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Research article

On characterizations of general type-2 fuzzy regular grammars and its languages

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Abstract: The purpose of this paper is to introduce and study the notion of various important forms of general type-2 fuzzy grammars, namely, the general type-2 fuzzy grammar in normal form, general type-2 fuzzy linear grammar, general type-2 fuzzy left linear grammar, and general type-2 fuzzy right linear grammar. We establish that each of these forms is equivalent to a general type-2 fuzzy regular grammar (except for an empty string) in the sense that they induce the same class of general type-2 fuzzy languages. These results not only generalizes classical equivalence theorems from fuzzy grammar theory but also provide a theoretical basis for the analysis and future development of general type-2 fuzzy grammar theory.

Mathematics Subject Classification: 03E72, 68Q42, 68Q45, 68Q70
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1 Introduction

In classical formal language theory, grammars operate under a binary framework where a string either belongs to a language or does not. However, natural languages inherently exhibit vagueness, ambiguity, and graded membership, which cannot be adequately captured by such crisp structures. To bridge this gap between formal languages and linguistic reality, Lee and Zadeh [8] introduced the notion of fuzzy languages and fuzzy grammars by incorporating the concept of fuzziness into formal grammatical systems. This framework is particularly useful in areas such as natural language processing, pattern recognition, and approximate reasoning, where uncertainty plays a central role.

Following this development, substantial research has been carried out to analyze the structural properties of fuzzy grammatical systems. In particular, the study of normal forms and restricted classes such as linear, left-linear, and right-linear fuzzy grammars has attracted considerable attention, along with investigations into their equivalence properties. In this context, Chaudhari and Komejwar [3] examined fuzzy regular grammars and established important structural results. Further extensions of the classical framework have also been explored, including fuzzy multiset regular grammars by Sharma et. al. [14], as well as intuitionistic fuzzy regular grammars by Rajasekar and Thilagavathi [12]. These studies demonstrate that significant efforts have been made to generalize classical grammatical models within various fuzzy frameworks in order to handle different aspects of uncertainty and imprecision.

Despite these advancements, most of the existing models are based on type-1 fuzzy sets introduced by Zadeh [17], where the degree of membership is represented by a precise numerical value $[0, 1]$. However, in many real-world problems, there exists an additional level of uncertainty associated with the membership values themselves, which cannot be effectively captured within the type-1 framework. This limitation motivates the use of type-2 fuzzy sets, introduced by Zadeh [18], wherein the membership grades are themselves fuzzy, providing a more expressive and flexible representation of uncertainty. Following this concept, Mizumoto and Tanaka [11] were the first to introduce the concept of fuzzy–fuzzy automata, also referred to as type-2 fuzzy automata. Subsequently, motivated by the work of Ying [16], Jiang and Tang [6] extended this framework to the interval type-2 setting, thereby proposing interval type-2 fuzzy automata along with interval type-2 fuzzy pushdown automata. Later, Sharan et al. [13], inspired by Jiang and Tang's approach, introduced the notion of interval type-2 fuzzy grammars and examined their relationship with the corresponding automata. In continuation of these efforts, Jing [5] generalized the interval type-2 fuzzy set-based model of Jiang and Tang [6] into a more comprehensive general type-2 fuzzy environment.

Although interval type-2 fuzzy grammars provide a richer framework than type-1 fuzzy grammars, they are restricted to interval-valued secondary membership functions and therefore cannot fully represent complex uncertainties arising in many real-world situations. In practical applications such as natural language processing, decision-making, pattern recognition, and knowledge representation, uncertainty may possess highly irregular and non-uniform characteristics that cannot be adequately modeled by interval type-2 structures. General type-2 fuzzy sets offer a more comprehensive representation of uncertainty by allowing arbitrary secondary membership functions, thereby capturing a wider range of imprecision and ambiguity. Despite this expressive capability, the structural theory of general type-2 fuzzy grammars remains largely unexplored compared with the extensive studies available for type-1 and interval type-2 fuzzy grammars (cf. [15], for more details). In particular, the existence of equivalent normal and linear forms and their effect on language generation have not been systematically investigated. This motivates the present study, which aims to establish a theoretical foundation for general type-2 fuzzy grammars and extend classical equivalence results to this broader framework.

In this direction, the present work focuses on general type-2 fuzzy grammars and investigates their structural properties. Specifically, we introduce several significant forms of general type-2 fuzzy grammars, including the normal form, linear form, left-linear form, and right-linear form of general type-2 fuzzy grammars. It is shown that each of these forms is equivalent to general type-2 fuzzy regular grammars (except for an empty string) in the sense that they generate the same class of general type-2 fuzzy languages. This equivalence result indicates that the structural constraints imposed by these grammatical forms do not alter their generative capacity. Consequently, these results not only generalizes classical equivalence theorems from fuzzy grammar theory but also provide a theoretical basis for the analysis and future development of general type-2 fuzzy grammar theory.

2 Preliminaries

In this section, we recall some concepts related to GT_2F sets, GT_2F automata and its languages. Throughout, $\mathbb{X}$ is a nonempty set, $\mathcal{I} = [0,1]$ and $[\mathcal{I}] = \{[a,b] : a \leq b, a, b \in \mathcal{I}\}$.

2.1 General type-2 fuzzy sets

We begin with the following:

Definition 2.1: [10] *A GT_2F sets, denoted by $\tilde{\mathcal{A}}$, whose membership function is type-2 fuzzy sets, defined as,*

$$\tilde{\mathcal{A}} = \{((x,v), \mu_{\tilde{\mathcal{A}}}(x,v)) : x \in \mathbb{X}, v \in J_x \subseteq \mathcal{I}\},$$

where $\mu_{\tilde{\mathcal{A}}}$ is type-2 membership function $\mu_{\tilde{\mathcal{A}}} : \mathbb{X} \times J_x \to \mathcal{I}, \forall x \in \mathbb{X}, J_x \subseteq \mathcal{I}$ in which $\mu_{\tilde{\mathcal{A}}} \in [0,1]$ can also be expressed as

$$\tilde{\mathcal{A}} = \int_{x \in X} \int_{u \in J_x} \frac{\mu_{\tilde{\mathcal{A}}}(x,v)}{(x,v)}, J_x \subseteq \mathcal{I},$$

where $\int \int$ denotes the union over all admissible x and u. For discrete universes of discourse $\int$ is replaced by $\sum$.

Throughout, $\mathscr{F}_{GT_2}(\mathbb{X})$ will denote the set of all GT_2F subsets of $\mathbb{X}$.

Definition 2.2: [10] *Let $\tilde{\mathcal{A}} \in \mathscr{F}_{GT_2}(\mathbb{X})$. Then for each value of $x \in \mathbb{X}$, say $x = x'$, the two-dimensional plane whose axes are $v \in J_{x'}$ and $\mu_{\tilde{\mathcal{A}}}(x', v)$ is called **vertical slice** of $\mu_{\tilde{\mathcal{A}}}(x,v)$. A secondary membership function is a vertical slice of $\mu_{\tilde{\mathcal{A}}}(x,u)$, i.e.,*

$$\mu_{\tilde{\mathcal{A}}}(x') \equiv \mu_{\tilde{\mathcal{A}}}(x = x', v) = \int_{v \in J_{x'}} \frac{g_{x'}(v)}{v}, J_{x'} \subseteq I, g_{x'}(v) \in \mathcal{I}.$$

Definition 2.3: *The domain of a secondary membership function is called the **primary membership** of x, which we also called as **secondary set**.*

In view of the concept of secondary sets, we can re-express $\tilde{\mathcal{A}} \in \mathscr{F}_{GT_2}(\mathbb{X})$ in terms of vertical slice as

$$\tilde{\mathcal{A}} = \{(x, \mu_{\tilde{\mathcal{A}}}(x)) : \forall x \in \mathbb{X}\}$$

$$or, as\ \tilde{\mathcal{A}} = \int_{x \in \mathbb{X}} \frac{\mu_{\tilde{\mathcal{A}}}(x)}{x} = \int_{x \in \mathbb{X}} \left[\int_{v \in J_x} \frac{g_x(v)}{v} \right] / x, \quad where\ J_x \subseteq \mathcal{I}.$$

If $\mathbb{X}$ and J_x are both discrete, then $\tilde{\mathscr{A}} \in \mathscr{F}_{GT_2}(\mathbb{X})$ can be expressed as follows

$$\begin{aligned}
\tilde{\mathscr{A}} &= \sum_{x\in\mathbb{X}}\left[\sum_{v\in J_x}\frac{g_x(v)}{v}\right]/x = \sum_{i=1}^{N}\left[\sum_{v\in J_{x_i}}\frac{g_x(v)}{v}\right]/x_i \\
&= \left[\sum_{k=1}^{M_1}\frac{g_x(v)}{v_{1k}}\right]/x_1 + \ldots + \left[\sum_{k=1}^{M_i}\frac{f_x(v)}{v_{ik}}\right]/x_i + \ldots \\
&\quad + \left[\sum_{k=1}^{M_N}\frac{g_x(v)}{v_{Nk}}\right]/x_N,
\end{aligned}$$

Throughout this paper, we consider both $\mathbb{X}$ and J_x are discrete.

Now, we recall some operations on GT_2F sets from [9].

Definition 2.4: *[9] Let $\tilde{\mathscr{A}}, \tilde{\mathscr{B}} \in \mathscr{F}_{GT_2}(\mathbb{X})$ be two GT_2F sets, where*

$$\tilde{\mathscr{A}} = \int_{x\in\mathbb{X}}\frac{\mu_{\tilde{\mathscr{A}}}(x)}{x} = \int_{x\in\mathbb{X}}\left[\int_{v\in J_x^v}\frac{g_x(v)}{v}\right]/x, \quad where\ J_x^v \subseteq \mathcal{I},$$

$$\tilde{\mathscr{B}} = \int_{x\in\mathbb{X}}\frac{\mu_{\tilde{\mathscr{B}}}(x)}{x} = \int_{x\in\mathbb{X}}\left[\int_{v\in J_x^v}\frac{h_x(v)}{v}\right]/x, \quad where\ J_x^v \subseteq \mathcal{I}.$$

Then $\forall x \in \mathbb{X}$,

*(i) the **union** of $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ is defined as*

$$\tilde{\mathscr{A}} \cup \tilde{\mathscr{B}} = \int_{x\in\mathbb{X}}\frac{\mu_{\tilde{\mathscr{A}}\cup\tilde{\mathscr{B}}}(x)}{(x)}, where\ \ \mu_{\tilde{\mathscr{A}}\cup\tilde{\mathscr{B}}}(x) = \int_{v\in J_x^v}\int_{w\in J_x^w}\frac{g_x(v) \blacklozenge h_x(w)}{(v \vee w)},$$

where $\blacklozenge$ is the minimum operation and $\vee$ is the maximum operation.

*(ii) the **intersection** of $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ is defined as*

$$\tilde{\mathscr{A}} \cap \tilde{\mathscr{B}} = \int_{x\in\mathbb{X}}\frac{\mu_{\tilde{\mathscr{A}}\cap\tilde{\mathscr{B}}}(x)}{(x)}, where\ \ \mu_{\tilde{\mathscr{A}}\cap\tilde{\mathscr{B}}}(x) = \int_{v\in J_x^v}\int_{w\in J_x^w}\frac{g_x(v) \blacklozenge h_x(w)}{(v \wedge w)},$$

where $\wedge$ is the minimum operation. We denote

$$\tilde{\mathscr{A}}(x) \cap \tilde{\mathscr{B}}(x) = \int_{v\in J_x^v}\int_{w\in J_x^w}\frac{g_x(v) \blacklozenge h_x(w)}{(v \wedge w)}.$$

A GT_2F set is said to be empty if its membership function is identically $1/0$ on $\mathbb{X}$. Then $\tilde{\mathscr{A}}(x) \cap \phi(x) = \phi(x) \cap \tilde{\mathscr{A}}(x) = \phi(x) = 1/0$, and $\tilde{\mathscr{A}}(x) \cup \phi(x) = \phi(x) \cup \tilde{\mathscr{A}}(x) = \tilde{\mathscr{A}}(x)$

*(iii) the **complement** of $\tilde{\mathcal{A}}$ is defined as*

$$\tilde{\mathcal{A}}^c = \int_{x\in\mathbb{X}} \frac{\mu_{\tilde{\mathcal{A}}^c}(x)}{(x)}, \text{where } \mu_{\tilde{\mathcal{A}}^c}(x) = \int_{v\in J_x^v} \frac{g_x(v)}{(1-v)}.$$

*(iv) the **scale product** of $\tilde{\mathcal{A}}$ with κ is defined as,*

$$(\kappa\cdot\tilde{\mathcal{A}})(x) = \int_{\sigma\in J}\int_{v\in J_x^v} \frac{g(\sigma)\wedge g_x(v)}{(\sigma\wedge v)}, \text{where } \kappa = \int_{\sigma\in J}\frac{g(\sigma)}{\sigma},$$

$g(0)\in\mathcal{I}$, i.e., κ is a fixed fuzzy number in $\mathcal{I}$ and $J\subseteq\mathcal{I}$ is a closed sub-interval of $\mathcal{I}$.

*(v) the **height** of $\tilde{\mathcal{A}} = \left[\sum_{k=1}^{M_1} g_{x_1}(v_{1k})/v_{1k}\right]/x_1 + \ldots$ $+ \left[\sum_{k=1}^{M_N} g_{x_N}(v_{Nk})/v_{Nk}\right]/x_N$ is denoted and defined by*

$$\tilde{\mathscr{H}}(\tilde{\mathcal{A}}) = \frac{\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\ldots\sum_{h=1}^{M_N}(g_{x_1}(v_{1i})\wedge g_{x_2}(v_{2j})\wedge\ldots\wedge g_{x_N}(v_{Nh}))}{(v_{1i}\vee v_{2j}\vee\ldots\vee v_{Nh})}$$

2.2 General Type-2 fuzzy automata and its languages

In this subsection, we recall the concept of GT_2F finite automata and its languages from [5].

Definition 2.5: *A GT_2F automaton is a quintuple $\tilde{\mathscr{M}} = (S, V, \tilde{\delta}, s_0, \tilde{\mathcal{A}})$, where S and V are finite set of states and finite set of input alphabets respectively, $s_0\in S$ is an initial state and $\tilde{\delta}$ and $\tilde{\mathcal{A}}$ are characterized as follows:*

1. *$\tilde{\delta} : S\times V\to\mathscr{F}_{GT_2}(S)$, called the transition map s.t., for any $s\in S$ and $a\in V$, $\tilde{\delta}(s,a)\in\mathscr{F}_{GT_2}(S)$, and it may be seen as the possibility distribution of the states that the automaton in state s and with input a can enter. Formally, $\tilde{\delta}(s_i,a)(s_j)$ and $\tilde{\delta}(s_i,a)$ are defined as follows:*

$$\tilde{\delta}(s_i,a)(s_j) = \sum_{k=1}^{M_j}\frac{f_{s_j}(u_{jk})}{u_{jk}} = \frac{f_{s_j}(u_{j1})}{u_{j1}} + \frac{f_{s_j}(u_{j2})}{u_{j2}} + \ldots + \frac{f_{s_j}(u_{jM_j})}{u_{jM_j}}, \text{where } u_{jk}\in J_x\subseteq I,\ \ 0\le j\le N,\ 1\le k\le M_j.$$

$$\tilde{\delta}(s_i,a) = \frac{\left[\sum_{k=1}^{M_0}\frac{f_{s_0}(u_{0k})}{u_{0k}}\right]}{s_0} + \ldots + \frac{\left[\sum_{k=1}^{M_i}\frac{f_{s_i}(u_{ik})}{u_{ik}}\right]}{s_i} + \ldots + \frac{\left[\sum_{k=1}^{M_N}\frac{f_{s_N}(u_{Nk})}{u_{Nk}}\right]}{s_N}, \text{where } u_{ik}\in J_x\subseteq I,\ \ 0\le i\le N.$$

2. *$\tilde{\mathcal{A}}\in\mathscr{F}_{GT_2}(S)$ and for each $s_i\in S$, $\tilde{\mathcal{A}}(s_i)$ indicates intuitively the degree to which s_i is a final state. That is, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}(s_i)$ are define as follows:*

$$\tilde{\mathcal{A}} = \frac{\left[\sum_{k=1}^{M_0}\frac{1}{u_{1k}}\right]}{s_1} + \ldots + \frac{\left[\sum_{k=1}^{M_i}\frac{1}{u_{ik}}\right]}{s_i} + \ldots + \frac{\left[\sum_{k=1}^{M_N}\frac{1}{u_{Nk}}\right]}{s_N}, \text{where},\ u_{ik}\in J_x\subseteq I,\ \ 1\le i\le N.$$

$$\tilde{\mathcal{A}}(s_i) = \frac{1}{\{u_{i1},u_{i2},\ldots,u_{iMi}\}} = \frac{1}{[u_{i1},u_{iMi}]},\ \text{where},\ u_{jk}\in J_x\subseteq I,\quad 1\le i\le N,\ \ 1\le k\le M_j.$$

Remark 2.1: *GT_2F finite automata can be considered as a generalizations of fuzzy finite automata defined in [16] as GT_2F finite automata are based on GT_2F sets while fuzzy finite automata are based on T_1 fuzzy sets.*

Definition 2.6: *[6] Let $\tilde{\mathscr{M}} = (S, V, \tilde{\delta}, s_0, \tilde{\mathcal{A}})$ be a GT_2F automaton.*

1. *The transition function $\tilde{\delta}$ is extended to $\tilde{\delta}^* : S \times V^* \to \mathscr{F}_{GT_2}(S)$, where*

$$\tilde{\delta}(s, \tilde{\delta}) = 1/1/s$$

$$\tilde{\delta}^*(s, \theta a) = \bigcup_{p \in S} [\tilde{\delta}^*(s, \theta)(p).\tilde{\delta}(p, a)]$$

   for all $\theta \in V^*$ and $a \in V$, where $1/1/s$ is a singleton in S. i.e., the GT_2F subset of S with membership 1 (i.e., $1/1$ at s and $1/0$ membership for all the other elements of S). In addition, $\tilde{\delta}(s, \theta)(p).\tilde{\delta}(p, a)$ stands for the scale product of GT_2F set $\tilde{\delta}(p, a)$ with the parameter $\tilde{\delta}(s, \theta)(p)$.

2. *For any $\theta \in V^*$, the degree to which θ is accepted by $\tilde{\mathscr{M}}$ is*

$$\tilde{\mathscr{L}}(\tilde{\mathscr{M}}, \theta) = (\tilde{\delta}^*(s_0, \theta) \cap \tilde{\mathcal{A}}).$$

3. *A GT_2F language accepted by $\tilde{\mathscr{M}}$, denoted as $\tilde{\mathscr{L}}(\tilde{\mathscr{M}})$, defined by*

$$\tilde{\mathscr{L}}(\tilde{\mathscr{M}})(\theta) = \tilde{\mathscr{L}}(\tilde{\mathscr{M}}, \theta),\ \forall\ \theta \in V^*,\ where\ \mathscr{F}_{GT_2}(V^*)\ denotes\ the\ set\ of\ all\ GT_2F\ sets\ in\ V^*.$$

Example 2.1: *Let $\tilde{\mathscr{M}} = (S, V, \tilde{\delta}, s_0, \tilde{\mathcal{A}})$ be a $\mathcal{GT}_2\mathcal{F}$ automaton, where $S = \{s_0, s_1, s_2\}, V = \{a, b\}, \tilde{\mathcal{A}} = 0.05/0.01/s_1 + 0.03/0.02/s_1 + 0.06/0.03/s_1 + 0.04/0.06/s_2 + 0.06/0.07/s_2 + 0.05/0.08/s_2 + 0.07/1/s_2$ and $\tilde{\delta}$ is given by*

$$\begin{aligned}
\tilde{\delta}(s_0, a) &= 0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.06/0.04/s_1, \\
\tilde{\delta}(s_1, a) &= 0.01/0.06/s_1 + 0.03/0.07/s_1 + 0.07/0.08/s_1 + 0.06/0.02/s_2 + 0.03/0.03/s_2, \\
\tilde{\delta}(s_2, a) &= 0.05/0.04/s_2 + 0.02/0.05/s_2 + 0.07/0.06/s_2, \\
\tilde{\delta}(s_1, b) &= 0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.1/0.03/s_2 + 0.02/0.05/s_2, \\
\tilde{\delta}(s_2, b) &= 0.08/0.01/s_1 + 0.01/0.02/s_1 + 0.05/0.04/s_1 + 0.03/0.06/s_2 + 0.02/0.07/s_2.
\end{aligned}$$

Then, we have the following:

$$\tilde{\delta}(s_0, a) = 0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.06/0.04/s_1,$$

$$\begin{aligned}
\tilde{\delta^*}(s_0, a^2) &= (0.03/0.02 + 0.04/0.03 + 0.06/0.04).\tilde{\delta}(s_1, a) \\
&= (0.03/0.02 + 0.04/0.03 + 0.06/0.04).(0.01/0.06/s_1 + 0.03/0.07/s_1 \\
&\quad +0.07/0.08/s_1 + 0.06/0.02/s_2 + 0.03/0.03/s_2) \\
&= 0.01/0.04/s_1 + 0.03/0.04/s_1 + 0.06/0.04/s_1 + 0.04/0.02/s_2 \\
&\quad +0.03/0.03/s_2
\end{aligned}$$

$$\begin{aligned}
\tilde{\delta^*}(s_0, a^3) &= \tilde{\delta^*}(s_0, aa)(s_1).\tilde{\delta}(s_0, a) \cup \tilde{\delta^*}(s_0, aa)(s_2).\tilde{\delta}(s_2, a) \\
&= (0.01/0.04 + 0.03/0.04 + 0.06/0.04).(0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.06/0.04/s_1) \\
&\quad \cup(0.04/0.02 + 0.03/0.03).(0.05/0.04/s_2 + 0.02/0.05/s_2 + 0.07/0.06/s_2) \\
&= 0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.06/0.04/s_1 + 0.04/0.03/s_2 + 0.02/0.03/s_2 \\
&\quad +0.04/0.03/s_2
\end{aligned}$$

$$\begin{aligned}
\tilde{\delta^*}(s_0, a^3 b) &= \tilde{\delta^*}(s_0, aaa)(s_1).\tilde{\delta}(s_1, b) \cup \tilde{\delta^*}(s_0, aaa)(s_2).\tilde{\delta}(s_2, b) \\
&= (0.03/0.02 + 0.04/0.03 + 0.06/0.04).(0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.1/0.03/s_2 \\
&\quad +0.02/0.05/s_2) \cup \ (0.04/0.03 + 0.02/0.03 + 0.04/0.03).(0.08/0.01/s_1 + 0.01/0.02/s_1 \\
&\quad +0.05/0.04/s_1 + 0.03/0.06/s_2 + 0.02/0.07/s_2) \\
&= (0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.1/0.03/s_2 + 0.02/0.04/s_2) \cup \ (0.04/0.01/s_1 + \\
&\quad 0.01/0.02/s_1 + 0.04/0.03/s_1 + 0.03/0.03/s_2 + 0.02/0.03/s_2) \\
&= 0.04/0.03/s_1 + 0.01/0.03/s_1 + 0.04/0.03/s_1 + 0.02/0.04/s_2 + 0.02/0.04/s_2
\end{aligned}$$

$$\begin{aligned}
\tilde{\delta^*}(s_0, a^3 b) \cap \tilde{\mathcal{A}} &= 0.04/0.03/s_1 + 0.01/0.03/s_1 + 0.04/0.03/s_1 + 0.02/0.04/s_2 + 0.02/0.04/s_2 \\
&\quad \cap\ 0.05/0.01/s_1 + 0.03/0.02/s_1 + 0.06/0.03/s_1 + 0.04/0.06/s_2 + 0.06/0.07/s_2 \\
&\quad +0.05/0.08/s_2 + 0.07/1/s_2 \\
&= 0.04/0.01/s_1 + 0.03/0.02/s_1 + 0.04/0.03/s_1 + 0.02/0.04/s_2 + 0.02/0.04/s_2 + \\
&\quad 0.02/0.04/s_2 + 0.02/0.04/s_2.
\end{aligned}$$

The degree to which string a^3b is accepted by $\tilde{\mathscr{M}}$ is

$$\begin{aligned}
\tilde{\mathscr{L}}(\tilde{\mathscr{M}}, a^3b) &= \tilde{\mathscr{H}}(\tilde{\delta}^*(s_0, a^3b) \cap \tilde{\mathcal{A}}) \\
&= \tilde{\mathscr{H}}(0.04/0.01/s_1 + 0.03/0.02/s_1 + 0.04/0.03/s_1 + \\
&\quad 0.02/0.04/s_2 + 0.02/0.04/s_2 + 0.02/0.04/s_2 + 0.02/0.04/s_2) \\
&= (0.04 \wedge 0.02)/(0.01 \vee 0.04) + (0.04 \wedge 0.02)/(0.01 \vee 0.04) + \\
&\quad (0.04 \wedge 0.02)/(0.01 \vee 0.04) + (0.04 \wedge 0.02)/(0.01 \vee 0.04) + \\
&\quad (0.03 \wedge 0.02)/(0.02 \vee 0.04) + (0.03 \wedge 0.02)/(0.02 \vee 0.04) + \\
&\quad (0.03 \wedge 0.02)/(0.02 \vee 0.04) + (0.03 \wedge 0.02)/(0.02 \vee 0.04) + \\
&\quad (0.04 \wedge 0.02)/(0.01 \cup 0.04) + (0.04 \wedge 0.02)/(0.01 \vee 0.04) + \\
&\quad (0.04 \wedge 0.02)/(0.01 \vee 0.04) + (0.04 \wedge 0.02)/(0.01 \vee 0.04) \\
&= 0.02/0.04.
\end{aligned}$$

3 General Type-2 fuzzy regular grammar and its languages

In this section we introduce and study the concept of GT_2F grammar and establish their relationship with GT_2F languages.

Definition 3.1: *A GT_2F grammar is a quadruple $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$, where N and T are finite alphabet of nonterminal and terminal symbols respectively, $I \in N$ is a starting nonterminal symbol, and $\tilde{\mathcal{R}}$ is characterize as follows:*

1. *$\tilde{\mathcal{R}}$ is a finite collections of GT_2F productions over $N \cup T$ such that $\tilde{\mathcal{R}} = \{\alpha \xrightarrow{\tilde{\sigma}} \beta | \alpha, \beta \in (N \cup T)^*\}$, where $\tilde{\sigma}$ is a mapping from $(N \cup T)^* \times (N \cup T)^*$ to $\mathscr{F}_{GT_2}(T^*)$, called GT_2F transition function.*

We may interpret $\tilde{\sigma}(\alpha, \beta)$ as the grade of membership that α will be replaced by β, denoted by $\tilde{\sigma}(\alpha, \beta) = \tilde{\sigma}(\alpha \to \beta)$. For the sake of convenience, $\alpha \xrightarrow{\tilde{\sigma}} \beta$ is sometimes written as $\alpha \to \beta$ in $\tilde{\mathcal{R}}$.

Remark 3.1: *The above definition can be viewed as a natural extension of the existing fuzzy grammar framework. In the classical fuzzy grammar, each production rule $\alpha \to \beta$ is associated with a type-1 membership degree in $[0, 1]$, representing the certainty with which the replacement of α by β takes place. In the proposed GT_2F grammar, this single-valued membership degree is generalized to a GT_2F membership function $\tilde{\sigma}(\alpha, \beta)$, thereby allowing uncertainty not only in the applicability of a production rule but also in the membership grade itself. Thus, the algebraic structure of the grammar $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ remains unchanged, while the uncertainty model becomes more expressive. Moreover, whenever the secondary membership grades reduce to singleton values, GT_2F grammar collapses to the classical fuzzy grammar, showing that the latter is a special case of the proposed model.*

Definition 3.2: *Let $\alpha \xrightarrow{\tilde{\sigma}} \beta$ be a production of the GT_2F grammar $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$, and let γ, $\tilde{\delta} \in (N \cup T)^*$. Then the string $\gamma\beta\tilde{\delta}$ is said to be **directly derivable** from $\gamma\alpha\tilde{\delta}$, which we shall denote by*

$$\gamma\alpha\tilde{\delta}\overset{\tilde{\sigma}}{\Longrightarrow}\gamma\beta\tilde{\delta}.$$

Now, let $\alpha_1, \alpha_2, \dots, \alpha_m \in (N \cup T)^*$ such that $\alpha_i \overset{\tilde{\sigma}_i}{\Longrightarrow} \alpha_{i+1}, \qquad i = 1, 2, \dots, m-1$. Then α_m is said to be **derivable** from α_1 in the grammar $\tilde{\mathcal{G}}$, and we write

$$\alpha_1 \tilde{\mathcal{G}} \overset{\tilde{\sigma}^*}{\Longrightarrow} \alpha_m.$$

The sequence

$$\alpha_1 \overset{\tilde{\sigma}_1}{\Longrightarrow} \alpha_2 \overset{\tilde{\sigma}_2}{\Longrightarrow} \alpha_3 \cdots \overset{\tilde{\sigma}_{m-1}}{\Longrightarrow} \alpha_m$$

is called the **derivation chain** of α_m from α_1. The overall GT_2F derivation degree of the chain is defined by

$$\tilde{\sigma} = \tilde{\mathscr{H}}(\tilde{\sigma}_1 \cap \tilde{\sigma}_2 \cap \cdots \cap \tilde{\sigma}_{m-1}).$$

Definition 3.3: *Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a GT_2F grammar. Then $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) \in \mathscr{F}_{GT_2}(T^*)$ is defined as*

$$\tilde{\mathscr{L}}(\tilde{\mathcal{G}}, \theta) = \tilde{\mathscr{H}}\{\tilde{\sigma} \,:\, I \underset{\tilde{\mathcal{G}}}{\overset{\tilde{\sigma}^*}{\Longrightarrow}} \theta \mid \theta \in T^*\},$$

where $\mathscr{F}_{GT_2}(T^*)$ denotes the set of all GT_2F sets in T^*.

Example 3.1: *Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a $\mathcal{GT}_2\mathcal{F}$ grammar, where $N = (I, A, B)$, $T = \{a, b\}$ and $\tilde{\mathcal{R}} = \{I \xrightarrow{0.04/0.03} aI, I \xrightarrow{0.05/0.02} aA, A \xrightarrow{0.06/0.04} aA, A \xrightarrow{0.05/0.06} aB, B \xrightarrow{0.03/0.05} b\}$. If the derivation of $\theta = (aaab)$ in $\tilde{\mathcal{G}}$ are as follows*

1. $I \xrightarrow{0.04/0.03} aI \xrightarrow{0.05/0.02} aaA \xrightarrow{0.05/0.06} aaaB \xrightarrow{0.03/0.05} aaab$
2. $I \xrightarrow{0.05/0.02} aA \xrightarrow{0.06/0.04} aaA \xrightarrow{0.05/0.06} aaaB \xrightarrow{0.03/0.05} aaab$ *, then,*

$$\begin{aligned} \tilde{\mathscr{L}}(\tilde{\mathcal{G}}, \theta) &= \tilde{\mathscr{H}}\{0.04/0.03 \cap 0.05/0.02 \cap 0.05/0.06 \cap 0.03/0.05 \\ &\quad \cap 0.05/0.02 \cap 0.06/0.04 \cap 0.05/0.06 \cap 0.03/0.05\} \\ &= \vee\{0.04, 0.05, 0.05, 0.03, 0.05, 0.06, 0.05, 0.03\}/ \\ &\quad \vee\{0.03, 0.02, 0.06, 0.05, 0.02, 0.04, 0.06, 0.05\} \\ &= 0.06/0.06. \end{aligned}$$

Definition 3.4: *Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a GT_2F grammar. Then $\tilde{\mathcal{G}}$ is called **regular** if every production $\alpha \xrightarrow{\tilde{\sigma}} \beta \in \tilde{\mathcal{R}}$ satisfies $\tilde{\sigma}(\alpha, \beta) \neq 1/0$, and is one of the following forms: $A \xrightarrow{\tilde{\sigma}} aB, A \xrightarrow{\tilde{\sigma}} a, I \xrightarrow{\tilde{\sigma}} \mu$, where $A, B \in N$, $a \in T$, and $\tilde{\sigma}(A, \beta)$ denotes the GT_2F degree of the production. Accordingly, $\tilde{\mathscr{L}}_{\tilde{\mathcal{G}}}$ is called GT_2F regular language.*

Definition 3.5: *Let $\tilde{\mathcal{G}}'$ and $\tilde{\mathcal{G}}''$ be Two GT_2F grammars. Then $\tilde{\mathcal{G}}'$ and $\tilde{\mathcal{G}}''$ are said to be equivalent ($\approx$) if they generate the same class of GT_2F languages, i.e., $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}') = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}'')$.*

Now we recall the following result from [15].

Proposition 3.1: *Let $\underset{\sim}{\mathcal{G}}$ be a GT_2F regular grammar, then $\exists$ $\mathcal{GT}_2\mathcal{F}$ automata $\underset{\sim}{\mathscr{M}}$ such that $\underset{\sim}{\mathscr{L}}(\underset{\sim}{\mathscr{M}}) = \underset{\sim}{\mathscr{L}}(\underset{\sim}{\mathcal{G}})$.*

Definition 3.6: *A GT_2F grammar $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ is said to be in normal form if every production of $\tilde{\mathcal{R}}$ is restricted to one of the following forms: $A \xrightarrow{\tilde{\sigma}} aB$; $A \xrightarrow{\tilde{\sigma}} a$; $A \xrightarrow{\tilde{\sigma}} \mu$, when $A = I$, where, $A, B \in N$, $a \in T$, $\tilde{\sigma}$ is GT_2F membership grade with $\tilde{\sigma} \neq 1/0$, and μ denotes the empty string.*

Example 3.2: *Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a GT_2 grammar, where, $N = \{I, A, B\}, T = \{a, b\}$. The set of GT_2F productions $\tilde{\mathcal{R}}$ is given by*

$I \xrightarrow{0.4/0.2} aA,$

$A \xrightarrow{0.6/0.3} aB,$

$B \xrightarrow{0.5/0.4} b.$

Clearly, all productions of $\tilde{\mathcal{G}}$ are in GT_2F normal form. Thus, the string $\theta = aab$ is derived in $\tilde{\mathcal{G}}$ as:

$I \xrightarrow{0.4/0.2} aA,$

$A \xrightarrow{0.6/0.3} aaB,$

$B \xrightarrow{0.5/0.4} aab.$

Proposition 3.2: *For every GT_2F regular grammar $\tilde{\mathcal{G}}$, there is a GT_2F grammar $\tilde{\mathcal{G}}'$ in normal form such that $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}')$.*

Proof: Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a GT_2F regular grammar. Then every production in $\tilde{\mathcal{R}}$ is of the form $A \xrightarrow{\tilde{\sigma}} \beta$, where $A \in N$, $\beta \in T^*B$, with $B \in N \cup \{\mu\}$, $\tilde{\sigma} \in \mathscr{F}_{GT_2}(T^*)$, $\tilde{\sigma} \neq 1/0$. In order to prove that there exists a GT_2F grammar, $\tilde{\mathcal{G}}' = (N', T, \tilde{\mathcal{R}}', I)$ in normal form, we have to prove that $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}')$. Let $A \xrightarrow{\tilde{\sigma}} a_1 a_2 \cdots a_k B,\ k \geq 2$, be any production in $\tilde{\mathcal{R}}$. Consider new non-terminal symbols, $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{k-1} \notin N$.

Then we define $N' = N \cup \{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{k-1}\}$. Now, replace the single production $A \xrightarrow{\tilde{\sigma}} a_1 a_2 \cdots a_k B$ by the following sequence of productions:

$$\begin{aligned} A &\xrightarrow{\tilde{\sigma}} a_1\mathcal{D}_1, \\ \mathcal{D}_1 &\xrightarrow{\tilde{\sigma}} a_2\mathcal{D}_2, \\ &\vdots \\ \mathcal{D}_{k-2} &\xrightarrow{\tilde{\sigma}} a_{k-1}\mathcal{D}_{k-1}, \\ \mathcal{D}_{k-1} &\xrightarrow{\tilde{\sigma}} a_k B. \end{aligned}$$

Here, all productions are now in Normal Form and the same GT_2F membership $\tilde{\sigma}$ is assigned to each production. In the original grammar, the production contributes the membership $\tilde{\sigma}$. In the transformed grammar, the derivation chain uses k productions, each labeled by $\tilde{\sigma}$. By Definition 3.2, the derivation membership is given by

$$\tilde{\mathscr{H}}(\tilde{\sigma} \cap \tilde{\sigma} \cap \cdots \cap \tilde{\sigma}) = \tilde{\sigma}.$$

Hence, there is no change in the GT_2F derivation degree. If $A \xrightarrow{\tilde{\sigma}} a_1 a_2 \cdots a_k$, then replace it by

$$\begin{aligned} A &\xrightarrow{\tilde{\sigma}} a_1\mathcal{D}_1, \\ \mathcal{D}_1 &\xrightarrow{\tilde{\sigma}} a_2\mathcal{D}_2, \\ &\vdots \\ \mathcal{D}_{k-1} &\xrightarrow{\tilde{\sigma}} a_k. \end{aligned}$$

Again, the grammar is in Normal Form and the GT_2F membership is preserved and the production $I \xrightarrow{\tilde{\sigma}} \mu$ already belongs to the Normal Form and is therefore kept unchanged. Lastly, every derivation in $\tilde{\mathcal{G}}$ has a corresponding derivation in $\tilde{\mathcal{G}}'$, the generated strings are identical, and the GT_2F membership aggregation via $\tilde{\mathscr{H}}$ is preserved. Moreover, the supremum over all derivations remains unchanged. Thus, $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}')$. $\square$

Proposition 3.3: *For every GT_2F grammar in normal form, there is a GT_2F finite automaton $\tilde{\mathscr{M}}$ such that $\tilde{\mathscr{L}}(\tilde{\mathscr{M}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}})$.*

Proof: Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a GT_2F grammar in normal form. Then $A \xrightarrow{\tilde{\sigma}} aB, \quad A \xrightarrow{\tilde{\sigma}} a, \quad I \xrightarrow{\tilde{\sigma}} \mu$. We construct a GT_2F automaton $\tilde{\mathscr{M}} = (Q, T, \tilde{\delta}, q_0, \tilde{\mathcal{F}})$ as follows:

- $Q = N$,
- $q_0 = I$,
- for each production $A \xrightarrow{\tilde{\sigma}} aB$, define $\tilde{\delta}(A, a, B) = \tilde{\sigma}$,

- for each production $A \xrightarrow{\tilde{\sigma}} \mu$, define $\tilde{\mathcal{F}}(A) = \tilde{\sigma}$.

Let $\theta = a_1 a_2 \cdots a_n \in T^*$. Suppose that

$$I \xrightarrow{\tilde{\sigma}_1} a_1 A_1 \xrightarrow{\tilde{\sigma}_2} a_1 a_2 A_2 \cdots \xrightarrow{\tilde{\sigma}_n} \theta$$

is a derivation of θ in $\tilde{\mathcal{G}}$. Then there exists a corresponding path in $\tilde{\mathscr{M}}$:

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n,$$

where $q_i = A_i$ for $i = 1, 2, \dots, n$. Then, the membership degree of this derivation is

$$\tilde{\mathscr{H}}(\tilde{\sigma}_1 \cap \tilde{\sigma}_2 \cap \cdots \cap \tilde{\sigma}_n),$$

which coincides with the acceptance degree of w by $\tilde{\mathscr{M}}$, that is,

$$\tilde{\mathscr{L}}(\tilde{\mathscr{M}}, \theta) = \tilde{\mathscr{H}}\left(\delta^*(q_0, \theta) \cap \tilde{\mathcal{F}}\right).$$

Hence,

$$\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) = \tilde{\mathscr{L}}(\tilde{\mathscr{M}}).$$

Conversely, Let $\tilde{\mathscr{M}} = (Q, T, \tilde{\delta}, q_0, \tilde{\mathcal{F}})$ be a GT_2F automaton. We construct a GT_2F grammar, $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ as follows:

- $N = Q$,
- $I = q_0$,
- for each transition $\tilde{\delta}(q, a, p) = \tilde{\sigma}$, include the production $q \xrightarrow{\tilde{\sigma}} ap$ in $\tilde{\mathcal{R}}$,
- for each final state $q \in Q$, include the production $q \xrightarrow{\tilde{F}(q)} \mu$.

All productions in $\tilde{\mathcal{R}}$ are in normal form. Each accepting path in $\tilde{\mathscr{M}}$ corresponds to a derivation in $\tilde{\mathcal{G}}$, and the membership aggregation is governed by the same operator $\tilde{\mathscr{H}}$. Therefore, $\tilde{\mathscr{L}}(\tilde{\mathscr{M}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}})$. $\square$

Definition 3.7: *A GT_2F grammar $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ is said to be a GT_2F* linear grammar *if every production in $\tilde{\mathcal{R}}$ is of one of the following forms:*

$$A \xrightarrow{\tilde{\sigma}} xB \quad \text{or} \quad A \xrightarrow{\tilde{\sigma}} x,$$

where,$A, B \in N$, $x \in T^*$, $\tilde{\sigma} \in \mathscr{F}_{GT_2}(T^*)$ with $\tilde{\sigma} \neq 1/0$.

If x appears only to the left *of the non-terminal B, then $\tilde{\mathcal{G}}$ is called a* right linear *GT_2F grammar. If x appears only to the* right *of B, then $\tilde{\mathcal{G}}$ is called a* left linear *GT_2F grammar.*

Definition 3.8: *A GT_2F language $\tilde{\mathscr{L}} \in \mathscr{F}_{GT_2}(T^*)$ is called a GT_2F* linear language*, if there exists a GT_2F linear grammar, $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ such that $\tilde{\mathscr{L}} = \tilde{\mathscr{L}}(\tilde{\mathcal{G}})$.*

Remark 3.2: *By the definition of GT_2F linear grammars, it is evident that the class of GT_2F regular languages is contained within the class of GT_2F linear languages.*

Proposition 3.4: *Every GT_2F left linear grammar is equivalent to a GT_2F right linear grammar, they generate the same GT_2F language.*

Proof: Let $\tilde{\mathcal{G}}_{\tilde{\mathscr{L}}} = (N, T, \tilde{\mathcal{R}}_{\tilde{\mathscr{L}}}, I)$ be a GT_2F left linear grammar. Then each production in $\tilde{\mathcal{R}}_{\tilde{\mathscr{L}}}$ is of one of the forms $A \xrightarrow{\tilde{\sigma}} Ba \quad \text{or} \quad A \xrightarrow{\tilde{\sigma}} a$, where $A, B \in N$, $a \in T$, and $\tilde{\sigma} \in \mathscr{F}_{GT_2}(T^*)$. We construct a GT_2F right linear grammar $\tilde{\mathcal{G}}_{\tilde{\mathcal{R}}} = (N, T, \tilde{\mathcal{R}}_{\tilde{\mathcal{R}}}, I)$ by transforming each production in $\tilde{\mathcal{R}}_{\tilde{\mathscr{L}}}$ as follows:

- For each production $A \xrightarrow{\tilde{\sigma}} Ba$ in $\tilde{\mathcal{R}}_{\tilde{\mathscr{L}}}$, include the production $A \xrightarrow{\tilde{\sigma}} aB$ in $\tilde{\mathcal{R}}_{\tilde{\mathcal{R}}}$.
- For each production $A \xrightarrow{\tilde{\sigma}} a$ in $\tilde{\mathcal{R}}_{\tilde{\mathscr{L}}}$, include the production $A \xrightarrow{\tilde{\sigma}} a$ in $\tilde{\mathcal{R}}_{\tilde{\mathcal{R}}}$.

Let $\theta = a_1 a_2 \cdots a_n \in T^*$, a derivation of θ in $\tilde{\mathcal{G}}_{\tilde{\mathscr{L}}}$ corresponds to a derivation of θ in $\tilde{\mathcal{G}}_{\tilde{\mathcal{R}}}$ obtained by reversing the order of non-terminal expansion. Since each corresponding production carries the same general type-2 fuzzy membership $\tilde{\sigma}$, the derivation membership in both grammars is $\mathscr{H}(\tilde{\sigma}_1 \cap \tilde{\sigma}_2 \cap \cdots \cap \tilde{\sigma}_n)$, which is identical. Hence,

$$\tilde{\mathscr{L}}(\tilde{\mathcal{G}}_{\tilde{\mathscr{L}}})(\theta) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}_{\tilde{\mathcal{R}}})(\theta) \quad \text{for all } \theta \in T^*.$$

Therefore,

$$\tilde{\mathscr{L}}(\tilde{\mathcal{G}}_{\tilde{\mathscr{L}}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}_{\tilde{\mathcal{R}}}).$$

Conversely, every GT_2F Right Linear Grammar is equivalent to a GT_2F Left Linear Grammar, can be proved analogously. Hence, GT_2F left linear and right linear grammars are equivalent. □

Proposition 3.5: *Every GT_2F Right Linear Language can be generated by a GT_2F Grammar in Normal Form.*

Proof: Let $\tilde{\mathscr{L}}$ be a GT_2F Right Linear Language. Then there exists a GT_2F Right Linear Grammar $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ such that $\tilde{\mathscr{L}} = \tilde{\mathscr{L}}(\tilde{\mathcal{G}})$. By definition, every production in $\tilde{\mathcal{R}}$ is of one of the forms $A \xrightarrow{\tilde{\sigma}} xB \quad \text{or} \quad A \xrightarrow{\tilde{\sigma}} x$, where $A, B \in N$, $x \in T^*$, and $\tilde{\sigma} \in \mathscr{F}_{GT_2}(T^*)$ with $\tilde{\sigma} \neq 1/0$. If $|x| = 1$, the production is already in Normal Form. Let $|x| = k \geq 2$ and write $x = a_1 a_2 \cdots a_k$. Introduce fresh non-terminals

$$\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{k-1} \notin N,$$

and replace the production

$$A \xrightarrow{\tilde{\sigma}} a_1 a_2 \cdots a_k B$$

by the sequence

$$A \xrightarrow{\tilde{\sigma}} a_1\mathcal{D}_1, \quad \mathcal{D}_1 \xrightarrow{\tilde{\sigma}} a_2\mathcal{D}_2, \quad \dots \quad \mathcal{D}_{k-1} \xrightarrow{\tilde{\sigma}} a_kB.$$

A similar construction applies to productions of the form $A \xrightarrow{\tilde{\sigma}} x$. All resulting productions are in Normal Form. Hence, $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}')$, where A' is the resulting Normal Form grammar. □

Example 3.3: *Let $\tilde{\mathcal{G}} = (N, T, \tilde{\mathcal{R}}, I)$ be a GT$_2$F Right Linear Grammar, where $N = \{I, A\}$, $T = \{a, b\}$, and $\tilde{\mathcal{R}} = \left\{ I \xrightarrow{\tilde{\sigma}_1} abA,\ A \xrightarrow{\tilde{\sigma}_2} b \right\}$. The production $I \xrightarrow{\tilde{\sigma}_1} abA$ is right linear but not in Normal Form. Introduce a new non-terminal $\mathcal{D}_1 \notin N$ and define*

$$N' = N \cup \{\mathcal{D}_1\}.$$

Replace the production $I \xrightarrow{\tilde{\sigma}_1} abA$ by $I \xrightarrow{\tilde{\sigma}_1} a\mathcal{D}_1, \quad \mathcal{D}_1 \xrightarrow{\tilde{\sigma}_1} bA$. Thus, the transformed grammar $\tilde{\mathcal{G}}' = (N', T, \tilde{\mathcal{R}}', I)$ has productions

$$\tilde{\mathcal{R}}' = \left\{ I \xrightarrow{\tilde{\sigma}_1} a\mathcal{D}_1,\ \mathcal{D}_1 \xrightarrow{\tilde{\sigma}_1} bA,\ A \xrightarrow{\tilde{\sigma}_2} b \right\},$$

which are all in Normal Form. Both grammars generate the same GT$_2$F Language, that is, $\tilde{\mathscr{L}}(\tilde{\mathcal{G}}) = \tilde{\mathscr{L}}(\tilde{\mathcal{G}}')$.

Proposition 3.6: *The following classes of GT$_2$F grammars are equivalent (except for the empty string Λ)*

(i) GT$_2$F regular grammar,
(ii) GT$_2$F left linear grammar,
(iii) GT$_2$F right linear grammar,
(iv) GT$_2$F grammar in normal form.

4 Conclusion

In this study we examines general type-2 fuzzy grammars with an emphasis on their internal structure and properties. Specifically, we have introduced several key variants of these general type-2 fuzzy grammars, namely the normal form, linear form, left-linear form, and right-linear form. We demonstrates with suitable examples that all these variants are equivalent to general type-2 fuzzy regular grammars, as they produce the same family of general type-2 fuzzy languages. This result shows that imposing such structural restrictions does not affect their expressive or generative power. We hope that the equivalence results obtained here are especially suitable for modeling sequential and pattern-based systems under higher-order uncertainty.

These forms are expected to play an important role in emerging applications such as intelligent decision-making, where general type-2 fuzzy frameworks provide improved capability for handling higher-order uncertainties compared to classical models (cf., [1,2], for more details). Furthermore, with the increasing interest in general type-2 fuzzy logic for complex system modeling, these grammatical forms may contribute to future developments in image processing, natural language processing, and hybrid learning systems, where multi-layered uncertainty and linguistic ambiguity are inherent [7].

As a direction for future work, the authors plans to study the topological and categorical aspects of automata and its language theory in the context of type-2 fuzzy sets. This approach is expected to provide clearer insights into their structural properties and contribute to a more coherent and well-defined theoretical framework (as done, e.g., in [4]).

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