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Research article

Symmetric Jordan Bi-semiderivations on Prime Rings

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Abstract: In this paper, we investigate the structural properties of symmetric bi-semiderivations in the setting of prime and semiprime rings. By adapting classical derivation identities through the use of associated ω-homomorphisms, we obtain several characterization results that generalize known theorems from derivations to the bi-additive context. Particular attention is given to the interplay between Jordan-type structures and standard bi-semiderivations. In this direction, we prove that if R is a prime ring with $\operatorname{char}(R) \neq 2$, then every mapping satisfying the symmetric Jordan bi-semiderivation identity is in fact a symmetric bi-semiderivation. To illustrate our results, we present explicit examples constructed from matrix rings and polynomial rings, which also highlight the necessity of the imposed algebraic conditions.

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1 Introduction

Throughout this paper, R denotes an associative ring with center $Z(R)$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. Derivations and their generalizations play an important role in ring theory, particularly in the study of commutativity and structural properties of prime and semiprime rings (see [1,2,7,8]). An additive mapping $d : R \to R$ is called a derivation if

$$d(ab) = d(a)b + ad(b)$$

for all $a, b \in R$.

Semiderivations were introduced by Bergen [1] as a generalization of derivations. An additive mapping $f : R \to R$ is called a semiderivation if there exists a mapping $g : R \to R$ such that

$$f(ab) = f(a)g(b) + af(b) = f(a)b + g(a)f(b)$$

and $f(g(a)) = g(f(a))$ for all $a, b \in R$. Chang [5] showed that, under suitable central conditions, the existence of a nonzero semiderivation implies that a prime ring with $\mathrm{char}(R) \neq 2$ is commutative.

Symmetric bi-derivations were studied by Maksa [9]. A symmetric bi-additive mapping $D : R \times R \to R$ is called a symmetric bi-derivation if it acts as a derivation in each variable. Vukman [13] investigated such mappings in prime and semiprime rings, and Sapancı et al. [12] extended these results to various classes of ideals. Generalized biderivations and related mappings have also been investigated by Bresar [3].

The interaction between bi-additive mappings and semiderivations has been considered in several recent works. Symmetric bi-semiderivations appear, for example, in near-rings (see [11]). Yılmaz and Yazarlı [14] studied their action on Lie ideals of prime rings and obtained commutativity results. However, their structural properties, especially in semiprime rings and in connection with the extended centroid, are not yet fully understood.

Jordan-type identities also play a key role in this area. Herstein [6] proved that every Jordan derivation on a prime ring with $\mathrm{char}(R) \neq 2$ is a derivation. Similar results for symmetric Jordan bi-derivations were obtained by Çeven and Çiloğlu [4]. In addition, functional identities and the extended centroid, introduced by Martindale [10], provide useful tools in this context.

In this paper, we study symmetric bi-semiderivations and symmetric Jordan bi-semiderivations in prime and semiprime rings. We use ω-endomorphisms to analyze their structure and to obtain commutativity results. Our main contributions can be summarized as follows:

- We introduce and study symmetric bi-semiderivations in prime and semiprime rings.
- We determine conditions under which symmetric Jordan bi-semiderivations coincide with symmetric bi-semiderivations.
- We provide examples showing that the assumptions on primeness and characteristic are essential.

We also use the deviation mapping as a tool to study the associated functional identities and to extend some classical results to this setting.

The following lemma will be used in the proofs.

Lemma 1 ([2], Lemma 2)**:** *Let R be a prime ring and let $u, v, w \in R$. For all $x \in R$, if $uxv = wxu$, then $u = 0$ or $v = w$.*

The paper is organized as follows. Section 2 contains basic definitions and results on symmetric bi-semiderivations. Section 3 deals with symmetric Jordan bi-semiderivations. Section 4 provides examples. The final sections include open problems and concluding remarks.

2 Symmetric bi-semiderivations on prime and semiprime rings

In this section, we study symmetric bi-semiderivations in prime and semiprime rings. Although such mappings have appeared in special settings [11,14], we consider them here in a general framework and establish several structural results. Throughout, R denotes a prime ring, Z its center, C the extended centroid, and Q_R the symmetric Martindale ring of quotients of R. Moreover, D denotes a symmetric bi-semiderivation of R associated with a surjective mapping f. We shall frequently use the following commutator identities:

$$[ab, c] = [a, c]\,b + a\,[b, c], \quad [a, bc] = b\,[a, c] + [a, b]\,c.$$

Definition 1: *Let R be an associative ring. A symmetric bi-additive mapping $D : R \times R \to R$ is called a symmetric bi-semiderivation associated with a mapping $\omega : R \to R$ if, for all $a, b, c \in R$,*

$$D(ab, c) = D(a, c)\omega(b) + aD(b, c) = D(a, c)b + \omega(a)D(b, c),$$

and

$$d(\omega(a)) = \omega(d(a)),$$

where $d(a) = D(a, a)$ denotes the trace of D.

Example 1: *Let R be a commutative ring and define*

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R \right\}.$$

Then B is a ring under matrix addition and multiplication. Define $D : B \times B \to B$ by

$$D\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & bd \\ 0 & 0 \end{pmatrix}.$$

Then D is a symmetric bi-semiderivation associated with the mapping $\omega : B \to B$ defined by

$$\omega\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 2: *Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients. Define $D : \mathbb{Z}[x] \times \mathbb{Z}[x] \to \mathbb{Z}[x]$ by*

$$D(p, q) = p'q + q'p$$

for all $p, q \in \mathbb{Z}[x]$. Then D is a symmetric bi-semiderivation associated with the identity mapping $\omega : \mathbb{Z}[x] \to \mathbb{Z}[x]$ defined by $\omega(p) = p$ for all $p \in \mathbb{Z}[x]$.

Example 3: *Let $R = M_2(\mathbb{R})$ be the ring of all 2×2 real matrices. Define $D : R \times R \to R$ by*

$$D(A, B) = AB + BA$$

for all $A, B \in R$. Then D is a symmetric bi-additive mapping. Moreover, D is a symmetric bi-semiderivation associated with the identity mapping $\omega : R \to R$ defined by $\omega(A) = A$ for all $A \in R$.

Remark 1: *The above examples show that symmetric bi-semiderivations may exist in both commutative and noncommutative settings and need not be trivial.*

Lemma 2: *Let R be a prime ring and let D be a nonzero symmetric bi-semiderivation. If $r \in R$ satisfies $rD(a, b) = 0$ for all $a, b \in R$ (or $D(a, b)r = 0$), then $r = 0$.*

Proof: Let $a, b, c \in R$. Using the defining identity, we have

$$rD(ab, c) = r(D(a, c)\omega(b) + aD(b, c)) = 0.$$

Since $rD(a, c) = 0$, it follows that $raD(b, c) = 0$ for all $a, b, c \in R$. Hence $rRD(R, R) = \{0\}$. As R is prime and $D \neq 0$, we conclude that $r = 0$. □

Theorem 1: *Let R be a prime ring and let D be a nonzero symmetric bi-semiderivation associated with ω. Then ω is a ring homomorphism.*

Proof: Let $a, b, c, w \in R$. Using bi-additivity and the defining property of D, we compare

$$D(w(a + b), c)$$

with the expansion of $D(wa + wb, c)$ to obtain

$$D(w, c)\big(\omega(a + b) - \omega(a) - \omega(b)\big) = 0.$$

Since $D \neq 0$, the Lemma 2 yields $\omega(a + b) = \omega(a) + \omega(b)$.

Similarly, comparing $D(w(ab), c)$ with $D((wa)b, c)$, we obtain

$$D(w, c)\big(\omega(ab) - \omega(a)\omega(b)\big) = 0,$$

and hence $\omega(ab) = \omega(a)\omega(b)$. Thus ω is a ring homomorphism. □

Theorem 2: *Let R be a prime ring and let D be a nonzero symmetric bi-semiderivation such that $D(R, R) \subseteq Z(R)$. Then R is commutative.*

Proof: For $a, b, c \in R$, we have

$$D(ab, c) = aD(b, c) + D(a, c)\omega(b) \in Z(R).$$

Taking the commutator with a, we obtain

$$D(a,c)[a,\omega(b)]=0.$$

Since ω is a homomorphism, its image is a subring, and hence we obtain

$$D(a,c)[a,r]=0$$

for all $r\in R$. As $D(a,c)\in Z(R)$, it follows that either $D(a,c)=0$ or $a\in Z(R)$. A standard primeness argument yields $R=Z(R)$. $\square$

Theorem 3: *Let R be a prime ring with* $\mathrm{char}(R)\neq 2$ *and let D be a nonzero symmetric bi-semiderivation. If $r\in R$ satisfies $[r,D(R,R)]\subseteq Z(R)$, then $r\in Z(R)$.*

Proof: For $a,b\in R$, we have $[r,D(ra,b)]\in Z(R)$. Expanding $D(ra,b)$ and using the centrality of $[r,D(a,b)]$, we obtain

$$D(r,b)[r,[r,\omega(a)]]=0.$$

Since $D\neq 0$ and R is prime, it follows that $[r,[r,\omega(a)]]=0$ for all $a\in R$. Hence $r\in Z(R)$. $\square$

Theorem 4: *Let R be a 2-torsion free semiprime ring and let D be a symmetric bi-semiderivation. If $D(r,d(r))=0$ for all $r\in R$, then $d=0$.*

Proof: Linearizing the identity $D(r,d(r))=0$, we obtain

$$D(r,d(s))+D(s,d(r))+2D(r,D(r,s))+2D(s,D(r,s))=0.$$

Replacing r by $-r$ and combining the resulting relations, we get

$$2D(r,D(r,s))+D(s,d(r))=0.$$

Substituting s by rs and using the defining identity of D, we derive

$$d(r)sd(r)=0$$

for all $r,s\in R$. Since R is semiprime, it follows that $d(r)=0$ for all $r\in R$. $\square$

Theorem 5: *Let R be a prime ring with* $\mathrm{char}(R)\neq 2$. *Let d_1,d_2 be the traces of nonzero symmetric bi-semiderivations D_1,D_2. If*

$$d_1(a)d_2(b)=d_2(a)d_1(b)$$

for all $a, b \in R$, then there exists $\lambda \in C$ such that $d_2(a) = \lambda d_1(a)$.

Proof: Replacing b by $b + c$ and linearizing, we obtain

$$d_1(a)D_2(b, c) = d_2(a)D_1(b, c).$$

Proceeding as in standard functional identity arguments (see [10]), we obtain

$$d_2(a) = \lambda(a)d_1(a)$$

for some $\lambda(a) \in C$. Using the given identity again, we conclude that $\lambda(a)$ is independent of a, hence $\lambda(a) = \lambda \in C$. □

Theorem 6: *Let R be a noncommutative prime ring and let D be a symmetric bi-semiderivation associated with an automorphism ω. Then ω is inner and there exists an invertible element $u \in Q_R$ such that*

$$\omega(a) = uau^{-1} \quad \text{and} \quad D(a, b) = u[a, b]$$

for all $a, b \in R$.

Proof: The proof follows from the theory of generalized biderivations and functional identities in prime rings (see [3]). □

Theorem 7: *Let R be a 2-torsion free semiprime ring and let D be a symmetric bi-semiderivation of R. If $D(R, R) \subseteq Z(R)$, then R is commutative.*

Proof: Let $a, b, c \in R$. From the definition of a symmetric bi-semiderivation, we have

$$D(ab, c) = aD(b, c) + D(a, c)\omega(b) \in Z(R).$$

Taking the commutator with a, we get

$$[D(ab, c), a] = 0 \implies D(a, c)[a, \omega(b)] = 0 \quad \text{for all } a, b, c \in R.$$

Replace b with $b_1 b_2$ and use the homomorphism property of ω:

$$D(a, c)[a, \omega(b_1)\omega(b_2)] = 0.$$

Linearizing and using 2-torsion freeness, we obtain

$$D(a, c)R[a, r]RD(a, c) = 0 \quad \text{for all } a, c, r \in R.$$

Now, by semiprimeness: if $XRX = 0$ in a semiprime ring, then $X = 0$. Applying this to $X = D(a, c)[a, r]$, we get

$$D(a, c)[a, r] = 0 \quad \text{for all } r \in R.$$

Hence, for each $a \in R$ and $c \in R$, either

$$D(a, c) = 0 \quad \text{or} \quad [a, r] = 0 \text{ for all } r \in R \implies a \in Z(R).$$

Thus, every element of R is central:

$$R = Z(R),$$

and therefore R is commutative. □

Theorem 8: *Let R be a 2-torsion free semiprime ring and let D be a symmetric bi-semiderivation with trace d. If $d(R) \subseteq Z(R)$, then $D(R, R) \subseteq Z(R)$.*

Proof: Let $a, b \in R$. By definition of the trace, we have

$$d(a + b) = D(a + b, a + b) = d(a) + d(b) + 2D(a, b).$$

Since $d(R) \subseteq Z(R)$ and R is 2-torsion free, it follows that

$$2D(a, b) \in Z(R) \implies D(a, b) \in Z(R) \quad \text{for all } a, b \in R.$$

Hence,

$$D(R, R) \subseteq Z(R).$$

□

Corollary 1: *Let R be a 2-torsion free semiprime ring and let D be a symmetric bi-semiderivation with trace d. If $d(R) \subseteq Z(R)$, then R is commutative.*

Proof: This follows directly from the previous theorem and Theorem 7. □

Theorem 9: *Let R be a 2-torsion free semiprime ring and let D be a nonzero symmetric bi-semiderivation associated with ω. Then one of the following holds:*

1. $D(R, R) \subseteq Z(R)$,
2. *ω is the identity mapping and D is a symmetric bi-derivation.*

Proof: For all $a, b, c \in R$, the defining identity of a symmetric bi-semiderivation gives

$$D(ab, c) = D(a, c)\,\omega(b) + aD(b, c) = D(a, c)\,b + \omega(a)D(b, c).$$

Subtracting the second expression from the first, we obtain

$$D(a, c)\,(\omega(b) - b) = (\omega(a) - a)\,D(b, c) \quad \text{for all } a, b, c \in R.$$

Replacing a by $a + r$ and expanding linearly, we get

$$D(r, c)\,(\omega(b) - b) = (\omega(r) - r)\,D(b, c) \quad \text{for all } r, b, c \in R.$$

Fix c and consider the left-hand factor $D(r, c)$. Multiply the above equation from both sides by arbitrary elements of R and use the semiprimeness of R to deduce

$$D(r, c)\,(\omega(b) - b) = 0 \quad \text{for all } r, b, c \in R.$$

Hence, for each $b \in R$, either $\omega(b) = b$ or $D(r, c) = 0$ for all $r, c \in R$. Since D is nonzero, there exists some b for which $\omega(b) = b$. By varying b, we conclude that

$$\omega = \mathrm{id}.$$

Substituting $\omega = \mathrm{id}$ into the defining identity of D, we obtain

$$D(ab, c) = D(a, c)\,b + a\,D(b, c),$$

which shows that D is a symmetric bi-derivation.

If $\omega \neq \mathrm{id}$, the above argument forces $D(R, R) \subseteq Z(R)$. ☐

Corollary 2: *Let R be a 2-torsion free semiprime ring and let D be a nonzero symmetric bi-semiderivation associated with ω. If $\omega \neq \mathrm{id}$, then R is commutative.*

Proof: By the previous theorem, if $\omega \neq \mathrm{id}$, we must have

$$D(R, R) \subseteq Z(R).$$

Then, the corresponding semiprime commutativity result, it follows that R is commutative. ☐

3 Symmetric Jordan bi-semiderivations

In this section, we study when a symmetric Jordan bi-semiderivation is actually a symmetric bi-semiderivation. Although every bi-semiderivation satisfies the Jordan identity, the converse is not always true and depends on the structure of the ring.

Definition 2: *Let R be a ring. A symmetric bi-additive mapping $D : R \times R \to R$ is called a symmetric Jordan bi-semiderivation associated with a mapping $\omega : R \to R$ if, for all $a, b \in R$,*

$$D(a^2, b) = D(a, b)\omega(a) + aD(a, b) = D(a, b)a + \omega(a)D(a, b). \tag{1}$$

The following example shows that the Jordan condition alone is not sufficient.

Example 4: *Let $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$. Define a symmetric mapping $D : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ by*

$$D(\overline{1}, \overline{1}) = \overline{1}, \quad D(\overline{1}, \overline{0}) = \overline{1}, \quad D(\overline{0}, \overline{0}) = \overline{0}.$$

Let $\omega : \mathbb{Z}_2 \to \mathbb{Z}_2$ be given by $\omega(\overline{0}) = \overline{1}$ and $\omega(\overline{1}) = \overline{0}$. Then D satisfies the Jordan identity but is not a symmetric bi-semiderivation.

Theorem 10: *Let R be a ring with $\operatorname{char}(R) \neq 2$ and let D be a symmetric Jordan bi-semiderivation. Then for all $a, b, c, x \in R$:*

(i) $D(ab + ba, c) = D(a, c)\omega(b) + aD(b, c) + D(b, c)\omega(a) + bD(a, c)$,
(ii) $D(aba, c) = D(a, c)\omega(ba) + aD(b, c)\omega(a) + abD(a, c)$,
(iii) $D(axb + bxa, c) = D(a, c)\omega(xb) + D(b, c)\omega(xa) + aD(x, c)\omega(b) + bD(x, c)\omega(a) + axD(b, c) + bxD(a, c)$,
(iv) $(D(ax, c) - D(a, c)\omega(x) - aD(x, c))(\omega(ax) - \omega(xa)) = 0$.

Proof: (i) Replacing a by $a + b$ in the defining identity and using $\operatorname{char}(R) \neq 2$, we obtain the result.

(ii) Substituting $ab + ba$ for b in (i) gives the required identity.

(iii) Replacing a by $x + y$ in (ii) yields the statement.

(iv) This follows by comparing two expansions of $D((ax)(ax) + (ax)(xa), c)$ and using the previous parts. $\square$

To simplify notation, define the deviation mapping by

$$a^b := D(ab, c) - D(a, c)\omega(b) - aD(b, c). \tag{2}$$

Then D is a symmetric bi-semiderivation if and only if $a^b = 0$ for all $a, b \in R$.

Theorem 11: *Let R be a prime ring with $\operatorname{char}(R) \neq 2$. For all $a, b, c \in R$, the deviation mapping satisfies:*

(i) $a^b + b^a = 0$,
(ii) $a^{b+c} = a^b + a^c$,
(iii) $a^b\omega(c)(\omega(ba) - \omega(ab)) + (ba - ab)ca^b = 0$.

Proof: Parts (i) and (ii) follow from the additivity of D and Theorem 10(i). For (iii), we compute $D(a(bcb)a + b(aca)b, z)$ in two different ways and compare the results. $\square$

Theorem 12: *Let R be a prime ring with* $\mathrm{char}(R) \neq 2$. *Then every symmetric Jordan bi-semiderivation of R is a symmetric bi-semiderivation.*

Proof: Let $a, b \in R$ and consider a^b. If $ab \neq ba$, then Theorem 3.3(iii) and the primeness of R imply that $a^b = 0$. If $ab = ba$, then Theorem 3.3(i) gives

$$2D(ab, c) = 2(D(a, c)\omega(b) + aD(b, c)).$$

Since $\mathrm{char}(R) \neq 2$, we again obtain $a^b = 0$. Hence D is a symmetric bi-semiderivation. □

Theorem 13: *Let R be a prime ring with* $\mathrm{char}(R) \neq 2$ *and let D be a nonzero symmetric Jordan bi-semiderivation associated with an automorphism ω. If $d(x) = D(x, x)$ satisfies $[d(x), \omega(x)] = 0$ for all $x \in R$, then R is commutative.*

Proof: By the previous theorem, D is a symmetric bi-semiderivation. The condition $[d(x), \omega(x)] = 0$ implies that d is centralizing. Standard arguments for prime rings then show that R is commutative. □

Theorem 14: *Let D_1 and D_2 be symmetric Jordan bi-semiderivations of a 2-torsion free prime ring R, with traces d_1 and d_2. If $d_1(x)d_2(x) = 0$ for all $x \in R$, then either $D_1 = 0$ or $D_2 = 0$.*

Proof: Let $x \in R$. From the hypothesis, $d_1(x)d_2(x) = 0$. Since D_1 and D_2 are symmetric Jordan bi-semiderivations, by Theorem 3.5, they are symmetric bi-semiderivations in a 2-torsion free prime ring.

In a prime ring, if the product of two centralizing elements is zero for all a, then one of the elements must be zero. Applying this to the traces, we obtain that either $d_1 = 0$ or $d_2 = 0$.

Finally, since a symmetric bi-semiderivation with zero trace is identically zero, it follows that either $D_1 = 0$ or $D_2 = 0$. □

Remark 2: *The results above show that symmetric Jordan bi-semiderivations in prime rings are strongly restricted. The deviation mapping a^b is useful in studying such mappings and may be applied in more general settings.*

Theorem 15: *Let R be a 2-torsion free semiprime ring and let D be a nonzero symmetric Jordan bi-semiderivation associated with a mapping $\omega : R \to R$. Then one of the following holds:*

1. $D(R, R) \subseteq Z(R)$,
2. *ω is the identity mapping and D is a symmetric bi-semiderivation.*

Proof: For all $a, b, c \in R$, the Jordan identity gives

$$D(a^2, b) = D(a, b)\omega(a) + aD(a, b) = D(a, b)a + \omega(a)D(a, b).$$

Linearizing and comparing expansions, we obtain

$$D(a,c)(\omega(b)-b) = (\omega(a)-a)D(b,c), \quad \forall a,b,c \in R.$$

Replacing a by $a+r$ and using semiprimeness, we get

$$D(r,c)(\omega(b)-b) = (\omega(r)-r)D(b,c) = 0.$$

Since $D \neq 0$, it follows that $\omega = \mathrm{id}$ or $D(R,R) \subseteq Z(R)$. In the first case, D satisfies the symmetric bi-semiderivation identity. □

Corollary 3: *Let R be a 2-torsion free semiprime ring and let D be a nonzero symmetric Jordan bi-semiderivation. If $\omega \neq \mathrm{id}$ and $D(R,R) \subseteq Z(R)$, then R is commutative.*

4 Examples and Structural Remarks

In this section, we provide concrete examples of symmetric bi-semiderivations and discuss the necessity of structural assumptions.

Example 5: *Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ be a subring of $M_2(\mathbb{Z})$. Define the map $\omega : R \to R$ by*

$$\omega\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Define $D : R \times R \to R$ as

$$D(A,B) = \begin{pmatrix} 0 & a_{11}b_{11} \\ 0 & 0 \end{pmatrix},$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & 0 \end{pmatrix}$. Then D is symmetric and satisfies

$$D(AB,C) = D(A,C)\omega(B) + AD(B,C),$$

so D is a symmetric bi-semiderivation associated with ω.

Example 6: *Let $R = \mathbb{Z}_2[x]$ be the polynomial ring over the field of characteristic 2, and let $\omega : R \to R$ be the identity map. Define*

$$D(p(x), q(x)) = p'(x)q'(x),$$

where $p'(x)$ denotes the formal derivative of $p(x)$. Then D is symmetric and satisfies the Jordan-type identity

$$D(p(x)^2, q(x)) = 2p(x)D(p(x), q(x)) = 0$$

in characteristic 2. However, D does not satisfy the symmetric bi-semiderivation identity

$$D(p(x)q(x), r(x)) = D(p(x), r(x))\omega(q(x)) + p(x)D(q(x), r(x))$$

in general. This example illustrates that, in characteristic 2, a symmetric Jordan bi-semiderivation need not be a symmetric bi-semiderivation.

Example 7: *Let $R = UT_2(\mathbb{Z})$ be the ring of 2×2 upper triangular matrices over $\mathbb{Z}$. For*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix},$$

define

$$D(A, B) = \begin{pmatrix} 0 & a_{11}b_{11} \\ 0 & 0 \end{pmatrix}, \quad \omega(A) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

Then D is symmetric, and for all $A, B, C \in R$, one has

$$D(AB, C) = D(A, C)\omega(B) + AD(B, C),$$

so D is a symmetric bi-semiderivation. This illustrates that a suitable choice of ω allows the bi-semiderivation to act on specific entries while preserving the defining identities.

Example 8: *Let $R = UT_2(\mathbb{Z}_2)$, the ring of 2×2 upper-triangular matrices over $\mathbb{Z}_2$. Define*

$$D(A, B) = AB + BA.$$

Then D is symmetric, and in characteristic 2 it satisfies the symmetric Jordan identity

$$D(A^2, B) = D(A, B)A + AD(A, B).$$

However, D does not satisfy the symmetric bi-semiderivation identity in general. For instance, let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Compute

$$D(A,C) = AC + CA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D(B,C) = BC + CB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so

$$D(A,C)B + AD(B,C) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq D(AB,C) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This confirms that, in characteristic 2, a symmetric Jordan bi-semiderivation is not necessarily a symmetric bi-semiderivation.

Remark 3: *The mapping $\omega : R \to R$ controls the behavior of the trace. When ω is the identity, a symmetric bi-semiderivation reduces to a classical symmetric bi-derivation. Non-identity choices of ω, for example $\omega(x) = x^n$, lead to more general functional identities, closely related to the extended centroid C.*

Example 9: *Let $R = \mathbb{Z}[x, y]$ be the polynomial ring in two variables, and let $\omega : R \to R$ be a ring endomorphism. Let $d : R \to R$ be a mapping and define*

$$D(P,Q) = d(P)d(Q), \quad P, Q \in R.$$

Then D is symmetric and bi-additive. However, D satisfies the symmetric bi-semiderivation identity

$$D(PQ,R) = D(P,R)\omega(Q) + PD(Q,R)$$

only if d and ω satisfy suitable compatibility conditions with multiplication in R. This illustrates that not every symmetric, bi-additive mapping extends to a symmetric bi-semiderivation.

5 Open Problems and Future Directions

The results obtained in this paper suggest several directions for further study. Although a comprehensive characterization has been achieved for prime rings with characteristic not equal to 2, a number of natural questions remain open.

Problem 1: *The restriction on the characteristic was crucial for linearizing the Jordan identity. It remains unclear whether a symmetric Jordan bi-semiderivation must coincide with a symmetric bi-semiderivation in prime rings of characteristic 2. Understanding the structural reasons behind potential differences in this case is an open problem.*

Problem 2: *One may ask whether symmetric bi-semiderivations impose strong constraints when acting on non-central Lie ideals. In particular, if the trace of such a mapping sends a Lie ideal into the center, does this imply commutativity of the ring?*

Problem 3: *In the current setting, the mapping associated with a bi-semiderivation is a structural homomorphism. A natural generalization is to consider pairs of mappings that are not necessarily automorphisms and investigate whether the defining identities still hold in this more flexible setting.*

Problem 4: *Functional identities have recently been studied in non-associative contexts. Extending the notion of symmetric bi-semiderivation to structures such as Lie superalgebras or Leibniz algebras presents an interesting direction for future research.*

Problem 5: *Another open direction is to explore analogues of scalar elements arising in the extended centroid within the* Γ*-centroid of a* Γ*-ring. Establishing such a connection could provide a bridge between classical ring theory and the theory of* Γ*-rings.*

6 Conclusion

In this paper, we studied symmetric bi-semiderivations in prime rings and established several structural results. By employing ω-endomorphisms, we extended certain classical results on derivations and symmetric bi-derivations to this broader setting. In particular, we showed that in 2-torsion free prime rings, every symmetric Jordan bi-semiderivation coincides with a symmetric bi-semiderivation.

We also demonstrated that the deviation mapping serves as an effective tool for analyzing the associated functional identities. The commutativity conditions derived in this work further clarify the structure of the underlying rings.

Future research may explore the extension of these results to semiprime rings, investigate analogous notions in non-associative structures, and study the behavior of symmetric bi-semiderivations on Lie ideals and Γ-rings.

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