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Research article

Grundy Locating-Hop Domination Sequences in Graphs

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Abstract: Locating-hop domination refines hop domination by requiring vertices outside a hop dominating set to have nonempty and pairwise distinct distance-two signatures. Dominating sequences, on the other hand, measure how long domination can be built through legal vertex choices. This paper introduces Grundy locating-hop domination sequences, in which a vertex choice is legal when it either footprints a previously undominated vertex through a closed hop neighborhood or strictly reduces an ambiguity potential that counts indistinguishable outside vertex pairs. The associated invariant is denoted by $\gamma_{\mathrm{gr}}^{\ell h}(G)$. General bounds are established, including $\gamma_{\ell h}(G) \leq \gamma_{\mathrm{gr}}^{\ell h}(G) \leq n(G)$. The parameter is additive over disjoint unions, and a hop-graph reduction identifies $\gamma_{\mathrm{gr}}^{\ell h}(G)$ with the corresponding locating-dominating sequence parameter on the hop graph $G^{(2)}$. Exact values are obtained for complete graphs and stars. In particular, $\gamma_{\mathrm{gr}}^{\ell h}(K_n) = n$ and $\gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = n$ for $n \geq 2$. Stars also give an infinite separation family: for $n \geq 3$, $\gamma_{\mathrm{gr}}^{h}(K_{1,n}) = 2$ while $\gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = n$.

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1 Introduction

Domination in graphs is a central area of graph theory, with many variants designed to encode coverage, identification, and extremal selection processes. A dominating set guarantees that every vertex is either selected or adjacent to a selected vertex. Several refinements replace adjacency by other distance constraints or impose additional distinguishing conditions on the vertices outside the selected set. These refinements are useful because they separate the task of covering a graph from the task of identifying vertices by their neighborhoods.

One sequential direction is the study of dominating sequences. In a dominating sequence, vertices are selected one at a time, and each noninitial choice must contribute a new dominated vertex. The longest possible length of such a sequence is the Grundy domination number, introduced and developed in [1,2]. Hop domination is a distance-two analogue of domination. Instead of adjacency, it uses vertices at distance two, and the corresponding Grundy hop domination number has been studied in [6,7].

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Locating conditions give a different refinement. A locating-type set not only dominates vertices outside the set but also distinguishes them by their intersections with the selected set. Locating-dominating sets in the adjacency setting originate in the work of Slater [8]; further bounds and extremal results for locating-dominating and identifying-type codes appear, for example, in [3,5]. In the distance-two setting, Canoy and Salasalan introduced locating-hop domination and studied its basic properties [4].

The purpose of this paper is to combine these two perspectives. We define a sequence parameter for locating-hop domination in which legality may be certified in two ways. A step is legal if it newly hop-dominates a vertex, or if it makes strict progress toward the locating condition by reducing the number of indistinguishable pairs among the vertices that remain outside the current prefix. Thus, the parameter records not only domination growth but also locating progress.

The main contributions are as follows. First, an ambiguity potential is introduced to measure unresolved hop-signature collisions among outside vertices. Second, Grundy locating-hop domination sequences and the invariant $\gamma_{\mathrm{gr}}^{\ell h}(G)$ are defined. Third, general bounds and additivity under disjoint union are proved. Fourth, a hop-graph reduction shows that the new parameter is equivalent to a locating-dominating sequence parameter on the graph $G^{(2)}$ whose edges represent distance-two pairs in G. Finally, exact values are obtained for complete graphs and stars, and stars are shown to form an infinite family in which the locating-hop sequence parameter is arbitrarily larger than the Grundy hop domination number.

2 Preliminaries

All graphs considered in this paper are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively, and let $n(G) = |V(G)|$. The distance between vertices $u, v \in V(G)$ is denoted by $\mathrm{dist}_G(u, v)$.

The notation is chosen to remain close to standard graph-theoretic usage. The symbol $N_2^G(v)$ denotes the set of vertices at distance two from v, and $N_2^G[v]$ denotes its closed hop neighborhood. For a selected set S, the signature $\sigma_S(x)$ records the selected vertices at distance two from x. The potential $\Phi_G(S)$ counts unordered pairs of outside vertices that still have the same signature. The superscript ℓh stands for locating-hop, and the subscript gr stands for Grundy.

2.1 Hop neighborhoods and locating-hop domination

Definition 1 (Hop neighborhood)**:** *Let G be a graph and let $v \in V(G)$. The hop neighborhood of v in G is*

$$N_2^G(v) = \{u \in V(G) : \mathrm{dist}_G(u, v) = 2\}.$$

The closed hop neighborhood of v is $N_2^G[v] = N_2^G(v) \cup \{v\}$. For $S \subseteq V(G)$, define

$$N_2^G[S] = \bigcup_{s \in S} N_2^G[s].$$

Definition 2 (Hop dominating and locating-hop dominating sets)**:** *A set $S \subseteq V(G)$ is a hop dominating set if $N_2^G[S] = V(G)$. For $x \in V(G) \setminus S$, the hop signature of x with respect to S is*

$$\sigma_S(x) = N_2^G(x) \cap S.$$

A hop dominating set S is a locating-hop dominating set if, for all distinct $x, y \in V(G) \setminus S$, one has $\sigma_S(x) \neq \sigma_S(y)$ and $\sigma_S(x) \neq \emptyset$. The minimum cardinality of a locating-hop dominating set in G is denoted by $\gamma_{\ell h}(G)$.

2.2 Hop domination sequences

Definition 3 (Closed hop neighborhood sequence)**:** *Let G be a graph. A sequence $\mathbf{s} = (v_1, \dots, v_k)$ of distinct vertices of G is a legal closed hop neighborhood sequence if, for each $i \geq 2$,*

$$N_2^G[v_i] \setminus \bigcup_{j<i} N_2^G[v_j] \neq \emptyset.$$

If $\{v_1, \dots, v_k\}$ is a hop dominating set, then $\mathbf{s}$ is a hop dominating sequence. The maximum length of a hop dominating sequence in G is the Grundy hop domination number, denoted by $\gamma_{\mathrm{gr}}^h(G)$.

3 Grundy locating-hop domination sequences

3.1 Ambiguity potential

Definition 4 (Hop-signature partition)**:** *Let G be a graph and let $S \subseteq V(G)$. Define an equivalence relation on $V(G) \setminus S$ by*

$$x \sim_S y \quad \Longleftrightarrow \quad \sigma_S(x) = \sigma_S(y).$$

Let $\mathcal{P}_S$ be the partition of $V(G) \setminus S$ into equivalence classes under $\sim_S$.

Definition 5 (Ambiguity potential)**:** *Let G be a graph and let $S \subseteq V(G)$. The ambiguity potential of S in G is*

$$\Phi_G(S) = \sum_{C \in \mathcal{P}_S} \binom{|C|}{2}.$$

Thus $\Phi_G(S)$ is the number of unordered pairs of vertices outside S that have the same hop signature with respect to S.

Lemma 1 (Monotonicity)**:** *If $S \subseteq T \subseteq V(G)$, then $\Phi_G(T) \leq \Phi_G(S)$.*

Proof: Fix a $\sim_T$-equivalence class $C \subseteq V(G) \setminus T$. If $x, y \in C$, then $\sigma_T(x) = \sigma_T(y)$. Hence

$$\sigma_S(x) = \sigma_T(x) \cap S = \sigma_T(y) \cap S = \sigma_S(y).$$

Therefore C is contained in a $\sim_S$-equivalence class. Each $\sim_S$-class is partitioned into some $\sim_T$-classes, possibly after vertices of $T \setminus S$ have been removed from the outside set. Since $\sum_i \binom{a_i}{2} \le \binom{\sum_i a_i}{2}$ for nonnegative integers a_i, the total contribution to $\Phi_G(T)$ from the parts of each $\sim_S$-class is no larger than its contribution to $\Phi_G(S)$. Summing over the $\sim_S$-classes gives $\Phi_G(T) \le \Phi_G(S)$. □

The ambiguity potential is a convenient certificate for locating progress. Each class $C \in \mathcal{P}_S$ contributes $\binom{|C|}{2}$ unresolved pairs, so a strict decrease of Φ_G records a strict reduction in the number of outside vertex pairs not yet distinguished by their hop signatures.

Remark 1: *A strict decrease $\Phi_G(S \cup \{v\}) < \Phi_G(S)$ may occur because adding v splits a non-singleton signature class, because v is removed from a non-singleton class of outside vertices, or by both mechanisms.*

Remark 2: *The equality $\Phi_G(S) = 0$ holds if and only if hop signatures are injective on $V(G) \setminus S$. If S is hop dominating, then $\sigma_S(x) \neq \emptyset$ for every $x \in V(G) \setminus S$.*

3.2 Legal locating-hop sequences

Definition 6 (Legal locating-hop sequence)**:** *Let G be a graph. A sequence $\mathbf{s} = (v_1, \dots, v_k)$ of distinct vertices is a legal locating-hop sequence if for each $i \ge 2$, writing $S_{i-1} = \{v_1, \dots, v_{i-1}\}$ and $S_i = S_{i-1} \cup \{v_i\}$, at least one of the following conditions holds:*

(L1) $N_2^G[v_i] \setminus N_2^G[S_{i-1}] \neq \emptyset$;

(L2) $\Phi_G(S_i) < \Phi_G(S_{i-1})$.

If, in addition, $S_k = \{v_1, \dots, v_k\}$ is a locating-hop dominating set, then $\mathbf{s}$ is a Grundy locating-hop dominating sequence.

Definition 7 (Grundy locating-hop domination number)**:** *For a graph G, the Grundy locating-hop domination number $\gamma_{\mathrm{gr}}^{\ell h}(G)$ is the maximum length of a Grundy locating-hop dominating sequence in G.*

Proposition 1: *Every legal closed hop neighborhood sequence is a legal locating-hop sequence.*

Proof: A legal closed hop neighborhood sequence satisfies condition (L1) at every noninitial step. Hence it is a legal locating-hop sequence. □

Corollary 1: *For every graph G,*

$$\gamma_{\ell h}(G) \le \gamma_{\mathrm{gr}}^{\ell h}(G) \le n(G).$$

Proof: The upper bound follows because a legal locating-hop sequence uses distinct vertices.

For the lower bound, let D be a minimum locating-hop dominating set, so $|D| = \gamma_{\ell h}(G)$. We construct an ordering of D that is a legal locating-hop sequence. Set $S_0 = \emptyset$. Suppose $S_t \subsetneq D$ has been chosen and put $R = D \setminus S_t$.

If $N_2^G[S_t] \neq V(G)$, choose $x \in V(G) \setminus N_2^G[S_t]$. Since D is hop dominating, there exists $r \in D$ such that $x \in N_2^G[r]$. This vertex r cannot belong to S_t, and therefore $r \in R$. Adding r satisfies (L1).

Assume now that $N_2^G[S_t] = V(G)$. If $\Phi_G(S_t) > 0$, then some $\sim_{S_t}$-class $C \subsetneq V(G) \setminus S_t$ has cardinality at least two. If $C \cap R \neq \emptyset$, choose $d \in C \cap R$. The class C contributes $\binom{|C|}{2}$ to $\Phi_G(S_t)$. Passing to $S_t \cup \{d\}$ removes d from the outside set, and all further refinements can only decrease ambiguity. Hence

$$\Phi_G(S_t \cup \{d\}) \leq \Phi_G(S_t) - \left(\binom{|C|}{2} - \binom{|C|-1}{2}\right) = \Phi_G(S_t) - (|C| - 1) < \Phi_G(S_t),$$

so (L2) holds.

It remains to consider the case $C \cap R = \emptyset$. Then $C \cap D = \emptyset$, and hence $C \subsetneq V(G) \setminus D$. Choose distinct $x, y \in C$. Since D is locating-hop dominating, $\sigma_D(x) \neq \sigma_D(y)$. Thus some vertex $d \in D$ belongs to exactly one of $N_2^G(x)$ and $N_2^G(y)$. Such a vertex d is not in S_t, because $\sigma_{S_t}(x) = \sigma_{S_t}(y)$. Therefore $d \in R$. Adding d separates x and y, so the class C is nontrivially refined. By Lemma 1, this refinement yields $\Phi_G(S_t \cup \{d\}) < \Phi_G(S_t)$, and (L2) holds.

If $N_2^G[S_t] = V(G)$ and $\Phi_G(S_t) = 0$, then S_t is already locating-hop dominating by Remark 2. This implies $|S_t| \geq \gamma_{\ell h}(G) = |D|$, contradicting $S_t \subsetneq D$. Therefore a vertex of R can always be added legally until all vertices of D have been chosen. Hence $\gamma_{\mathrm{gr}}^{\ell h}(G) \geq |D| = \gamma_{\ell h}(G)$. □

4 Disjoint unions

For graphs G and H with disjoint vertex sets, let $G \sqcup H$ denote their disjoint union.

Definition 8: *For a graph G and $S \subsetneq V(G)$, define*

$$Z_G(S) = \{x \in V(G) \setminus S \,:\, \sigma_S(x) = \emptyset\}.$$

Equivalently, $Z_G(S) = V(G) \setminus N_2^G[S]$.

Lemma 2: *Let G and H be graphs with disjoint vertex sets, and let $S \subsetneq V(G \sqcup H)$. Write $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then*

$$\Phi_{G \sqcup H}(S) = \Phi_G(S_G) + \Phi_H(S_H) + |Z_G(S_G)||Z_H(S_H)|.$$

Proof: Signatures are computed componentwise. If $x \in V(G) \setminus S_G$ and $y \in V(H) \setminus S_H$, then $\sigma_S(x) \subsetneq V(G)$ and $\sigma_S(y) \subsetneq V(H)$. Hence $\sigma_S(x) = \sigma_S(y)$ if and only if both signatures are empty. Thus the empty-signature class in $G \sqcup H$ consists of $Z_G(S_G) \cup Z_H(S_H)$, and its cross pairs contribute exactly $|Z_G(S_G)||Z_H(S_H)|$ in addition to the within-component ambiguity contributions. □

Theorem 1 (Additivity under disjoint union)**:** *For any graphs G and H with disjoint vertex sets,*

$$\gamma_{\mathrm{gr}}^{\ell h}(G \sqcup H) = \gamma_{\mathrm{gr}}^{\ell h}(G) + \gamma_{\mathrm{gr}}^{\ell h}(H).$$

Proof: Let $\mathbf{s}_G$ and $\mathbf{s}_H$ be maximum Grundy locating-hop dominating sequences in G and H, inducing final sets D_G and D_H, respectively. Concatenate them to obtain a sequence in $G \sqcup H$. A step in one component satisfying (L1) also satisfies (L1) in the disjoint union. If a step in G satisfies (L2) in G, then Φ_G strictly decreases, Φ_H is unchanged, $|Z_H|$ is unchanged, and $|Z_G|$ is nonincreasing. Lemma 2 gives a strict decrease of $\Phi_{G\sqcup H}$. The same argument applies to steps in H. The final set $D_G \cup D_H$ is locating-hop dominating in $G \sqcup H$, because hop domination and nonempty signatures hold within each component, and nonempty signatures from different components cannot be equal. Therefore

$$\gamma_{\mathrm{gr}}^{\ell h}(G \sqcup H) \geq \gamma_{\mathrm{gr}}^{\ell h}(G) + \gamma_{\mathrm{gr}}^{\ell h}(H).$$

Conversely, let $\mathbf{s}$ be a Grundy locating-hop dominating sequence in $G \sqcup H$, and let S_i be its prefix sets. Restrict $\mathbf{s}$ to the vertices of G, and write $S_i^G = S_i \cap V(G)$. If a selected vertex $v_i \in V(G)$ satisfies (L1) in $G \sqcup H$, then it satisfies (L1) in G, since hop neighborhoods do not cross components. If it satisfies (L2) in $G \sqcup H$, then the H-term $\Phi_H(S_i^H)$ and the factor $\left|Z_H(S_i^H)\right|$ in Lemma 2 are unchanged. Hence the strict decrease of $\Phi_{G\sqcup H}$ comes either from $\Phi_G(S_i^G) < \Phi_G(S_{i-1}^G)$, which gives (L2) in G, or from $\left|Z_G(S_i^G)\right| < \left|Z_G(S_{i-1}^G)\right|$, which means that adding v_i newly hop-dominates a vertex of G and therefore gives (L1) in G. Thus the restriction to G is a legal locating-hop sequence whose final set is locating-hop dominating in G. The number of vertices chosen from G is at most $\gamma_{\mathrm{gr}}^{\ell h}(G)$. The same argument applies to H, and the reverse inequality follows. □

5 Hop-graph reduction

Definition 9 (Hop graph)**:** *Let G be a graph. The hop graph of G is the graph $G^{(2)}$ with vertex set $V(G)$ in which distinct vertices u and v are adjacent if and only if $\mathrm{dist}_G(u,v) = 2$.*

Remark 3: *For every $v \in V(G)$,*

$$N_{G^{(2)}}(v) = N_2^G(v) \quad \textit{and} \quad N_{G^{(2)}}[v] = N_2^G[v].$$

Moreover, for $x \notin S$, the hop signature $N_2^G(x) \cap S$ in G equals the adjacency signature $N_{G^{(2)}}(x) \cap S$ in $G^{(2)}$.

Definition 10 (Adjacency analogue)**:** *Let H be a graph. For $S \subseteq V(H)$ and $x \in V(H) \setminus S$, set $\tau_S(x) = N_H(x) \cap S$. Let $\Phi_H^{\ell d}(S)$ be the ambiguity potential obtained from the partition of $V(H) \setminus S$ by equality of the signatures $\tau_S(\cdot)$.*

A set $S \subseteq V(H)$ is a locating-dominating set if $N_H[S] = V(H)$ and, for all distinct $x, y \in V(H) \setminus S$, one has $\tau_S(x) \neq \tau_S(y)$ and $\tau_S(x) \neq \emptyset$.

A sequence $\mathbf{s} = (v_1, \ldots, v_k)$ is a legal locating-dominating sequence in H if for each $i \geq 2$, with $S_{i-1} = \{v_1, \ldots, v_{i-1}\}$ and $S_i = S_{i-1} \cup \{v_i\}$, at least one of the following conditions holds:

(A1) $\quad N_H[v_i] \setminus N_H[S_{i-1}] \neq \emptyset;$

(A2) $\quad \Phi_H^{\ell d}(S_i) < \Phi_H^{\ell d}(S_{i-1}).$

If S_k is a locating-dominating set in H, then **s** *is a Grundy locating-dominating sequence. The maximum length of such a sequence is denoted by $\gamma_{\rm gr}^{\ell d}(H)$.*

Proposition 2 (Hop-graph reduction)**:** *For every graph G,*

$$\gamma_{\rm gr}^{\ell h}(G) = \gamma_{\rm gr}^{\ell d}(G^{(2)}).$$

Proof: By Remark 3, $N_2^G[S] = N_{G^{(2)}}[S]$ for every $S \subseteq V(G)$, and the hop signatures in G coincide with the adjacency signatures in $G^{(2)}$. Thus (L1) in G is identical to (A1) in $G^{(2)}$, and (L2) in G is identical to (A2) in $G^{(2)}$. A set is locating-hop dominating in G if and only if it is locating-dominating in $G^{(2)}$. Therefore legal locating-hop dominating sequences in G correspond bijectively to legal locating-dominating sequences in $G^{(2)}$, while preserving sequence length. □

Corollary 2: *For disjoint graphs H_1 and H_2,*

$$\gamma_{\rm gr}^{\ell d}(H_1 \sqcup H_2) = \gamma_{\rm gr}^{\ell d}(H_1) + \gamma_{\rm gr}^{\ell d}(H_2).$$

Proof: The proof follows from the same componentwise argument used in Theorem 1. Replace closed hop neighborhoods by ordinary closed neighborhoods, hop signatures by adjacency signatures, locating-hop dominating sets by locating-dominating sets, and Φ by $\Phi^{\ell d}$. □

6 Exact values for graph families

6.1 Complete graphs

Theorem 2: *For $n \geq 1$,*

$$\gamma_{\rm gr}^{\ell h}(K_n) = n.$$

Proof: In K_n, distinct vertices are at distance one. Hence

$$K_n^{(2)} = \overline{K_n}.$$

In the edgeless graph $\overline{K_n}$, a locating-dominating set must contain every vertex, because any vertex outside the selected set has empty adjacency signature. The sequence listing all n vertices is legal: at each noninitial step, the selected vertex footprints itself through its closed neighborhood. Hence $\gamma_{\rm gr}^{\ell d}(\overline{K_n}) = n$, and the result follows from Proposition 2. □

6.2 Stars

Let $K_{1,n}$ have center c and leaves $\ell_1, \ldots, \ell_n$.

Lemma 3: *For $n \geq 2$,*

$$(K_{1,n})^{(2)} \cong K_n \sqcup K_1,$$

where K_n is induced by the leaves and the isolated vertex corresponds to the center.

Proof: Two distinct leaves of $K_{1,n}$ are at distance two through the center, so they are adjacent in the hop graph. The center has no vertex at distance two from it, so it is isolated in the hop graph. □

Lemma 4: *For $n \geq 2$, $\gamma_{\mathrm{gr}}^{\ell d}(K_n) = n - 1$, while $\gamma_{\mathrm{gr}}^{\ell d}(K_1) = 1$.*

Proof: The assertion for K_1 is immediate. Let $n \geq 2$. Any locating-dominating set in K_n has size at least $n - 1$, because if two vertices remain outside the set, then both have the selected set as their adjacency signature.

Let $\mathbf{s} = (v_1, v_2, \dots, v_{n-1})$ be any sequence of $n - 1$ distinct vertices. For $1 \leq t \leq n - 2$, put $S_t = \{v_1, \dots, v_t\}$. All vertices in $V(K_n) \setminus S_t$ have the same adjacency signature, namely S_t. Adding v_{t+1} removes one vertex from this non-singleton class, and therefore strictly decreases $\Phi_{K_n}^{\ell d}$. Hence every noninitial step in $\mathbf{s}$ is legal by (A2). The final set S_{n-1} is locating-dominating because exactly one vertex remains outside. Thus $\gamma_{\mathrm{gr}}^{\ell d}(K_n) \geq n - 1$.

A sequence of length n is impossible. After selecting $n - 1$ vertices, the graph is dominated and the outside set has size one, so the ambiguity potential is zero. Adding the last vertex satisfies neither (A1) nor (A2). Therefore $\gamma_{\mathrm{gr}}^{\ell d}(K_n) = n - 1$. □

Theorem 3: *For $n \geq 2$,*

$$\gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = n.$$

Proof: By Lemma 3, Proposition 2, Corollary 2, and Lemma 4,

$$\gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = \gamma_{\mathrm{gr}}^{\ell d}(K_n \sqcup K_1) = \gamma_{\mathrm{gr}}^{\ell d}(K_n) + \gamma_{\mathrm{gr}}^{\ell d}(K_1) = (n-1) + 1 = n.$$

□

7 A separation family

Proposition 3: *For $n \geq 3$,*

$$\gamma_{\mathrm{gr}}^{h}(K_{1,n}) = 2 \quad \text{and} \quad \gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = n.$$

Proof: The equality $\gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = n$ follows from Theorem 3. It remains to compute $\gamma_{\mathrm{gr}}^{h}(K_{1,n})$. For any leaf ℓ_i,

$$N_2^{K_{1,n}}[\ell_i] = \{\ell_1, \dots, \ell_n\},$$

while

$N_2^{K_{1,n}}[c] = \{c\}.$

Thus (c, ℓ_1) is a hop dominating sequence of length two. After one leaf has been chosen, every other leaf has closed hop neighborhood contained in the set already hop-dominated by that leaf. After both c and one leaf have been chosen, the graph is hop dominated, and no remaining vertex can satisfy the hop-footprint condition. Hence no longer hop dominating sequence exists, and $\gamma_{\mathrm{gr}}^{h}(K_{1,n}) = 2$. □

8 Conclusions

This paper introduced Grundy locating-hop domination sequences and the invariant $\gamma_{\mathrm{gr}}^{\ell h}(G)$. The legality condition combines two forms of progress: ordinary hop-footprinting and strict reduction of unresolved hop-signature ambiguity among outside vertices. The ambiguity potential was shown to be monotone under enlargement of the selected set, which gives a useful way to certify locating progress. The parameter satisfies the general bounds $\gamma_{\ell h}(G) \leq \gamma_{\mathrm{gr}}^{\ell h}(G) \leq n(G)$ and is additive over disjoint unions. The hop-graph reduction $\gamma_{\mathrm{gr}}^{\ell h}(G) = \gamma_{\mathrm{gr}}^{\ell d}(G^{(2)})$ connects the parameter to a corresponding adjacency-based locating-dominating sequence invariant. Exact values were obtained for complete graphs and stars. In particular, $\gamma_{\mathrm{gr}}^{\ell h}(K_n) = n$ and $\gamma_{\mathrm{gr}}^{\ell h}(K_{1,n}) = n$ for $n \geq 2$. Since $\gamma_{\mathrm{gr}}^{h}(K_{1,n}) = 2$ for $n \geq 3$, stars show that locating-progress legality can produce an unbounded separation from the classical Grundy hop domination number.

The results also suggest several directions in which the new invariant may be developed further. The most immediate questions concern exact values for standard graph families, structural characterizations of extremal cases, behavior under graph operations, and algorithmic complexity. Since the hop-graph reduction translates the problem into an adjacency-based locating-dominating sequence problem, future work may also compare the invariant with known domination, locating-domination, and zero-forcing-type parameters. The following problems are proposed for further investigation:

OP1: Determine $\gamma_{\mathrm{gr}}^{\ell h}(G)$ for paths, cycles, wheels, complete bipartite graphs, complete multipartite graphs, and trees with prescribed diameter.

OP2: Characterize graphs G for which $\gamma_{\mathrm{gr}}^{\ell h}(G) = n(G)$ or $\gamma_{\mathrm{gr}}^{\ell h}(G) = n(G) - 1$.

OP3: Determine sharp upper and lower bounds for $\gamma_{\mathrm{gr}}^{\ell h}(G)$ in terms of order, diameter, maximum degree, minimum degree, twin classes, and the locating-hop domination number $\gamma_{\ell h}(G)$.

OP4: Study $\gamma_{\mathrm{gr}}^{\ell h}$ under graph operations such as join, corona, lexicographic product, Cartesian product, strong product, and graph powers.

OP5: Determine the computational complexity of deciding whether $\gamma_{\mathrm{gr}}^{\ell h}(G) \geq k$, and study fixed-parameter tractability with respect to k, treewidth, vertex cover number, or the number of twin classes.

OP6: Develop connected, total, restrained, or independent versions of Grundy locating-hop domination sequences and compare them with the corresponding nonsequential parameters.

OP7: Investigate whether the ambiguity-potential method can be used for other locating-type sequence parameters, including metric-locating, identifying-code, and resolving-domination variants.

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