\to+\infty} L_h^k = \lim_{k\to+\infty} \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h e^{-\mu_h kT}}{1-(1-p_h)e^{-\mu_h(k+1)T}}\right) = \frac{\Lambda_h}{\mu_h}$$

and

$$\lim_{k\to+\infty} U_h^k = \lim_{k\to+\infty} \frac{\Lambda_h}{\mu_h}\left(1 - \frac{p_h e^{-\mu_h(k+1)T}}{1-(1-p_h)e^{-\mu_h(k+1)T}}\right) = \frac{\Lambda_h}{\mu_h}.$$

Thus,

$$\lim_{k\to+\infty} L_h^k = \lim_{k\to+\infty} U_h^k = \frac{\Lambda_h}{\mu_h}.$$

We also have

$$\lim_{k\to+\infty} L_v^k = \lim_{k\to+\infty} \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v e^{-\mu_v kT}}{1-(1-p_v)e^{-\mu_v(k+1)T}}\right) = \frac{\Lambda_v}{\mu_v},$$

and

$$\lim_{k\to+\infty} U_v^k = \lim_{k\to+\infty} \frac{\Lambda_v}{\mu_v}\left(1 - \frac{p_v e^{-\mu_v(k+1)T}}{1-(1-p_v)e^{-\mu_v(k+1)T}}\right) = \frac{\Lambda_v}{\mu_v}.$$

Hence,

$$\lim_{k\to+\infty} L_v^k = \lim_{k\to+\infty} U_v^k = \frac{\Lambda_v}{\mu_v}.$$

Therefore,

$$L_h^0 < U_h^0 < L_h^1 < U_h^1 < \cdots < L_h^{k_1} < U_h^{k_1} < \cdots < L_h^{k_2} < U_h^{k_2} < \cdots < \frac{\Lambda_h}{\mu_h} \tag{10}$$

and

$$L_v^0 < U_v^0 < L_v^1 < U_v^1 < \cdots < L_v^{k_1} < U_v^{k_1} < \cdots < L_v^{k_2} < U_v^{k_2} < \cdots < \frac{\Lambda_v}{\mu_v}. \tag{11}$$

If $(k_1+1)T$-PPS and $(k_2+1)T$-PPS of system (5) coexist, then we would have

$$L_h^{k_1} < S_{hT} \le U_h^{k_1},\;\; L_h^{k_2} < S_{hT} \le U_h^{k_2},\;\; L_v^{k_1} < S_{vT} \le U_v^{k_1},\;\; L_v^{k_2} < S_{vT} \le U_v^{k_2},$$

which contradicts equations (10) and (11). Thus, the $(k_1+1)T$-PPS and $(k_2+1)T$-PPS cannot coexist for all $(k_1, k_2) \in \mathbb{N}^2$, with $k_1 \ne k_2$. Then, we state the following result.

Proposition 1: *If $S_{hT} \leq U_h^0$ and $S_{vT} \leq U_v^0$ holds, system* (5) *has a unique* $(0+1)T$*-PPS* $(S_{hT}^{(0,1)}(t), S_{vT}^{(0,1)}(t))$*. Generally, for any positive integer k, if $L_h^k < S_{hT} \leq U_h^k$ and $L_v^k < S_{vT} \leq U_v^k$, there exists a unique* $(k+1)T$*-PPS* $(S_{hT}^{(k,1)}(t), S_{vT}^{(k,1)}(t))$ *of system* (5).

The existence of the $(k+1)T$-PPS $(S_{hT}^{(k,1)}(t), S_{vT}^{(k,1)}(t))$ of system (5) means that there is an order-1 $(k+1)T$-DFPS $(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0)$ of system (4).

3.2 Stability analysis of the order-1 disease-free periodic solution

In this part, we study the stability of the $(k+1)T$-DFPS $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$ of system (4). Assuming that $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$ exist, thus there is a neighborhood region of (S_{1h}^k, S_{1v}^k), denoted by $N_{S_{1h}^k} \times N_{S_{1v}^k}$, such that solutions of system (4) with $(S_h(t), S_v(t))$ starting in $N_{S_{1h}^k} \times N_{S_{1v}^k}$ satisfy $\forall t \in]nT, (n+k+1)T]$ with $n \in \mathbb{N}$, the following system

$$
\begin{cases}
\left.\begin{aligned}
\frac{dS_h(t)}{dt} &= \Lambda_h - \frac{\beta S_h I_v}{1+c_1 I_v} - \mu_h S_h \\
\frac{dI_h(t)}{dt} &= \frac{\beta S_h I_v}{1+c_1 I_v} - (\mu_h+\gamma+d) I_h \\
\frac{dS_v(t)}{dt} &= \Lambda_v - \frac{\xi S_v I_h}{1+c_2 I_h} - \mu_v S_v \\
\frac{dI_v(t)}{dt} &= \frac{\xi S_v I_h}{1+c_2 I_h} - \mu_v I_v
\end{aligned}\right\} t \neq (n+k+1)T, n \in \mathbb{N}, \\
\\
\left.\begin{aligned}
S_h(t^+) &= (1-p_h) S_h(t) \\
I_h(t^+) &= (1-q_h) I_h(t) \\
S_v(t^+) &= (1-p_v) S_v(t) \\
I_v(t^+) &= (1-p_v) I_v(t)
\end{aligned}\right\} t = (n+k+1)T, n \in \mathbb{N}.
\end{cases}
\tag{12}
$$

Let $\mathcal{M}$ be the jacobian matrix associated to (12) at the point $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$. Thus,

$$
\mathcal{M} = \begin{pmatrix}
-\mu_h & 0 & 0 & -\beta S_{hT}^{(k,1)}(t) \\
0 & -(\mu_h+\gamma+d) & 0 & \beta S_{hT}^{(k,1)}(t) \\
0 & -\xi S_{vT}^{(k,1)}(t) & -\mu_v & 0 \\
0 & \xi S_{vT}^{(k,1)}(t) & 0 & -\mu_v
\end{pmatrix}.
$$

Let

$$\begin{cases} S_h(t) = U_1(t) + S_{hT}^{(k,1)}(t), \\ I_h(t) = V_1(t), \\ S_v(t) = U_2(t) + S_{vT}^{(k,1)}(t), \\ I_v(t) = V_2(t), \end{cases}$$

where $U_1(t)$, $U_2(t)$, $V_1(t)$ and $V_2(t)$ are small perturbations with

$$\begin{pmatrix} U_1(t) \\ V_1(t) \\ U_2(t) \\ V_2(t) \end{pmatrix} = \varphi(t) \begin{pmatrix} U_1(0) \\ V_1(0) \\ U_2(0) \\ V_2(0) \end{pmatrix}, 0 \leq t < (k+1)T$$

where the matrix $\varphi(t)$ satisfy

$$\frac{d\varphi(t)}{dt} = \mathcal{M}\varphi(t), \tag{13}$$

and $\varphi(0) = I$ is the identity matrix. It follows that

$$\varphi((k+1)T) = \begin{pmatrix} e^{-\mu_h (k+1)T} & \varphi_{12} & 0 & \varphi_{14} \\ 0 & \varphi_{22} & 0 & \varphi_{24} \\ 0 & \varphi_{32} & e^{\int_0^{(k+1)T} \delta_1 \, \mathrm{d}t} & \varphi_{34} \\ 0 & \varphi_{42} & 0 & \varphi_{44} \end{pmatrix}$$

where,

$$
\begin{aligned}
\varphi_{12} &= \left(\frac{\beta\xi S_{vT}^{(k,1)}(t)S_{hT}^{(k,1)}(t)}{(\delta_2-\delta_1)(\mu_h+\delta_2)} - \frac{\beta\xi S_{vT}^{(k,1)}(t)S_{hT}^{(k,1)}(t)}{(\delta_2-\delta_1)(\mu_h+\delta_1)}\right)e^{-\mu_h(k+1)T} \\
&\quad + \frac{\beta\xi S_{vT}^{(k,1)}(t)S_{hT}^{(k,1)}(t)}{(\delta_2-\delta_1)(\mu_h+\delta_1)}e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t} - \frac{\beta\xi S_{vT}^{(k,1)}(t)S_{hT}^{(k,1)}(t)}{(\delta_2-\delta_1)(\mu_h+\delta_2)}e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t}, \\
\varphi_{14} &= \left(\frac{\beta S_{hT}^{(k,1)}(t)(\mu_v+\delta_2)}{(\delta_2-\delta_1)(\mu_h+\delta_1)} - \frac{\beta S_{hT}^{(k,1)}(t)(\mu_v+\delta_1)}{(\delta_2-\delta_1)(\mu_h+\delta_2)}\right)e^{-\mu_h(k+1)T} \\
&\quad + \frac{\beta S_{hT}^{(k,1)}(t)(\mu_v+\delta_1)}{(\delta_2-\delta_1)(\mu_h+\delta_2)}e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t} - \frac{\beta S_{hT}^{(k,1)}(t)(\mu_v+\delta_2)}{(\delta_2-\delta_1)(\mu_h+\delta_1)}e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t}, \\
\varphi_{22} &= \frac{\mu_v+\delta_2}{\delta_2-\delta_1}e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t} - \frac{\mu_v+\delta_1}{\delta_2-\delta_1}e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t}, \\
\varphi_{24} &= \frac{(\mu_v+\delta_1)(\mu_v+\delta_2)}{\xi S_{vT}^{(k,1)}(t)(\delta_2-\delta_1)}\left(e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t} - e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t}\right), \\
\varphi_{32} &= \frac{\xi S_{vT}^{(k,1)}(t)}{\delta_2-\delta_1}\left(e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t} - e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t}\right), \\
\varphi_{34} &= e^{-\mu_v(k+1)T} - \frac{\mu_v+\delta_2}{\delta_2-\delta_1}e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t} + \frac{\mu_v+\delta_1}{\delta_2-\delta_1}e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t}, \\
\varphi_{42} &= \frac{\xi S_{vT}^{(k,1)}(t)}{\delta_2-\delta_1}\left(e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t} - e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t}\right), \\
\varphi_{44} &= \frac{\mu_v+\delta_2}{\delta_2-\delta_1}e^{\int_0^{(k+1)T}\delta_1\,\mathrm{d}t} - \frac{\mu_v+\delta_1}{\delta_2-\delta_1}e^{\int_0^{(k+1)T}\delta_2\,\mathrm{d}t}, \\
\delta_1 &= \frac{-(\mu_v+\mu_h+\gamma+d) - \sqrt{(\mu_v-\mu_h-\gamma-d)^2 + 4\xi\beta S_{vT}^{(k,1)}(t)S_{hT}^{(k,1)}(t)}}{2}, \\
\delta_2 &= \frac{-(\mu_v+\mu_h+\gamma+d) + \sqrt{(\mu_v-\mu_h-\gamma-d)^2 + 4\xi\beta S_{vT}^{(k,1)}(t)S_{hT}^{(k,1)}(t)}}{2}.
\end{aligned}
$$

Then, we obtain

$$
\begin{pmatrix} U_1((n+k+1)T^+) \\ V_1((n+k+1)T^+) \\ U_2((n+k+1)T^+) \\ V_2((n+k+1)T^+) \end{pmatrix} = \begin{pmatrix} 1-p_h & 0 & 0 & 0 \\ 0 & 1-q_h & 0 & 0 \\ 0 & 0 & 1-p_v & 0 \\ 0 & 0 & 0 & 1-p_v \end{pmatrix}\begin{pmatrix} U_1((n+k+1)T) \\ V_1((n+k+1)T) \\ U_2((n+k+1)T) \\ V_2((n+k+1)T) \end{pmatrix}.
$$

Let the matrix

$$\mathcal{J} = \begin{pmatrix} 1-p_h & 0 & 0 & 0 \\ 0 & 1-q_h & 0 & 0 \\ 0 & 0 & 1-p_v & 0 \\ 0 & 0 & 0 & 1-p_v \end{pmatrix} \times \begin{pmatrix} e^{-\mu_h(k+1)T} & \varphi_{12} & 0 & \varphi_{14} \\ 0 & \varphi_{22} & 0 & \varphi_{24} \\ 0 & \varphi_{32} & e^{\int_0^{(k+1)T} \delta_1 \, \mathrm{d}t} & \varphi_{34} \\ 0 & \varphi_{42} & 0 & \varphi_{44} \end{pmatrix}.$$

Thus, with a bit of algebra we obtain

$$\mathcal{J} = \begin{pmatrix} (1-p_h)e^{-\mu_h(k+1)T} & (1-p_h)\varphi_{12} & 0 & (1-p_h)\varphi_{14} \\ 0 & (1-q_h)\varphi_{22} & 0 & (1-q_h)\varphi_{24} \\ 0 & (1-p_v)\varphi_{32} & (1-p_v)e^{\int_0^{(k+1)T} \delta_1 \, \mathrm{d}t} & (1-p_v)\varphi_{34} \\ 0 & (1-p_v)\varphi_{42} & 0 & (1-p_v)\varphi_{44} \end{pmatrix}.$$

The characteristic polynomial of $P_{\mathcal{J}}$ of $\mathcal{J}$ is given by

$$P_{\mathcal{J}} = (\lambda - \mathcal{J}_{11})(\lambda - \mathcal{J}_{33})(\lambda - \lambda_3)(\lambda - \lambda_4).$$

Clearly, the eigenvalues of $\mathcal{J}$ are given by

$$\begin{aligned} \lambda_1 &= (1-p_h)e^{-\mu_h(k+1)T}, \\ \lambda_2 &= (1-p_v)e^{\int_0^{(k+1)T} \delta_1 \, dt}, \\ \lambda_3 &= \frac{\mathcal{J}_{22} + \mathcal{J}_{44} - \sqrt{(\mathcal{J}_{22} - \mathcal{J}_{44})^2 + 4\mathcal{J}_{42}\mathcal{J}_{24}}}{2}, \\ \lambda_4 &= \frac{\mathcal{J}_{22} + \mathcal{J}_{44} + \sqrt{(\mathcal{J}_{22} - \mathcal{J}_{44})^2 + 4\mathcal{J}_{42}\mathcal{J}_{24}}}{2}, \end{aligned}$$

where

$$\mathcal{J}_{11} = (1-p_h)e^{-\mu_h(k+1)T}, \quad \mathcal{J}_{12} = (1-p_h)\varphi_{12}, \quad \mathcal{J}_{14} = (1-p_h)\varphi_{14}, \quad \mathcal{J}_{22} = (1-q_h)\varphi_{22},$$
$$\mathcal{J}_{24} = (1-q_h)\varphi_{24}, \quad \mathcal{J}_{32} = (1-p_v)\varphi_{32}, \mathcal{J}_{33} = (1-p_v)e^{\int_0^{(k+1)T}\delta_1 dt}, \quad \mathcal{J}_{34} = (1-p_v)\varphi_{34},$$
$$\mathcal{J}_{42} = (1-p_v)\varphi_{42}, \quad \mathcal{J}_{44} = (1-p_v)\varphi_{44}.$$

From Floquet theory ([7], [8]), if the moduli of the eigenvalues of $\mathcal{J}$ are less than one, then the $(k+1)T$-DFPS $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$ of system (4) is locally asymptotically stable.
We observe that $0 < \lambda_1, \lambda_2 < 1$ and therefore the stability of $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$ depends on the eigenvalues λ_3 and λ_4. For that, let

$$R_{k+1} = \max\{\mid \lambda_3 \mid, \mid \lambda_4 \mid\},$$

thus the following result holds.

Proposition 2: *For any positive integer k :*
i) if $R_{k+1} < 1$, then the $(k+1)T$-DFPS $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$ of system (4) is locally asymptotically stable,
ii) if $R_{k+1} > 1$, then the $(k+1)T$-DFPS $\left(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0\right)$ of system (4) is unstable.

3.3 Higher order disease-free periodic solution

Here, we study the existence of the order-m DFPS of system (4) with $m \geq 2$. To do so, we start with simple examples and the general case will be derive by mathematical induction. We start with $m = 2$ and we first investigate on the nonexistence of the $(0+2)T$-DFPS. Assuming that pulsed interventions are implemented at nT with $n \in \mathbb{N}$, thus from (6) and (7) we deduce that

$$S_h((n+2)T^+) = (1-p_h)\left[\frac{\Lambda_h}{\mu_h} + \left(S_h((n+1)T^+) - \frac{\Lambda_h}{\mu_h}\right)e^{-\mu_h T}\right].$$

Let $S_h(nT^+) = S_h((n+2)T^+)$, then

$$S_h(nT^+) = \frac{(1-p_h)\Lambda_h\left(1 - p_h e^{-\mu_h T} - (1-p_h)e^{-2\mu_h T}\right)}{\mu_h(1-(1-p_h)^2 e^{-2\mu_h T})}.$$

Since $1 - p_h e^{-\mu_h T} - (1-p_h)e^{-2\mu_h T} > 0$, it follows that

$$S_{2h}^0 = \frac{(1-p_h)\Lambda_h\left(1 - p_h e^{-\mu_h T} - (1-p_h)e^{-2\mu_h T}\right)}{\mu_h(1-(1-p_h)^2 e^{-2\mu_h T})},$$

is the unique positive fixed point of equation $S_h(nT^+) = S_h((n+2)T^+)$. Similarly, solving equation $S_v(nT^+) = S_v((n+2)T^+)$ gives us

$$S_v(nT^+) = \frac{(1-p_v)\Lambda_v\left(1-p_v e^{-\mu_v T}-(1-p_v)e^{-2\mu_v T}\right)}{\mu_v(1-(1-p_v)^2 e^{-2\mu_v T})}.$$

Further, using the fact that $1-p_v e^{-\mu_v T}-(1-p_v)e^{-2\mu_v T} > 0$, we obtain a unique positive fixed point given by

$$S_{2v}^0 = \frac{(1-p_v)\Lambda_v\left(1-p_v e^{-\mu_v T}-(1-p_v)e^{-2\mu_v T}\right)}{\mu_v(1-(1-p_v)^2 e^{-2\mu_v T})}.$$

System (4) has a $(0+2)T$-PPS if

$$S_h((n+1)T^+) \neq S_{2h}^0 \text{ and } S_v((n+1)T^+) \neq S_{2v}^0.$$

We easily verify that $S_h((n+1)T^+) = S_{2h}^0$ and $S_v((n+1)T^+) = S_{2v}^0$, so there is no $(0+2)T$-PPS of system (5). Thus, the following result holds.

Proposition 3: *There is no* $(0+2)T$*-PPS of system* (5)*, therefore there is no* $(0+2)T$*-DFPS of system* (4).

We now study the existence of $(1+m)T$-PPS of system (5) $\forall m \geq 2$. Assuming that no intervention is carried out at time $(n+1)T$ while preventive and control measures are applied at the appropriate times $(n+2)T, \cdots, (n+m+1)T$. Then,

$$S_h((n+l)T^+) = \begin{cases} \dfrac{\Lambda_h}{\mu_h} + \left(S_h(nT^+) - \dfrac{\Lambda_h}{\mu_h}\right)e^{-\mu_h T} \text{ for } l = 1, \\[2ex] \dfrac{\Lambda_h(1-p_h)}{\mu_h} - \dfrac{p_h\Lambda_h}{\mu_h}\displaystyle\sum_{j=1}^{l-2}(1-p_h)^j e^{-j\mu_h T} - \dfrac{\Lambda_h(1-p_h)^{l-1}e^{-l\mu_h T}}{\mu_h} \\ + (1-p_h)^{l-1}e^{-l\mu_h T}S_h(nT^+) \text{ for } 2 \leq l \leq m+1, \end{cases} \tag{14}$$

and

$$S_v((n+l)T^+) = \begin{cases} \dfrac{\Lambda_v}{\mu_v} + \left(S_v(nT^+) - \dfrac{\Lambda_v}{\mu_v} \right) e^{-\mu_v T} \text{ for } l = 1, \\ \\ \dfrac{\Lambda_v(1-p_v)}{\mu_v} - \dfrac{p_v\Lambda_v}{\mu_v} \displaystyle\sum_{j=1}^{l-2} (1-p_v)^j e^{-j\mu_v T} - \dfrac{\Lambda_v(1-p_v)^{l-1} e^{-l\mu_v T}}{\mu_v} \\ + (1-p_v)^{l-1} e^{-l\mu_v T} S_v(nT^+) \text{ for } 2 \le l \le m+1. \end{cases} \tag{15}$$

Equations (14) and (15) can be proved by mathematical induction. Firstly, for $l = 2$ and $l = 3$, equations (14) and (15) holds true. Assuming that expressions (14) and (15) holds true for $l \le m$. Compute that

$$\begin{aligned} S_h((n+l+1)T) = {} & \frac{\Lambda_h}{\mu_h} - \frac{p_h\Lambda_h}{\mu_h} \sum_{j=1}^{l-1} (1-p_h)^{j-1} e^{-j\mu_h T} - \frac{\Lambda_h(1-p_h)^{l-1} e^{-(l+1)\mu_h T}}{\mu_h} \\ & + (1-p_h)^{l-1} e^{-(l+1)\mu_h T} S_h(nT^+) \end{aligned}$$

and

$$\begin{aligned} S_v((n+l+1)T) = {} & \frac{\Lambda_v}{\mu_v} - \frac{p_v\Lambda_v}{\mu_v} \sum_{j=1}^{l-1} (1-p_v)^{j-1} e^{-j\mu_v T} - \frac{\Lambda_v(1-p_v)^{l-1} e^{-(l+1)\mu_v T}}{\mu_v} \\ & + (1-p_v)^{l-1} e^{-(l+1)\mu_v T} S_v(nT^+). \end{aligned}$$

Thus,

$$\begin{aligned} S_h((n+l+1)T^+) = {} & \frac{\Lambda_h(1-p_h)}{\mu_h} - \frac{p_h\Lambda_h}{\mu_h} \sum_{j=1}^{l-1} (1-p_h)^j e^{-j\mu_h T} - \frac{\Lambda_h(1-p_h)^l e^{-(l+1)\mu_h T}}{\mu_h} \\ & + (1-p_h)^l e^{-(l+1)\mu_h T} S_h(nT^+) \end{aligned}$$

and

$$\begin{aligned} S_v((n+l+1)T^+) = {} & \frac{\Lambda_v(1-p_v)}{\mu_v} - \frac{p_v\Lambda_v}{\mu_v} \sum_{j=1}^{l-1} (1-p_v)^j e^{-j\mu_v T} - \frac{\Lambda_v(1-p_v)^l e^{-(l+1)\mu_v T}}{\mu_v} \\ & + (1-p_v)^l e^{-(l+1)\mu_v T} S_v(nT^+). \end{aligned}$$

Therefore, equations (14) and (15) holds true for $l \in \{1, ..., m+1\}$.

Setting $S_h(nT^+) = S_h((n+m+1)T^+)$ and $S_v(nT^+) = S_v((n+m+1)T^+)$, we obtain

$$S_h(nT^+) = \frac{\Lambda_h\left(1 - p_h - p_h \sum_{j=1}^{m-1}(1-p_h)^j e^{-j\mu_h T} - (1-p_h)^m e^{-(m+1)\mu_h T}\right)}{\mu_h\left(1-(1-p_h)^m e^{-(m+1)\mu_h T}\right)}$$

and

$$S_v(nT^+) = \frac{\Lambda_v\left(1 - p_v - p_v \sum_{j=1}^{m-1}(1-p_v)^j e^{-j\mu_v T} - (1-p_v)^m e^{-(m+1)\mu_v T}\right)}{\mu_v\left(1-(1-p_v)^m e^{-(m+1)\mu_v T}\right)}.$$

Since

$$1 - p_h - p_h \sum_{j=1}^{m-1}(1-p_h)^j e^{-j\mu_h T} - (1-p_h)^m e^{-(m+1)\mu_h T} > 0$$

and

$$1 - p_v - p_v \sum_{j=1}^{m-1}(1-p_v)^j e^{-j\mu_v T} - (1-p_v)^m e^{-(m+1)\mu_v T} > 0,$$

thus the only positive fixed points of equations $S_h(nT^+) = S_h((n+m+1)T^+)$ and $S_v(nT^+) = S_v((n+m+1)T^+)$ are respectively

$$S_{hm}^1 = \frac{\Lambda_h\left(1 - p_h - p_h \sum_{j=1}^{m-1}(1-p_h)^j e^{-j\mu_h T} - (1-p_h)^m e^{-(m+1)\mu_h T}\right)}{\mu_h\left(1-(1-p_h)^m e^{-(m+1)\mu_h T}\right)}$$

and

$$S_{vm}^1 = \frac{\Lambda_v\left(1 - p_v - p_v \sum_{j=1}^{m-1}(1-p_v)^j e^{-j\mu_v T} - (1-p_v)^m e^{-(m+1)\mu_v T}\right)}{\mu_v\left(1-(1-p_v)^m e^{-(m+1)\mu_v T}\right)}.$$

Then, there exist a $(1+m)T$-PPS of system (5) denoted by $\left(S_{hT}^{(1,m)}(t), S_{vT}^{(1,m)}(t)\right)$, where

$$S_{hT}^{(1,m)}(t) = \begin{cases} \dfrac{\Lambda_h}{\mu_h} - \dfrac{\Lambda_h p_h \left(1 + \sum\limits_{j=1}^{m-1} (1-p_h)^j e^{-j\mu_h T}\right) e^{-\mu_h (t-nT)}}{\mu_h \left(1 - (1-p_h)^m e^{-(m+1)\mu_h T}\right)}, nT < t \le (n+2)T \\ \dfrac{\Lambda_h}{\mu_h} - \dfrac{\Lambda_h p_h}{\mu_h} \left(1 + \dfrac{\sum\limits_{j=1}^{l-2} (1-p_h)^j e^{-j\mu_h T} + \sum\limits_{j=l}^{m} (1-p_h)^j e^{-(j+1)\mu_h T}}{1-(1-p_h)^m e^{-(m+1)\mu_h T}} \right. \\ \left. + \dfrac{(1-p_h)^{l-1} e^{-l\mu_h T}}{1-(1-p_h)^m e^{-(m+1)\mu_h T}} \right) e^{-\mu_h (t-(n+l)T)}, 2 \le l \le m, \\ (n+l)T < t \le (n+l+1)T \end{cases} \tag{16}$$

and

$$S_{vT}^{(1,m)}(t) = \begin{cases} \dfrac{\Lambda_v}{\mu_v} - \dfrac{\Lambda_v p_v \left(1 + \sum\limits_{j=1}^{m-1} (1-p_v)^j e^{-j\mu_v T}\right) e^{-\mu_v (t-nT)}}{\mu_v \left(1 - (1-p_v)^m e^{-(m+1)\mu_v T}\right)}, nT < t \le (n+2)T \\ \dfrac{\Lambda_v}{\mu_v} - \dfrac{\Lambda_v p_v}{\mu_v} \left(1 + \dfrac{\sum\limits_{j=1}^{l-2} (1-p_v)^j e^{-j\mu_v T} + \sum\limits_{j=l}^{m} (1-p_v)^j e^{-(j+1)\mu_v T}}{1-(1-p_v)^m e^{-(m+1)\mu_v T}} \right. \\ \left. + \dfrac{(1-p_v)^{l-1} e^{-l\mu_v T}}{1-(1-p_v)^m e^{-(m+1)\mu_v T}} \right) e^{-\mu_v (t-(n+l)T)}, 2 \le l \le m, \\ (n+l)T < t \le (n+l+1)T. \end{cases} \tag{17}$$

By computation, we obtain

$$S_h((n+2)T) - S_h((n+3)T) = \frac{\Lambda_h p_h e^{-\mu_h T}}{\mu_h} \left(\frac{1 - e^{-\mu_h T}}{1-(1-p_h)^2 e^{-3\mu_h T}} \right) > 0$$

and

$$S_v((n+2)T) - S_v((n+3)T) = \frac{\Lambda_v p_v e^{-\mu_v T}}{\mu_v} \left(\frac{1 - e^{-\mu_v T}}{1-(1-p_v)^2 e^{-3\mu_v T}} \right) > 0.$$

Thus, $S_h((n+2)T) > S_h((n+3)T)$ and $S_v((n+2)T) > S_v((n+3)T)$.

For $l \geq 3$, we have

$$S_h((n+l)T) - S_h((n+l+1)T) = \frac{\Lambda_h p_h (1-p_h)^{l-2} e^{-(l-1)\mu_h T}}{\mu_h \left(1-(1-p_h)^l e^{-(l+1)\mu_h T}\right)} > 0$$

and

$$S_v((n+l)T) - S_v((n+l+1)T) = \frac{\Lambda_v p_v (1-p_v)^{l-2} e^{-(l-1)\mu_v T}}{\mu_v \left[1-(1-p_v)^l e^{-(l+1)\mu_v T}\right]} > 0.$$

Hence, $S_h((n+l)T) > S_h((n+l+1)T)$ and $S_v((n+l)T) > S_v((n+l+1)T)$.
Let $L_h^{(1,m)}$ and $L_v^{(1,m)}$ are the lower bounds of $S_{hT}^{(1,m)}(t)$ and $S_{vT}^{(1,m)}(t)$ respectively and $U_h^{(1,m)}$ and $U_v^{(1,m)}$ are the upper bounds of $S_{hT}^{(1,m)}(t)$ and $S_{vT}^{(1,m)}(t)$ respectively.
Thus, we have

$$\begin{aligned} L_h^{(1,m)} &= S_{hT}^{(1,m)}((n+1)T), \quad L_v^{(1,m)} = S_{vT}^{(1,m)}((n+1)T), \\ U_h^{(1,m)} &= S_{hT}^{(1,m)}((n+l+1)T), \quad U_v^{(1,m)} = S_{vT}^{(1,m)}((n+l+1)T). \end{aligned}$$

That is,

$$\begin{cases} L_h^{(1,m)} = \dfrac{\Lambda_h}{\mu_h} - \dfrac{\Lambda_h p_h \left(1 + \displaystyle\sum_{j=1}^{m-1} (1-p_h)^j e^{-j\mu_h T}\right)}{\mu_h (1-(1-p_h)^m e^{-(m+1)\mu_h T})}, \\ L_v^{(1,m)} = \dfrac{\Lambda_v}{\mu_v} - \dfrac{\Lambda_v p_v \left(1 + \displaystyle\sum_{j=1}^{m-1} (1-p_v)^j e^{-j\mu_v T}\right)}{\mu_v \left(1-(1-p_v)^m e^{-(m+1)\mu_v T}\right)}, \end{cases}$$

and

$$\begin{cases} U_h^{(1,m)} &= \dfrac{\Lambda_h}{\mu_h} - \dfrac{\Lambda_h p_h}{\mu_h}\left(1 + \dfrac{\sum\limits_{j=1}^{m-2}(1-p_h)^j e^{-j\mu_h T} + (1-p_h)^m e^{-(m+1)\mu_h T}}{1-(1-p_h)^m e^{-(m+1)\mu_h T}} \right. \\ & \left. + \dfrac{(1-p_h)^{m-1} e^{-m\mu_h T}}{1-(1-p_h)^m e^{-(m+1)\mu_h T}} \right) e^{-j\mu_h T}, \\ U_v^{(1,m)} &= \dfrac{\Lambda_v}{\mu_v} - \dfrac{\Lambda_v p_v}{\mu_v}\left(1 + \dfrac{\sum\limits_{j=1}^{m-2}(1-p_v)^j e^{-j\mu_v T} + (1-p_v)^m e^{-(m+1)\mu_v T}}{1-(1-p_v)^m e^{-(m+1)\mu_v T}} \right. \\ & \left. + \dfrac{(1-p_v)^{m-1} e^{-m\mu_v T}}{1-(1-p_v)^m e^{-(m+1)\mu_v T}} \right) e^{-j\mu_v T}. \end{cases}$$

Thus, the result about the existence of the $(1+m)T$-PPS of system (5) can be stated as follow.

Proposition 4: *$\forall m \geq 0$, if $L_h^{(1,m)} < S_{hT} \leq U_h^{(1,m)}$ and $L_v^{(1,m)} < S_{vT} \leq U_v^{(1,m)}$, then system (5) have a $(1+m)T$-PPS $\left(S_{hT}^{(1,m)}(t), S_{vT}^{(1,m)}(t)\right)$. Therefore, system (4) has a $(1+m)T$-DFPS denoted by $\left(S_{hT}^{(1,m)}(t), 0, S_{vT}^{(1,m)}(t), 0\right)$.*

Now, we derive the stability result of the $(1+2)T$-DFPS $\left(S_{hT}^{(1,2)}(t), 0, S_{vT}^{(1,2)}(t), 0\right)$. Let

$$\mathcal{H}_1 = \begin{pmatrix} (1-p_h)e^{-2\mu_h T} & \varphi_{12}^1 & 0 & \varphi_{14}^1 \\ 0 & \varphi_{22}^1 & 0 & \varphi_{24}^1 \\ 0 & \varphi_{32}^1 & (1-p_v)e^{\int_0^{2T} \delta_1 \, \mathrm{d}t} & \varphi_{34}^1 \\ 0 & \varphi_{42}^1 & 0 & \varphi_{44}^1 \end{pmatrix}$$

and

$$\mathcal{H}_2 = \begin{pmatrix} (1-p_h)e^{-\mu_h T} & \varphi_{12}^2 & 0 & \varphi_{14}^2 \\ \\ 0 & \varphi_{22}^2 & 0 & \varphi_{24}^2 \\ \\ 0 & \varphi_{32}^2 & (1-p_v)e^{\int_{2T}^{3T}\delta_1\,\mathrm{d}t} & \varphi_{34}^2 \\ \\ 0 & \varphi_{42}^2 & 0 & \varphi_{44}^2 \end{pmatrix}.$$

Define $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. Thus,

$$\mathcal{H} = \begin{pmatrix} (1-p_h)^2 e^{-3\mu_h T} & \psi_{12} & 0 & \psi_{14} \\ \\ 0 & \psi_{22} & 0 & \psi_{24} \\ \\ 0 & \psi_{32} & (1-p_v)^2 e^{\int_0^{3T}\delta_1\,\mathrm{d}t} & \psi_{34} \\ \\ 0 & \psi_{42} & 0 & \psi_{44} \end{pmatrix},$$

with

$$\begin{cases} \psi_{12} = (1-p_h)e^{-2\mu_h T}\varphi_{12}^2 + \varphi_{12}^1\varphi_{22}^2 + \varphi_{14}^1\varphi_{42}^2, \\ \psi_{14} = (1-p_h)e^{-2\mu_h T}\varphi_{14}^2 + \varphi_{12}^1\varphi_{24}^2 + \varphi_{14}^1\varphi_{44}^2, \\ \psi_{22} = \varphi_{22}^1\varphi_{22}^2 + \varphi_{24}^1\varphi_{42}^2, \\ \psi_{24} = \varphi_{22}^1\varphi_{24}^2 + \varphi_{24}^1\varphi_{44}^2, \\ \psi_{32} = \varphi_{32}^1\varphi_{22}^2 + (1-p_v)e^{\int_0^{2T}\delta_1\,\mathrm{d}t}\varphi_{32}^2 + \varphi_{34}^1\varphi_{42}^2, \\ \psi_{34} = \varphi_{32}^1\varphi_{24}^2 + (1-p_v)e^{\int_0^{2T}\delta_1\,\mathrm{d}t}\varphi_{34}^2 + \varphi_{34}^1\varphi_{44}^2, \\ \psi_{42} = \varphi_{42}^1\varphi_{22}^2 + \varphi_{44}^1\varphi_{42}^2, \\ \psi_{44} = \varphi_{42}^1\varphi_{24}^2 + \varphi_{44}^1\varphi_{44}^2. \end{cases}$$

We focus on the moduli of the eigenvalues of the matrix $\mathcal{H}$. Let

$$\begin{aligned} &\mathcal{H}_{11} = (1-p_h)^2 e^{-3\mu_h T},\ \mathcal{H}_{12} = \psi_{12},\ \mathcal{H}_{14} = \psi_{14}, \\ &\mathcal{H}_{22} = \psi_{22},\ \mathcal{H}_{24} = \psi_{24},\ \mathcal{H}_{32} = \psi_{32},\ \mathcal{H}_{33} = (1-p_v)^2 e^{\int_0^{3T}\delta_1\,\mathrm{d}t}, \\ &\mathcal{H}_{34} = \psi_{34},\ \mathcal{H}_{42} = \psi_{42},\ \mathcal{H}_{44} = \psi_{44} \end{aligned}$$

The eigenvalues of the matrix $\mathcal{H}$ are given by

$$\sigma_1 = (1-p_h)^2 e^{-3\mu_h T},$$
$$\sigma_2 = (1-p_v)^2 e^{\int_0^{3T} \delta_1 \, \mathrm{d}t},$$
$$\sigma_3 = \frac{\mathcal{H}_{22}+\mathcal{H}_{44}-\sqrt{(\mathcal{H}_{22}-\mathcal{H}_{44})^2+4\mathcal{H}_{42}\mathcal{H}_{24}}}{2},$$
$$\sigma_4 = \frac{\mathcal{H}_{22}+\mathcal{H}_{44}+\sqrt{(\mathcal{H}_{22}-\mathcal{H}_{44})^2+4\mathcal{H}_{42}\mathcal{H}_{24}}}{2}.$$

Clearly, $0<\sigma_1<1$, $0<\sigma_2<1$, therefore the stability of $\left(S_{hT}^{(1,2)}(t), 0, S_{vT}^{(1,2)}(t), 0\right)$ depends on the moduli of the eigenvalues σ_3 and σ_4. Let $R^{(1,2)} = \max\{|\sigma_3|, |\sigma_4|\}$, then the following result hold.

Proposition 5: *If $R^{(1,2)}<1$, the $(1+2)T$-DFPS $\left(S_{hT}^{(1,2)}(t), 0, S_{vT}^{(1,2)}(t), 0\right)$ of system (4) is locally asymptotically stable and when $R^{(1,2)}>1$, the $(1+2)T$-DFPS $\left(S_{hT}^{(1,2)}(t), 0, S_{vT}^{(1,2)}(t), 0\right)$ is unstable.*

4 Numerical simulations

In this section, we present the results of numerical simulations of model (4). We present two situations, namely when the disease goes extinct and when it persists in the community. In Figure 1, the initial conditions are varied and we observe that the number of cases of infections decreases and eventually reaches zero. This situation reflects the result obtained in Proposition 5. Figure 2 shows that the disease persist in the community. Here also, the initial conditions are varied and we observe long-term stability of the number of infections. In this case, the application of additional control measures is necessary to bring the epidemic back to a non-endemic state.

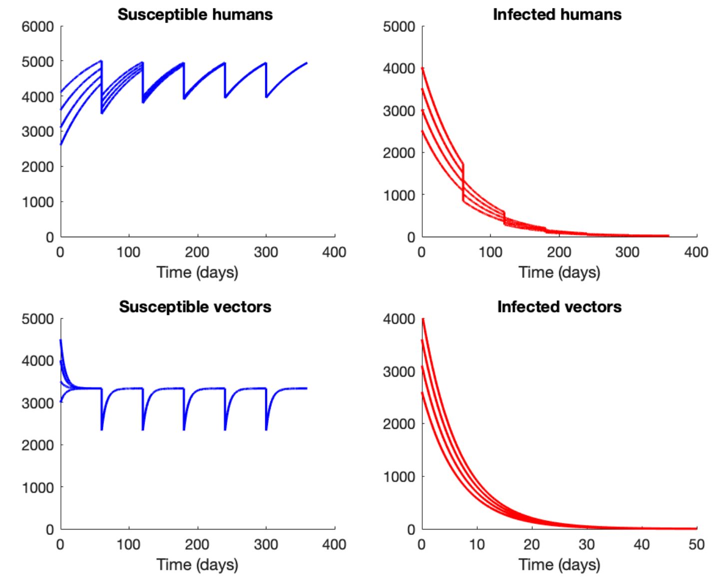

Figure 1: Disease extinction. Parameters used are $\Lambda_h = 80, \beta = 0.0002, \mu_h = 0.014, \gamma = 0.00012, \Lambda_v = 500, \xi = 0.00038, \mu_v = 0.15, d = 0.0001, c_1 = 4, c_2 = 5, p_h = 0.2, q_h = 0.22, p_v = 0.3$.

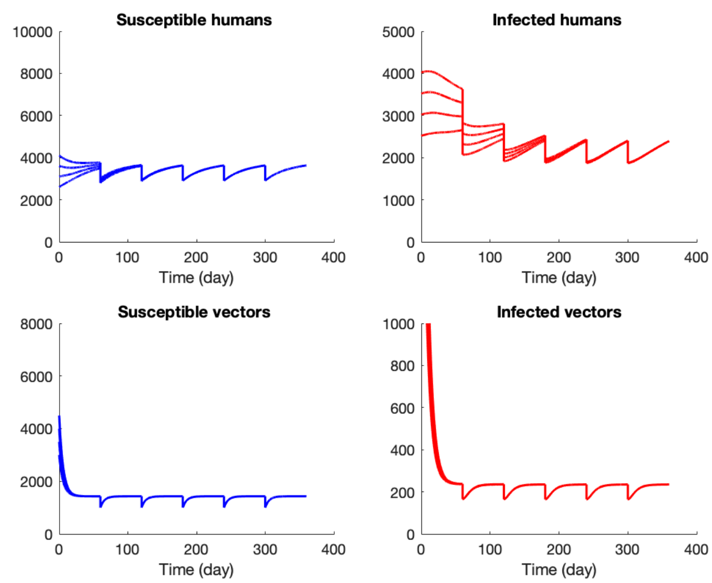

Figure 2: Disease persistence. The parameters are $\Lambda_h = 100, \beta = 0.0002, \mu_h = 0.014, \gamma = 0.00012, \Lambda_v = 250, \xi = 0.00038, \mu_v = 0.15, d = 0.001, c_1 = 0.012, c_2 = 0.015, p_h = 0.2, q_h = 0.22, p_v = 0.3$.

5 Conclusion

In this paper, we have presented a mathematical model of a vector-borne disease which highlights impulsive treatments at fixed times. This model describe susceptibles-triggered threshold policy at fixed monitoring timings to decide whether to administrate intervention or not. This modelling approach can effectively avoid the excessive use of prevention and control measures and waste of ressources. We defined a periodic solution containing k monitoring periods without implementing interventions and m monitoring periods with administrating interventions, which is denoted as $(k+m)T$-PS. Firstly, we investigate the dynamics of the disease-free periodic solution of the model. For any given nonnegative integer k, we discuss the existence of the $(k+1)T$-DFPS $(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0)$ of system (4) and prove that $(S_{hT}^{(k,1)}(t), 0, S_{vT}^{(k,1)}(t), 0)$ is locally asymptotically stable for $R_{k+1} < 1$, while it is unstable for $R_{k+1} > 1$. On the other hand, for any given m being positive integer, we study the existence of $(1+m)T$-DFPS $(S_{hT}^{(1,m)}(t), 0, S_{vT}^{(1,m)}(t), 0)$ of system (4), and we also investigate its stability with studying the stability of $(S_{hT}^{(1,2)}(t), 0, S_{vT}^{(1,2)}(t), 0)$ as an example. Furthermore, numerical simulations are carried out to support the theoretical results obtained. Several extensions of this work deserve consideration. On the one hand, it would be relevant to extend this impulsive framework to fractional-order models, combining the memory effects inherent to fractional operators with the discontinuous structure of impulses. On the other hand, introducing a stochastic component into the impulsive system would make it possible to account for random environmental fluctuations affecting vector dynamics.

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