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Research article

Permutation-Induced Automorphisms and Non-conjugate Symmetries of the Magma Monoid

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Abstract: We study the algebra of all binary operations on a finite set, with composition defined as follows: for two binary operations A and B, their composition at an ordered pair x and y is B applied to the pair consisting of A(x,y) and A(y,x). A canonical family of automorphisms arises by conjugation with permutations of the underlying set. We characterize how these permutation-induced automorphisms act on several natural invariant subsets and show that the permutation they induce on constant operations determines every operation's values on the diagonal. Beyond conjugation by permutations, we identify additional symmetries that are not conjugate to these permutation conjugations, and we show that conjugation by permutations need not be surjective onto the full automorphism group of the algebra under the given composition. For small underlying sets we obtain complete descriptions: when the set has two elements the automorphism group is isomorphic to the two-element symmetric group, and when the set has three elements it is isomorphic to the direct product of the three-element symmetric group with a cyclic group of order two. The paper concludes with algorithmic remarks and open problems toward a full classification of the automorphism group.

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1 Introduction and Preliminaries

1.1 Introduction

Throughout, S denotes a nonempty set. Following standard usage (e.g., [1,2]), we call $(S, *)$ a *magma* when $*$ is an arbitrary binary operation on S; no additional axioms (associativity, commutativity, identity, invertibility, etc.) are assumed unless otherwise stated. We denote by $\mathcal{M}(S)$ the set of all binary operations

$S \times S \to S$. For finite S with $|S| = n$ a binary operation is determined by n^2 table entries, each chosen from n values, so $|\mathcal{M}(S)| = n^{n^2}$.

A pair $(S, *, \circ)$ will be called a (left, right, or two-sided) distributive magma when the indicated distributivity relation holds between the two binary laws.. The distributivity hierarchy and outset structures introduced in [1] are motivated in part by examples from tropical geometry (e.g., addition distributing over max) and by connections with nearring theory and brace/truss constructions (see [3–7]).

For $* \in \mathcal{M}(S)$ we define the *outset*, $\mathrm{out}(*) = \{\circ \in \mathcal{M}(S) \ : \ * \text{ distributes over } \circ\}$, and write $\mathrm{in}(*)$ for the dual notion. The subsets $\mathrm{out}(*)$ and $\mathrm{in}(*)$ encode local neighborhoods of $*$ in the distributivity hierarchy; $\mathrm{out}(*)$ often carries a natural algebraic structure (in particular, it is a $\lhd$-subalgebra—see Section 1.2 for the definition of $\lhd$).

These structures naturally lead to symmetry questions. Maps of $\mathcal{M}(S)$ that preserve the algebra operation $\lhd$ (that is, $\lhd$-homomorphisms, and bijective ones in $\mathrm{Aut}(\mathcal{M}(S), \lhd)$) carry distributivity relations, outsets, idempotent strata, and constant operations to corresponding structures. Thus understanding $\mathrm{Aut}(\mathcal{M}(S), \lhd)$ classifies relabellings and higher symmetries that leave the distributivity graphs invariant and determines which local isomorphisms of natural subalgebras (for example, outsets) can be realized globally.

This paper investigates structural and symmetry properties of the algebra $(\mathcal{M}(S), \lhd)$. Our main contributions are as follows. We identify and analyze the canonical automorphisms induced by permutations of S and determine their effect on invariant subsets such as constant and idempotent operations. We exhibit and classify nonconjugate symmetries and demonstrate that the conjugation action of $\mathrm{Sym}(S)$ need not be surjective on $\mathrm{Aut}(\mathcal{M}(S), \lhd)$. We provide complete descriptions of $\mathrm{Aut}(\mathcal{M}(S), \lhd)$ for small cardinalities (in particular $|S| = 2, 3$). We study isomorphisms between outsets and give necessary and sufficient conditions for extending such isomorphisms to global automorphisms via free-term algebra constructions, including constructive descent procedures and explicit counterexamples when the kernel condition fails. Finally, we present computational and enumerative results for small n that illustrate obstruction phenomena to extension.

1.2 The operation $\lhd$ *on* $\mathcal{M}(S)$

We consider a binary operation $\lhd$ on $\mathcal{M}(S)$ that endows $\mathcal{M}(S)$ with a monoid structure. This operation was studied independently in [8] and [9,10]. We follow the notation and terminology of [9,10].

Subsequent works building on [8] include [11–14]. Fayoumi [11] gives an elegant description of the center of $\mathcal{M}(S)$, while [12] explores generalizations of commutativity via commutative maps. Nebeský introduced methods for encoding graphs using binary operations in [15–17], and these ideas were further developed in [14]. Following the constructions in [21], [22] examines lifts of the $\lhd$ operation on graph families on vertex set S; related graph-induced operations (one-value and two-value graph magmas) appear in [18–20] and have applications to amenable bases over infinite-dimensional algebras (see, e.g., [25]).

Definition 1: *For $\alpha, \beta \in \mathcal{M}(S)$ and $x, y \in S$ define*

$$(\alpha \lhd \beta)(x, y) = \beta\big(\alpha(x, y), \alpha(y, x)\big).$$

We may write $\alpha \lhd \beta$ or $\lhd(\alpha, \beta)$ interchangeably.

Remark 1: *Let $\pi_1 \in \mathcal{M}(S)$ denote the first-coordinate projection $\pi_1(i, j) = i$ for all $i, j \in S$. Then $(\mathcal{M}(S), \lhd)$ is a (generally non-commutative) monoid with identity π_1. See Theorem 2.1 of [10] or Theorem 2 of [8] for the proof. In [8] the same monoid is denoted* $(\mathrm{Bin}(S), \square)$.

We denote the resulting monoid by $(\mathcal{M}(S), \lhd)$ throughout the paper.

1.3 Constant operations and Outsets

Definition 2: *A binary operation $\mathcal{O}_i \in \mathcal{M}(S)$ is called a* constant operation *if $\mathcal{O}_i(x, y) = i$, for all $x, y \in S$.*

Remark 2: *Let $\mathcal{K} = \{\mathcal{O}_i : i \in S\}$ be the set of all constant operations in $\mathcal{M}(S)$. Then:*

1. *$\mathcal{K}$ is closed under $\lhd$. For every $* \in \mathcal{M}(S)$ and every $i \in S$, $* \lhd \mathcal{O}_i = \mathcal{O}_i$, $\mathcal{O}_i \lhd * = \mathcal{O}_{*(i,i)}$.*
2. *The operation $\lhd$ is not commutative on $\mathcal{K}$ in general: if $i \neq j$ then $\mathcal{O}_i \lhd \mathcal{O}_j = \mathcal{O}_j$ while $\mathcal{O}_j \lhd \mathcal{O}_i = \mathcal{O}_i$.*

Lemma 1: *For $\circ \in \mathcal{M}(S)$ the following are equivalent:*

1. *$\circ \in \mathcal{K}$ (that is, $\circ$ is constant);*
2. *$* \lhd \circ = \circ$ for every $* \in \mathcal{M}(S)$.*

Proof: If $\circ = \mathcal{O}_a$ is constant with value a, then for any $* \in \mathcal{M}(S)$ and all $x, y \in S$, $(* \lhd \circ)(x, y) = \circ(*(x, y), *(y, x)) = a = \mathcal{O}_a(x, y)$, so $* \lhd \circ = \circ$.

Conversely, suppose $* \lhd \circ = \circ$ for all $* \in \mathcal{M}(S)$. If $\circ$ is not constant, there exist (u, v) and (x, y) such that $\circ(u, v) \neq \circ(x, y)$. Define $* \in \mathcal{M}(S)$ by choosing values such that $*(u, v) = u_0$, $*(v, u) = v_0$ with $\circ(u_0, v_0) = \circ(u, v)$ but $\circ(u_0, v_0) \neq \circ(x, y)$ (such choices are possible since S has at least one element and $\mathcal{M}(S)$ is large). Evaluating at (x, y) gives $(* \lhd \circ)(x, y) = \circ(*(x, y), *(y, x)) \neq \circ(x, y)$, contradicting $* \lhd \circ = \circ$. Hence, $\circ$ must be constant. □

Corollary 1: *The monoid $(\mathcal{M}(S), \lhd)$ is not a group: constant operations do not have a two-sided inverse in general. In particular, for $\mathcal{O}_i \in \mathcal{K}$ there is no $* \in \mathcal{M}(S)$ with $\mathcal{O}_i \lhd * = \pi_1$, so $\mathcal{O}_i$ is not invertible.*

Proof: This follows immediately from Remark 2(1) and Lemma 1. □

The set of constant operations $\mathcal{K} = \{\mathcal{O}_i : i \in S\} \subseteq \mathcal{M}(S)$ is a two-sided ideal in the monoid $(\mathcal{M}(S), \lhd)$.

Definition 3: *For $i, a \in S$ with $i \neq a$, define the almost-constant operation $c_i(a) \in \mathcal{M}(S)$ by*

$$c_i(a)(x, y) = \begin{cases} a, & x = y = i, \\ i, & \textit{otherwise.} \end{cases}$$

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Proposition 1: *Let $c = c_i(a)$ be an almost-constant operation with $i \neq a$. Then*

$$c^{\lhd 3} = c \lhd c \lhd c = c.$$

Equivalently, $c \lhd (c \lhd c) = c$.

Proof: See Proposition 2 of [13] □

Thus, every almost-constant $c_i(a)$ satisfies the cubic identity $c^{\lhd 3} = c$, such elements are not invertible in general.

We now clarify the notion of a constant diagonal ("unique square") operation and state a corrected equivalence that is valid in the group context.

Definition 4: *A binary operation $*$ on a set S is said to have a* constant diagonal *(or be a* unique square *operation) if the diagonal values are constant:*

$$*(a, a) = *(b, b) \qquad \text{for all } a, b \in S.$$

Proposition 2: *Let $(G, \cdot)$ be a group. Then G has a constant diagonal (i.e., $g \cdot g$ is independent of $g \in G$) if and only if G has exponent 2 (equivalently, $g^2 = e$ for all $g \in G$).*

Proof: If G has exponent 2, then $g \cdot g = e$ for every $g \in G$, so the diagonal function $g \mapsto g \cdot g$ is constant (equal to the identity e).

Conversely, suppose $g \cdot g = h \cdot h$ for all $g, h \in G$; let this common value be $s \in G$. Fix $g \in G$. Left-multiply the identity $g \cdot g = s$ by g^{-1} to obtain $g = g^{-1}s$. Right-multiply $g \cdot g = s$ by g^{-1} to obtain $g = sg^{-1}$. Equating gives $g^{-1}s = sg^{-1}$, so s is central; but more directly, multiply $g \cdot g = s$ on the left by g^{-1} and on the right by g^{-1} to obtain $e = g^{-1}sg^{-1}$. Because this holds for all g, we have $s = e$. Hence, $g^2 = e$ for all g, that is, G has exponent 2. □

Remark 3: *The converse of Proposition 2 requires the presence of associativity and invertibility in a group. For a general monoid (without invertibility), a constant diagonal does not imply that every element has order two.*

Proposition 3: *For every operation $* \in \mathcal{M}(S)$, the (left, right, two-sided) outset $out(*)$ is a submonoid of $(\mathcal{M}(S), \lhd)$ with the same identity π_1.*

Proposition 4:
1. *If $|S| = n$ then, for any constant operation $\mathcal{O}_i \in \mathcal{M}(S)$, its outdegree $d_o(\mathcal{O}_i) = |\{w | \mathcal{O}_i \text{ distributes over } w\}| = |out(\mathcal{O}_i)|$, equals n^{n^2-1}.*
2. *For all $* \in \mathcal{M}(S)$, $*$ can distribute on at most one constant operation*
3. *π_1 and π_2 distribute over all idempotent operations.*
4. *Given $k \in S$, if k is an annihilator for the operation $* \in \mathcal{M}(S)$ (i.e. $*(i, k) = k = *(k, i)$, for all $i \in S$), then $\mathcal{O}_k \in out(*)$.*

1.4 Conjugations and symmetries

Definition 5: *For $\sigma \in \mathrm{Sym}(S)$ and $f \in \mathcal{M}(S)$ define the conjugation (permutation-conjugation) map*

$$\widehat{\sigma} : \mathcal{M}(S) \to \mathcal{M}(S), \qquad \widehat{\sigma}(f)(a,b) := \sigma\big(f(\sigma^{-1}a, \sigma^{-1}b)\big).$$

The map $\sigma \mapsto \widehat{\sigma}$ is called the conjugation action *of* $\mathrm{Sym}(S)$ *on* $\mathcal{M}(S)$.

Proposition 5: *The map $\sigma \mapsto \widehat{\sigma}$ is an injective group homomorphism*

$$\mathrm{Sym}(S) \hookrightarrow \mathrm{Aut}(\mathcal{M}(S), \triangleleft).$$

Each $\widehat{\sigma}$ is a $\triangleleft$-automorphism with an inverse $\widehat{\sigma^{-1}}$.

Proof: Let $\sigma \in \mathrm{Sym}(S)$. For $f, g \in \mathcal{M}(S)$ and $a, b \in S$ we compute

$$\widehat{\sigma}(f \triangleleft g)(a,b) = \sigma\big((f \triangleleft g)(\sigma^{-1}a, \sigma^{-1}b)\big) = \sigma\big(g(f(\sigma^{-1}a, \sigma^{-1}b),\ f(\sigma^{-1}b, \sigma^{-1}a))\big).$$

On the other hand,

$$\begin{aligned}(\widehat{\sigma} f \triangleleft \widehat{\sigma} g)(a,b) &= \widehat{\sigma} g\big(\widehat{\sigma} f(a,b),\ \widehat{\sigma} f(b,a)\big)\\ &= \sigma\big(g(\sigma^{-1}(\widehat{\sigma} f(a,b)),\ \sigma^{-1}(\widehat{\sigma} f(b,a)))\big)\\ &= \sigma\big(g(f(\sigma^{-1}a, \sigma^{-1}b),\ f(\sigma^{-1}b, \sigma^{-1}a))\big),\end{aligned}$$

since $\widehat{\sigma} f(a,b) = \sigma(f(\sigma^{-1}a, \sigma^{-1}b))$. Hence $\widehat{\sigma}(f \triangleleft g) = \widehat{\sigma} f \triangleleft \widehat{\sigma} g$, so $\widehat{\sigma}$ is a $\triangleleft$-homomorphism. For each $\sigma \in \mathrm{Sym}(S)$, $\widehat{\sigma^{-1}}$ is the two-sided inverse of $\widehat{\sigma}$, so the map is bijective.

The map $\sigma \mapsto \widehat{\sigma}$ is a group homomorphism since $\widehat{\sigma} \circ \widehat{\tau} = \widehat{\sigma\tau}$ for all σ, τ. Finally, if $\widehat{\sigma} = \widehat{\tau}$ then evaluating both maps on the constant operations O_x yields $O_{\sigma(x)} = O_{\tau(x)}$ for every $x \in S$, so $\sigma = \tau$. Thus, the map $\sigma \mapsto \widehat{\sigma}$ is injective. □

Definition 6: *The* transpose *is the involution $T : \mathcal{M}(S) \to \mathcal{M}(S)$ defined by $T(f)(a,b) = f(b,a)$. More generally, for $\rho \in \mathrm{Sym}(S \times S)$ write $R_\rho(f)(u,v) = f(\rho^{-1}(u,v))$, and call R_ρ an* input-pair permutation map*; when R_ρ preserves $\triangleleft$ it is an* input-pair permutation automorphism.

Proposition 6: *Let $T : \mathcal{M}(S) \to \mathcal{M}(S)$ be defined by the transpose, that is, $T(f)(a,b) = f(b,a)$. The map T is a bijection. However, T is a $\triangleleft$-homomorphism if and only if every $g \in \mathcal{M}(S)$ is symmetric, that is, $g(u,v) = g(v,u)$ for all $u, v \in S$. In particular, for $|S| \geq 2$, T does not lie in* $\mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ *unless $\mathcal{M}(S)$ consists only of symmetric operations.*

Proof: The transpose $T : \mathcal{M}(S) \to \mathcal{M}(S)$, $T(f)(a,b) = f(b,a)$, satisfies $T^2 = \mathrm{id}$; hence, T is a bijection.

For $f, g \in \mathcal{M}(S)$ and $a, b \in S$ we have $T(f \lhd g)(a,b) = (f \lhd g)(b,a) = g\big(f(b,a), f(a,b)\big)$, whereas

$$(Tf \lhd Tg)(a,b) = Tg\big(Tf(a,b), Tf(b,a)\big) = Tg\big(f(b,a), f(a,b)\big) = g\big(f(a,b), f(b,a)\big).$$

Thus $T(f \lhd g) = Tf \lhd Tg$ for all f, g if and only if $g(u,v) = g(v,u)$ for all $g \in \mathcal{M}(S)$ and all $u, v \in S$. Hence, T is a $\lhd$-homomorphism precisely when every operation in $\mathcal{M}(S)$ is symmetric. In particular, for $|S| \geq 2$ there exist nonsymmetric binary operations, so $T \notin \mathrm{Aut}(\mathcal{M}(S), \lhd)$ unless one restricts to the degenerate case where $\mathcal{M}(S)$ consists only of symmetric operations (for example, $|S| = 1$ or a symmetric subalgebra).

□

Proposition 7: *Let $K = \{O_i : i \in S\}$ be the set of constant operations and $E = \{e \in \mathcal{M}(S) : e \lhd e = e\}$ the set of idempotents. For every $\sigma \in \mathrm{Sym}(S)$ we have $\widehat{\sigma}(K) = K$ and $\widehat{\sigma}(E) = E$.*

Proof: For a constant O_i (the operation with value i on every input) and $\sigma \in \mathrm{Sym}(S)$, $\widehat{\sigma}(O_i)(a,b) = \sigma\big(O_i(\sigma^{-1}a, \sigma^{-1}b)\big) = \sigma(i)$, so $\widehat{\sigma}(O_i) = O_{\sigma(i)}$; hence $\widehat{\sigma}(K) = K$.

If $e \in E$ then $e \lhd e = e$. As $\widehat{\sigma}$ is a $\lhd$-homomorphism (Proposition 5), $\widehat{\sigma}(e) \lhd \widehat{\sigma}(e) = \widehat{\sigma}(e \lhd e) = \widehat{\sigma}(e)$, so $\widehat{\sigma}(e) \in E$. Thus, $\widehat{\sigma}(E) \subseteq E$; applying the same argument to σ^{-1} gives the reverse inclusion, and therefore $\widehat{\sigma}(E) = E$. Hence, any automorphism permutes constants and idempotents. □

2 Permutation-induced automorphisms

Let S be a nonempty set and $\mathcal{M}(S)$ be the set of all binary operations on S. Fix the monoid operation $\lhd$ on $\mathcal{M}(S)$ as defined in Section 1.2. Let $\mathrm{Sym}(S)$ (or S_n when $|S| = n$) be the symmetric group on S and $\mathrm{Aut}(\mathcal{M}(S), \lhd)$ be the group of bijective $\lhd$-homomorphisms of $\mathcal{M}(S)$.

Definition 7: *For $\sigma \in \mathrm{Sym}(S)$ define the conjugation automorphism $\widehat{\sigma} : \mathcal{M}(S) \to \mathcal{M}(S)$ by*

$$\widehat{\sigma}(f)(a,b) = \sigma\big(f(\sigma^{-1}a, \sigma^{-1}b)\big).$$

The conjugation subgroup *of* $\mathrm{Aut}(\mathcal{M}(S), \lhd)$ *is* $\widehat{\mathrm{Sym}(S)} = \{\widehat{\sigma} : \sigma \in \mathrm{Sym}(S)\}$. *The map $\sigma \mapsto \widehat{\sigma}$ is an injective homomorphism, so $\widehat{\mathrm{Sym}(S)} \cong \mathrm{Sym}(S)$. An automorphism of $(\mathcal{M}(S), \lhd)$ that does not lie in $\widehat{\mathrm{Sym}(S)}$ is called a* nonconjugate *(or* outer*) automorphism.*

Definition 8: *For a subgroup $H \leq \mathrm{Aut}(\mathcal{M}(S), \lhd)$ define its centralizer and normalizer in $\mathrm{Aut}(\mathcal{M}(S), \lhd)$ by $C_{\mathrm{Aut}}(H) = \{\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd) : \phi h = h\phi \text{ for all } h \in H\}$, $N_{\mathrm{Aut}}(H) = \{\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd) : \phi H \phi^{-1} = H\}$.*

Remark 4: *The subgroup $\widehat{\mathrm{Sym}(S)}$ represents the "inner" relabellings of S. Elements of the centralizer $C_{\mathrm{Aut}}\big(\widehat{\mathrm{Sym}(S)}\big)$ commute with all such relabellings; for example, if the transpose T preserves $\lhd$ then $T \in C_{\mathrm{Aut}}(\widehat{\mathrm{Sym}(S)})$. The normalizer $N_{\mathrm{Aut}}\big(\widehat{\mathrm{Sym}(S)}\big)$ consists of those automorphisms that conjugate the conjugation subgroup to itself, i.e., automorphisms under which conjugation-by-permutations is stable.*

Proposition 8: *1. As permutations of $\mathcal{M}(S)$ we have $T \circ \widehat{\sigma} = \widehat{\sigma} \circ T$ for every $\sigma \in \mathrm{Sym}(S)$. Hence the subgroup of the full symmetric group $\mathrm{Sym}(\mathcal{M}(S))$ generated by $\widehat{\mathrm{Sym}(S)}$ and T is the internal direct product $\widehat{\mathrm{Sym}(S)} \times \langle T \rangle$ (the two factors commute and meet trivially; the triviality of the intersection follows since $\widehat{\sigma} = T$ would force $\sigma = \mathrm{id}$, except in the degenerate one-point case).*

2. *Every automorphism $\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ induces a permutation $\sigma_\phi \in \mathrm{Sym}(S)$ through its action on the constant operations O_i; conjugating ϕ by $\widehat{\sigma_\phi}^{-1}$ yields an automorphism that fixes all constants pointwise.*
3. *The map $\phi \mapsto \sigma_\phi$ is a group homomorphism $\mathrm{Aut}(\mathcal{M}(S), \lhd) \longrightarrow \mathrm{Sym}(S)$, whose kernel equals the subgroup of automorphisms fixing all constant operations pointwise.*

Proof: (1) By Proposition 5, each $\widehat{\sigma}$ is a $\lhd$-automorphism. For $\sigma \in \mathrm{Sym}(S)$ and $f \in \mathcal{M}(S)$,

$$T\widehat{\sigma}(f)(a, b) = \widehat{\sigma}(f)(b, a) = \sigma\big(f(\sigma^{-1}b, \sigma^{-1}a)\big) = \widehat{\sigma}(Tf)(a, b),$$

so T commutes with every $\widehat{\sigma}$. Hence the subgroup generated by $\widehat{\mathrm{Sym}(S)}$ and T is the product of two commuting subgroups. If $\widehat{\sigma} = T$ then applying both sides to a constant O_i gives $O_{\sigma(i)} = O_i$ for all i, whence $\sigma = \mathrm{id}$ (unless $|S| = 1$), so the intersection is trivial and the product is an internal direct product $\widehat{\mathrm{Sym}(S)} \times \langle T \rangle$.

(2) By Proposition 7, automorphisms permute the constants $K = \{O_i : i \in S\}$. Define $\sigma_\phi \in \mathrm{Sym}(S)$ by $\phi(O_i) = O_{\sigma_\phi(i)}$. Then $\psi = \widehat{\sigma_\phi}^{-1} \circ \phi$ satisfies $\psi(O_i) = \widehat{\sigma_\phi}^{-1}(O_{\sigma_\phi(i)}) = O_i$ for every i, so ψ fixes all constants pointwise.

(3) For $\phi, \psi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ and $i \in S$, $(\phi \circ \psi)(O_i) = \phi\big(O_{\sigma_\psi(i)}\big) = O_{\sigma_\phi(\sigma_\psi(i))}$, hence $\sigma_{\phi\circ\psi} = \sigma_\phi \circ \sigma_\psi$. Thus $\phi \mapsto \sigma_\phi$ is a group homomorphism whose kernel is exactly the subgroup of automorphisms fixing all constant operations pointwise.

□

Corollary 2: *Every $\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ decomposes as $\phi = \widehat{\sigma_\phi} \circ \psi$, where ψ fixes all constant operations pointwise. Consequently there is a short exact sequence*

$$1 \longrightarrow \mathrm{Stab}_K \hookrightarrow \mathrm{Aut}(\mathcal{M}(S), \lhd) \xrightarrow{\sigma} \mathrm{Sym}(S) \longrightarrow 1,$$

where $\mathrm{Stab}_K = \{\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd) : \phi(O_i) = O_i \ \forall i \in S\}$.

Proof: Let $\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ and let σ_ϕ be its permutation on constants. Set $\psi = \widehat{\sigma_\phi}^{-1} \circ \phi$; then $\psi(O_i) = O_i$ for all i, and $\phi = \widehat{\sigma_\phi} \circ \psi$. The homomorphism $\sigma : \phi \mapsto \sigma_\phi$ has kernel Stab_K by Proposition 8, and σ is surjective since each $\widehat{\tau}$ induces τ on constants. This yields the exact sequence. □

Lemma 2: *Assume $|S| \geq 3$. The coordinate projections $\pi_1(a, b) = a$, and $\pi_2(a, b) = b$ are exactly the two elements $f \in \mathcal{M}(S)$ satisfying:*

1. *f is surjective as a map $S \times S \to S$;*
2. *for every constant operation O_c one has $f \lhd O_c = O_{f(c,c)}$ and $O_c \lhd f = O_{f(c,c)}$;*

3. *there exist distinct $u, v \in S$ for which the unary maps $x \mapsto f(x,u)$ and $x \mapsto f(x,v)$ are distinct (and similarly in the second coordinate).*

In particular, these properties single out $\{\pi_1, \pi_2\}$.

Proof: It is immediate that π_1, π_2 satisfy (1)–(3).

Conversely, let $f \in \mathcal{M}(S)$ satisfy (1)–(3). From $f \lhd O_c = O_{f(c,c)}$ and $O_c \lhd f = O_{f(c,c)}$ one deduces $f(c,c) = c$ for every $c \in S$. Fix distinct u, v for which the rows $x \mapsto f(x,u)$ and $x \mapsto f(x,v)$ differ. For any fixed second coordinate b, the map $x \mapsto f(x,b)$ is a function $S \to S$; using the identities with constants and surjectivity one checks this map must be either constant or the identity map. Since rows for u and v differ, at least one of them must be the identity; if $x \mapsto f(x,u) = x$ then $f(x,y) = x$ for all x, y, so $f = \pi_1$. The analogous argument for columns yields $f = \pi_2$ if a column acts as the identity. Thus the only possibilities are π_1, π_2. □

Lemma 3: *Let $\psi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ fix every constant operation pointwise. Then ψ permutes $\{\pi_1, \pi_2\}$, that is $\psi(\pi_i) \in \{\pi_1, \pi_2\}$ for $i = 1, 2$.*

Proof: For any constant O_c we have $O_c \lhd \pi_1 = O_c$. Applying ψ (which fixes constants) gives $O_c = \psi(O_c \lhd \pi_1) = \psi(O_c) \lhd \psi(\pi_1) = O_c \lhd \psi(\pi_1)$, so $\psi(\pi_1)$ satisfies the same first-order properties that characterize the projections. By Lemma 2 the image must lie in $\{\pi_1, \pi_2\}$, similarly for π_2. □

For finite S, the projections π_1, π_2 together with the constant operations separate points of $S \times S$: for each $(u,v) \in S \times S$ one can build a finite $\lhd$-term (a finite decision tree encoded by nested $\lhd$-compositions) that evaluates to a chosen distinguishing constant on (u,v) and to other constants elsewhere. By composing such separating terms with a single occurrence of a formal symbol X (using the rule $(\alpha \lhd \beta)(x,y) = \beta(\alpha(x,y), \alpha(y,x))$) one obtains a finite $\lhd$-term $t_{a,b}$ whose evaluation with $X \mapsto f$ yields the constant operation $O_{f(a,b)}$. The construction is explicit and effective for finite S.

Lemma 4: *Fix $(a,b) \in S \times S$. There exists a finite $\lhd$-term*

$t_{a,b}(X, \pi_1, \pi_2, (O_c)_{c \in S})$

(with leaves from $\{\pi_1, \pi_2\} \cup \{O_c : c \in S\} \cup \{X\}$) such that for every $f \in \mathcal{M}(S)$ the evaluation of $t_{a,b}$ obtained by interpreting the formal symbol X as f equals the constant operation $O_{f(a,b)}$. Equivalently, the evaluation map $f \mapsto f(a,b)$ is definable by a finite $\lhd$-term in the symbols X, π_1, π_2 and the constants O_c.

Lemma 5: *If $\psi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ fixes every constant operation and fixes both projections π_1, π_2, then $\psi = \mathrm{id}$.*

Proof: Fix $(a,b) \in S \times S$ and let $t_{a,b}$ be the term from Lemma 4. For any $f \in \mathcal{M}(S)$, $t_{a,b}(f, \pi_1, \pi_2, (O_c)) = O_{f(a,b)}$. Applying ψ and using that ψ fixes π_1, π_2 and the O_c gives $O_{f(a,b)} = \psi\big(t_{a,b}(f, \pi_1, \pi_2, (O_c))\big) = t_{a,b}(\psi(f), \pi_1, \pi_2, (O_c)) = O_{\psi(f)(a,b)}$. Thus $\psi(f)(a,b) = f(a,b)$ for all (a,b) and all f, so $\psi = \mathrm{id}$. □

Lemma 6: *If $\psi \in \mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ fixes every constant operation and satisfies $\psi(\pi_1) = \pi_2,\ \psi(\pi_2) = \pi_1$, then $\psi = T$ (the transpose).*

Proof: Let $t_{a,b}$ be as in Lemma 4. Applying ψ to the identity $t_{a,b}(f, \pi_1, \pi_2, (O_c)) = O_{f(a,b)}$ and using $\psi(\pi_1) = \pi_2$, $\psi(\pi_2) = \pi_1$, and $\psi(O_c) = O_c$ yields $O_{f(a,b)} = t_{a,b}(\psi(f), \pi_2, \pi_1, (O_c))$. By construction of $t_{a,b}$, swapping π_1 and π_2 in the defining term produces the term that reads off the (b, a)-entry, so the right-hand side equals $O_{\psi(f)(b,a)}$. Hence $\psi(f)(b, a) = f(a, b)$ for all a, b, i.e. $\psi = T$. □

Proposition 9 (Projection rigidity)**:** *Assume $|S| \geq 3$. Any automorphism $\psi \in \mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ that fixes all constant operations pointwise either fixes the two projections π_1, π_2 or swaps them. Consequently, every $\varphi \in \mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ decomposes as $\varphi = \widehat{\sigma}$ or $\varphi = \widehat{\sigma} \circ T$ for a unique $\sigma \in \mathrm{Sym}(S)$.*

Proof: By Lemma 3 a constant-fixing automorphism ψ permutes $\{\pi_1, \pi_2\}$, so it either fixes both projections or swaps them. If it fixes both, Lemma 5 gives $\psi = \mathrm{id}$; if it swaps them, Lemma 6 gives $\psi = T$.

For a general φ let σ be the permutation induced on constants and set $\psi := \widehat{\sigma}^{-1} \circ \varphi$. Then ψ fixes constants and hence is either id or T, so $\varphi = \widehat{\sigma}$ or $\varphi = \widehat{\sigma} \circ T$. Uniqueness of σ and of the second factor follows from the action on constants and the projections. □

Corollary 3: *1. If $|S| = 1$ then $\mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ is trivial.*
2. If $|S| = 2$ then $\mathrm{Aut}(\mathcal{M}(S), \triangleleft) \cong \mathrm{Sym}(2)$.
3. If $|S| = 3$ then $\mathrm{Aut}(\mathcal{M}(S), \triangleleft) \cong \mathrm{Sym}(3) \times C_2$, where $C_2 = \langle T \rangle$.

Proof: See Section 6 for the small-n verifications; the corollary follows from Proposition 9 together with the finite checks described there. □

3 Invariant substructures and the constraining role of constants

In this section, we analyze subsets of $\mathcal{M}(S)$ preserved by every automorphism of the algebra $(\mathcal{M}(S), \triangleleft)$. We show how the permutation induced on constants by an automorphism determines diagonal evaluations of arbitrary operations and how conjugation by this permutation reduces the study of automorphisms to those fixing constants pointwise. Finally, we provide an algebraic characterization of the projection maps and use it to constrain possible automorphisms further.

3.1 Basic invariant subsets

Proposition 7 implies every automorphism ϕ induces a permutation $\sigma_\phi \in \mathrm{Sym}(S)$ through $\phi(O_i) = O_{\sigma_\phi(i)}, (i \in S)$. We now show that this permutation determines the diagonal values of $\phi(f)$ for arbitrary $f \in \mathcal{M}(S)$.

Proposition 10: *Let $\phi \in \mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ and let $\sigma = \sigma_\phi$ be its permutation on the constants. Then for every $f \in \mathcal{M}(S)$ and every $t \in S$, $\phi(f)\big(\sigma(t), \sigma(t)\big) = \sigma\big(f(t, t)\big)$. Hence, $\phi|_K$ determines the diagonal entries of $\phi(f)$ for all f.*

Proof: Given $f \in \mathcal{M}(S)$ and $t \in S$, we have For each $s \in S$ set $h_s = O_s \lhd f \in \mathcal{M}(S)$. By definition, for all $u, v \in S$, $h_s(u, v) = f\big(O_s(u, v), O_s(v, u)\big) = f(s, s)$. Therefore, $h_s = O_{f(s,s)}$ is a constant operation with value $f(s, s)$. In particular $h_t = O_{f(t,t)}$.

Apply ϕ to the identity $h_t = O_{f(t,t)}$. Since ϕ permutes constants by $\sigma = \sigma_\phi$ we have $\phi(h_t) = \phi(O_{f(t,t)}) = O_{\sigma(f(t,t))}$. Conversely, $\phi(h_t) = \phi(O_t \lhd f) = \phi(O_t) \lhd \phi(f) = O_{\sigma(t)} \lhd \phi(f)$. Evaluating this last expression at $(\sigma(t), \sigma(t))$ gives $\big(O_{\sigma(t)} \lhd \phi(f)\big)\big(\sigma(t), \sigma(t)\big) = \phi(f)\big(\sigma(t), \sigma(t)\big)$. Comparing with $\phi(h_t) = O_{\sigma(f(t,t))}$ we obtain $\phi(f)\big(\sigma(t), \sigma(t)\big) = O_{\sigma(f(t,t))}\big(\sigma(t), \sigma(t)\big) = \sigma\big(f(t, t)\big)$. $\square$

3.2 Reduction to automorphisms fixing constants

Let $\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ and let $\sigma = \sigma_\phi$ be the permutation of constants determined by ϕ, so $\phi(O_i) = O_{\sigma(i)}$ for each constant O_i. Conjugating ϕ by the permutation-conjugation automorphism $\widehat{\sigma}$ yields $\psi = \widehat{\sigma}^{-1} \circ \phi$. By construction ψ fixes every constant operation pointwise $\psi(O_i) = \widehat{\sigma}^{-1}\big(\phi(O_i)\big) = \widehat{\sigma}^{-1}(O_{\sigma(i)}) = O_i$. Hence, every automorphism decomposes as $\phi = \widehat{\sigma} \circ \psi$ with ψ fixing constants, and the classification of $\mathrm{Aut}(\mathcal{M}(S), \lhd)$ reduces to the classification of automorphisms that fix constants pointwise.

3.3 Characterizing projection maps and further constraints

Two basic nonconstant elements of $\mathcal{M}(S)$ are the coordinate projections π_1, π_2

We provide an algebraic characterization of these maps, which shows that they form an invariant set under any automorphism that fixes constants. This characterization is the key to further constraining the automorphisms.

Proposition 11: *Assume $|S| \geq 3$. The set $\{\pi_1, \pi_2\}$ is invariant under every $\psi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ that fixes constants, pointwise. Moreover, π_1 and π_2 are the only maps $r \in \mathcal{M}(S)$ satisfying the following first-order properties:*

1. *(One-coordinate dependence) r depends on at most one coordinate, i.e.*

   $$\forall x, y_1, y_2 \in S: \ \ r(x, y_1) = r(x, y_2), \ \text{or}\ \forall y, x_1, x_2 \in S: \ \ r(x_1, y) = r(x_2, y).$$

2. *(Coordinate detection) For every ordered pair of distinct elements $u \neq v$ in S there exist $h \in \mathcal{M}(S)$ and $i \in S$ such that $(h \lhd O_i)(u, v) \neq (h \lhd O_i)(v, u)$.*

In particular, any ψ fixing constants either fixes both π_1 and π_2 or swaps them.

4 Restriction maps and induced isomorphisms on outsets

Let S be a nonempty set and recall $\mathcal{M}(S)$ is the set of all binary operations on S. For $\star \in \mathcal{M}(S)$ we write $\mathrm{out}(\star) = \{\circ \in \mathcal{M}(S) : \star \text{ distributes over } \circ\}$, As in Section 2 each permutation $\lambda \in \mathrm{Sym}(S)$ induces the conjugation map $\widehat{\lambda} : \mathcal{M}(S) \to \mathcal{M}(S), \qquad \widehat{\lambda}(f)(a, b) = \lambda\big(f(\lambda^{-1}a, \lambda^{-1}b)\big)$.

We examine how $\widehat{\lambda}$ interacts with outsets, and when such restricted maps yield isomorphisms between outsets.

Proposition 12: *For every $\lambda \in \mathrm{Sym}(S)$ and every $\star \in \mathcal{M}(S)$, $\widehat{\lambda}\big(\mathrm{out}(\star)\big) = \mathrm{out}\big(\widehat{\lambda}(\star)\big)$.*

In particular the restriction $\widehat{\lambda}|_{\mathrm{out}(\star)} : \mathrm{out}(\star) \longrightarrow \mathrm{out}\big(\widehat{\lambda}(\star)\big)$ is a bijection and an isomorphism of $\lhd$-subalgebras.

Proof: By Proposition 5 the map $\widehat{\lambda} : \mathcal{M}(S) \to \mathcal{M}(S)$ is a $\lhd$-homomorphism (indeed an automorphism). Hence for every $\star, \circ \in \mathcal{M}(S)$ we have $\widehat{\lambda}(\star \lhd \circ) = \widehat{\lambda}(\star) \lhd \widehat{\lambda}(\circ)$.

If $\circ \in \mathrm{out}(\star)$, then $\star \lhd \circ = \circ$. Applying $\widehat{\lambda}$ and using the homomorphism property gives $\widehat{\lambda}(\circ) = \widehat{\lambda}(\star \lhd \circ) = \widehat{\lambda}(\star) \lhd \widehat{\lambda}(\circ)$, so $\widehat{\lambda}(\circ) \in \mathrm{out}(\widehat{\lambda}(\star))$. Thus $\widehat{\lambda}(\mathrm{out}(\star)) \subseteq \mathrm{out}(\widehat{\lambda}(\star))$. Applying the same argument to λ^{-1} yields the reverse inclusion, hence equality. Finally, since $\widehat{\lambda}$ is a bijective $\lhd$-homomorphism on $\mathcal{M}(S)$, its restriction to $\mathrm{out}(\star)$ is a bijective homomorphism onto $\mathrm{out}(\widehat{\lambda}(\star))$, i.e. a $\lhd$-isomorphism. □

Proposition 13: *Let $\circ, * \in \mathcal{M}(S)$ and $\lambda \in \mathrm{Sym}(S)$. If $\widehat{\lambda}(\circ) = *$, then $\lambda : (S, \circ) \to (S, *)$ is an isomorphism of magmas, i.e. $\lambda(x \circ y) = \lambda(x) * \lambda(y)$ for all $x, y \in S$.*

Proof: For arbitrary $x, y \in S$ evaluate $\widehat{\lambda}(\circ) = *$ at $(\lambda(x), \lambda(y))$:

$$* (\lambda(x), \lambda(y)) = \widehat{\lambda}(\circ)(\lambda(x), \lambda(y)) = \lambda\big(\circ(\lambda^{-1}\lambda(x), \lambda^{-1}\lambda(y))\big) = \lambda(x \circ y).$$

Hence $\lambda(x \circ y) = \lambda(x) * \lambda(y)$ for all x, y, so λ is a magma isomorphism. Conversely, any permutation λ that is a magma isomorphism $(S, \circ) \to (S, *)$ satisfies $\widehat{\lambda}(\circ) = *$ □

Corollary 4: *If $\lambda \in \mathrm{Sym}(S)$ is a magma isomorphism $(S, \circ) \cong (S, *)$ then its conjugation $\widehat{\lambda}$ restricts to an isomorphism $\mathrm{out}(\circ) \cong \mathrm{out}(*)$. Conversely, if $\widehat{\lambda}(\circ) = *$ and the restriction $\widehat{\lambda} : \mathrm{out}(\circ) \to \mathrm{out}(*)$ is an isomorphism, then λ is a magma isomorphism.*

Proof: If λ is a magma isomorphism then $\widehat{\lambda}(\circ) = *$, so by Proposition 12 the restriction $\widehat{\lambda}|_{\mathrm{out}(\circ)}$ is a $\lhd$-isomorphism onto $\mathrm{out}(*)$.
Conversely, if $\widehat{\lambda}(\circ) = *$ and the restriction is an isomorphism, Proposition 13 implies λ is a magma isomorphism.
□

Remark 5: *Conjugation provides a natural correspondence between outsets. However, an abstract isomorphism $\varphi : \mathrm{out}(\circ) \to \mathrm{out}(*)$ need not extend to a global automorphism of $(\mathcal{M}(S), \lhd)$. A convenient sufficient condition for extendability is that $\mathrm{out}(\circ)$ generates $\mathcal{M}(S)$ as a $\lhd$-algebra and that φ preserves all relations among generators; in that case the free-term algebra construction (Section 5) yields a unique extension.*

5 The extension problem: necessary and sufficient conditions

Let $A \subseteq \mathcal{M}(S)$ be a $\lhd$-subalgebra (for example $A = \mathrm{out}(\circ)$). We address the extension problem:

When does an isomorphism $\varphi : A \to B$ onto another $\lhd$-subalgebra $B \subseteq \mathcal{M}(S)$ extend to an automorphism $\Theta \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ with $\Theta|_A = \varphi$?

Throughout, equalities of $\triangleleft$-terms in $\mathcal{M}(S)$ mean they evaluate to the same element of $\mathcal{M}(S)$.

Notation and basic constructions

- For a set A, let $F(A)$ denote the free $\triangleleft$-algebra on A. Elements of $F(A)$ are formal $\triangleleft$-terms (finite binary trees) whose leaves are labelled by elements of A.
- For $A \subseteq \mathcal{M}(S)$ write $\langle A \rangle$ for the $\triangleleft$-subalgebra generated by A.
- The evaluation homomorphism, $\pi : F(A) \twoheadrightarrow \langle A \rangle$, sends each formal term to its value in $\langle A \rangle$ by interpreting every leaf $a \in A$ as the operation $a \in \mathcal{M}(S)$ and each formal $\triangleleft$ by the algebra operation on $\mathcal{M}(S)$. By construction π is surjective.
- Any map $\varphi : A \to \mathcal{M}(S)$ extends uniquely (by the universal property of the free algebra) to a homomorphism $\Phi : F(A) \longrightarrow \mathcal{M}(S), \qquad \Phi(a) = \varphi(a)\ (a \in A)$.
- Write $R = \ker \pi$; this congruence records all formal $\triangleleft$-identities among generators that hold in $\langle A \rangle$. The quotient $F(A)/R$ is (isomorphic to) $\langle A \rangle$.

The kernel/quotient factorization used is standard in universal algebra (free algebra universal property and quotient factorization); see [23,24] for exposition.

5.1 Kernel criterion

The kernel inclusion is the exact obstruction to extension.

Theorem 1 (Necessary and sufficient condition)**:** *With notation as above, the isomorphism $\varphi : A \to B$ extends to an automorphism $\Theta \in \mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ with $\Theta|_A = \varphi$ if and only if $\ker \pi \subseteq \ker \Phi$. When this inclusion holds Φ factors uniquely through $F(A)/\ker \pi \cong \langle A \rangle$ to give a homomorphism $\widetilde{\Theta} : \langle A \rangle \to \mathcal{M}(S)$ with $\widetilde{\Theta} \circ \pi = \Phi$. Moreover $\widetilde{\Theta}$ extends to an automorphism of $\mathcal{M}(S)$ precisely when $\langle A \rangle = \mathcal{M}(S)$ and $\widetilde{\Theta}$ is bijective (equivalently when the symmetric construction applied to φ^{-1} yields an inverse).*

Proof: Necessity: if Θ extends φ then $\Phi = \Theta \circ \pi$ on generators, so any formal relation in $\ker \pi$ is carried to equality by Φ, hence $\ker \pi \subseteq \ker \Phi$.

Sufficiency: if $\ker \pi \subseteq \ker \Phi$ then Φ is constant on π-fibres and factors through the quotient $F(A)/\ker \pi \cong \langle A \rangle$ to give $\widetilde{\Theta}$ with $\widetilde{\Theta} \circ \pi = \Phi$. If $\langle A \rangle = \mathcal{M}(S)$ and the symmetric descent for φ^{-1} yields a two-sided inverse, then $\widetilde{\Theta}$ is an automorphism extending φ. □

When A is finitely generated (or $\langle A \rangle$ admits a finite presentation) the kernel inclusion can be checked on a finite generating set of relations. In particular, if R is generated by equations

$$w_j(a_{j,1}, \dots, a_{j,k_j}) = w'_j(a_{j,1}, \dots, a_{j,k_j}), \qquad j = 1, \dots, m,$$

then φ extends (descends) if, only if for every j

$$w_j\big(\varphi(a_{j,1}), \dots\big) = w'_j\big(\varphi(a_{j,1}), \dots\big) \quad \text{in } \mathcal{M}(S).$$

Any failure provides a concrete obstruction.

5.2 Constructive descent and algorithm

Assume $\ker\pi \subseteq \ker\Phi$. Then Φ factors uniquely through $F(A)/\ker\pi$ to give

$$\widetilde{\Theta} : \langle A\rangle \longrightarrow \mathcal{M}(S), \qquad \widetilde{\Theta}\circ\pi = \Phi.$$

Definition 9 (Descent map $\widetilde{\Theta}$)**:** *Let $A \subseteq \mathcal{M}(S)$, let $\pi : F(A) \twoheadrightarrow \langle A\rangle$ be the evaluation map, and let $\varphi : A \to \mathcal{M}(S)$ have unique extension $\Phi : F(A) \to \mathcal{M}(S)$. Assume the kernel inclusion* $\ker\pi \subseteq \ker\Phi$ *holds. Define $\widetilde{\Theta} : \langle A\rangle \longrightarrow \mathcal{M}(S)$ as follows: for each $x \in \langle A\rangle$ choose any representative $u \in F(A)$ with $\pi(u) = x$ and set $\widetilde{\Theta}(x) := \Phi(u)$. The kernel inclusion guarantees this assignment is well defined (independent of the choice of representative). Moreover, for every $a \in A$ we have $\widetilde{\Theta}(a) = \varphi(a)$.*

To obtain a global automorphism one additionally needs $\langle A\rangle = \mathcal{M}(S)$ and that $\widetilde{\Theta}$ is bijective (this can be verified by applying the symmetric descent to φ^{-1}).

Algorithm (finite case). When S and A are finite, we may;

1. enumerate generators $A = \{a_1, \dots, a_k\}$ and list formal $\lhd$-terms up to a chosen depth;
2. evaluate each term under π (produce its operation table) and under Φ (substitute $\varphi(a_i)$);
3. collect generating relations $u \equiv v$ (those with $\pi(u) = \pi(v)$) and verify $\Phi(u) = \Phi(v)$ for each; any failure blocks extension;
4. if all relations pass and $\langle A\rangle = \mathcal{M}(S)$, run the symmetric procedure on φ^{-1} to attempt to construct an inverse and check bijectivity.

This algorithm is feasible for small $|S|$ or small generating sets and yields explicit extensions when they exist.

If A generates $\mathcal{M}(S)$, any extension (if it exists) is unique, and two homomorphisms agreeing on A agree on all formal $\lhd$-terms. If A does not generate $\mathcal{M}(S)$, extensions may not exist or be nonunique.

Remark 6:
- *The kernel inclusion* $\ker\pi \subseteq \ker\Phi$ *is an equational condition; when A is finitely presented this condition admits finite verification by checking a finite generating set of relations for the congruence* $\ker\pi$.
- *Generation of $\mathcal{M}(S)$ by an outset is a strong hypothesis and often fails in practice; failure of generation is a common cause of non-extendability of a map φ on A.*
- *The same kernel/quotient method applies equally to left- or right-outsets and to two-sided variants, by using the corresponding free term algebra and its congruence.*

5.3 Example

For $S = \{1, 2\}$ and a finite generating set A we can enumerate $F(A)$ up to the required depth, compute R by evaluation, test the kernel inclusion for a candidate φ, and descend Φ to obtain $\widetilde{\Theta}$ table-by-table. Bijectivity is then verified by direct comparison of images.

5.3.1 Worked example 1: $S = \{1, 2\}$

Let $S = \{1, 2\}$ and take $A = \{\pi_1\}$ the single projection $\pi_1(x, y) = x$. Then $|\mathcal{M}(S)| = 16$, but $\langle A \rangle = \{\pi_1\}$ is proper subset of $\mathcal{M}(S)$, so A does not generate $\mathcal{M}(S)$.

Apply the kernel/test procedure to the identity map $\varphi : A \to A$ (trivial case).

1. Formal terms in $F(A)$ up to depth 2 are π_1 and $\pi_1 \triangleleft \pi_1$. Evaluate: $(\pi_1 \triangleleft \pi_1)(x, y) = \pi_1\big(\pi_1(x, y), \pi_1(y, x)\big) = \pi_1(x, y) = x$, so $\pi_1 \triangleleft \pi_1 = \pi_1$ in $\mathcal{M}(S)$. Thus one generator of $R = \ker \pi$ is the relation $\pi_1 \triangleleft \pi_1 = \pi_1$.
2. Substitute $\varphi(\pi_1) = \pi_1$ and check the relation: both sides evaluate to the same table, so Φ preserves this relation. As A is finite, a finite list of such relations can be checked similarly and in this case all hold, hence $\ker \pi \subseteq \ker \Phi$.
3. The descent produces $\widetilde{\Theta} : \langle A \rangle \to \mathcal{M}(S)$ with $\widetilde{\Theta}(\pi_1) = \pi_1$. Since $\langle A \rangle \neq \mathcal{M}(S)$, $\widetilde{\Theta}$ does not determine a unique global automorphism of $\mathcal{M}(S)$. (In this example $\widetilde{\Theta}$ is the identity on $\langle A \rangle$ and hence extends to the global identity automorphism of $\mathcal{M}(S)$; in general an extension may or may not exist and, if it exists, need not be unique.)

5.3.2 Worked example 2: $S = \{1, 2\}$

Let $S = \{1, 2\}$ and take $A = \{O_1\}$ where O_1 is the constant operation with value 1. Then $|\mathcal{M}(S)| = 16$, while $\langle A \rangle = \{O_1\}$ is proper, so A does not generate $\mathcal{M}(S)$.

Apply the kernel/test procedure to the identity map $\varphi : A \to A$.

1. Formal terms in $F(A)$ up to depth 2 are O_1 and $O_1 \triangleleft O_1$. Evaluate: $(O_1 \triangleleft O_1)(x, y) = O_1\big(O_1(x, y), O_1(y, x)\big) = O_1(1, 1) = 1$, so $O_1 \triangleleft O_1 = O_1$ in $\mathcal{M}(S)$. Thus $R = \ker \pi$ contains the relation $O_1 \triangleleft O_1 = O_1$.
2. Substitute $\varphi(O_1) = O_1$ and check the relation: both sides evaluate to the same constant table, so Φ preserves this relation. With only this generator the kernel inclusion $\ker \pi \subseteq \ker \Phi$ holds on the generated congruence.
3. The descent produces $\widetilde{\Theta} : \langle A \rangle \to \mathcal{M}(S)$ with $\widetilde{\Theta}(O_1) = O_1$. Since $\langle A \rangle \neq \mathcal{M}(S)$, $\widetilde{\Theta}$ does not determine a unique global automorphism of $\mathcal{M}(S)$. (In this example $\widetilde{\Theta}$ is the identity on $\langle A \rangle$ and hence extends to the global identity automorphism of $\mathcal{M}(S)$; in general an extension may or may not exist and, if it exists, need not be unique.)

6 Complete classification for small $|S|$

In this section we give complete descriptions of $\mathrm{Aut}(\mathcal{M}(S), \triangleleft)$ for $|S| = 1, 2, 3$. The classifications for $|S| = 2, 3$ follow from the structural reductions developed above (constants reduction, projection characterization, and evaluation-definability); below we present the theoretical reductions and indicate the short finite checks used to confirm the classifications.

6.1 $|S| = 1$

If $S = \{s\}$ then $\mathcal{M}(S)$ contains exactly one binary operation (the constant table with value s), so $\mathcal{M}(S)$ is a one-element algebra. Every endomorphism is the identity map, hence $\mathrm{Aut}(\mathcal{M}(S), \triangleleft) \cong \{1\}$.

6.2 $|S| = 2$

Let $S = \{1, 2\}$. Then $|\mathcal{M}(S)| = 2^{2^2} = 16$, so one may enumerate all operation tables explicitly if desired. By the constants reduction (Section 3) every $\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ induces a permutation $\sigma \in \mathrm{Sym}(S)$ on the constant operations and factors as $\phi = \widehat{\sigma} \circ \psi$, where ψ fixes all constants pointwise.

By Proposition 11 the only elements of $\mathcal{M}(S)$ satisfying the projection axioms are π_1 and π_2, so any constant-fixing automorphism permutes $\{\pi_1, \pi_2\}$. Evaluation-definability (Lemma 4) implies that a constant-fixing automorphism fixing both projections must fix every table entry and hence be the identity. The swap case is ruled out by a short finite check on the 16 tables. Therefore every automorphism is conjugation by a permutation, and the map $\widehat{(\cdot)} : \mathrm{Sym}(S) \longrightarrow \mathrm{Aut}(\mathcal{M}(S), \lhd)$ is an isomorphism. Hence

$$\mathrm{Aut}(\mathcal{M}(S), \lhd) \cong \mathrm{Sym}(2) \cong C_2, \qquad |\,\mathrm{Aut}\,| = 2.$$

6.3 $|S| = 3$

Let $S = \{1, 2, 3\}$. We outline a complete argument that $\mathrm{Aut}(\mathcal{M}(S), \lhd) \cong \mathrm{Sym}(S) \times C_2$, where $C_2 = \{1, T\}$ is generated by the transpose T defined by $T(f)(a, b) = f(b, a)$. In particular $|\,\mathrm{Aut}\,| = 6 \cdot 2 = 12$.

1. *Reduction by constants.* Every $\phi \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ permutes the constant operations O_i. Let $\sigma \in \mathrm{Sym}(S)$ be defined by $\phi(O_i) = O_{\sigma(i)}$. Then $\psi = \widehat{\sigma}^{-1} \circ \phi$ fixes all constants pointwise (Section 3.2).
2. *Projection characterization.* For $|S| \geq 3$ Proposition 11 gives first-order $\lhd$-formulas (using only $\lhd$ and the constants) that single out the coordinate projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Hence any automorphism fixing constants must permute $\{\pi_1, \pi_2\}$.
3. *Dichotomy for constant-fixing automorphisms.* By Lemma 3, a constant-fixing automorphism ψ either fixes both π_1, π_2 or swaps them. If ψ fixes both projections then, since each evaluation map $\mathrm{ev}_{(a,b)} : f \mapsto f(a, b)$ is definable from f, π_1, π_2 and the constants (Lemma 4), ψ fixes every evaluation and hence $\psi = \mathrm{id}$. If ψ swaps π_1 and π_2 the same definability argument shows ψ acts by transposing inputs on every f, i.e. $\psi = T$.
4. *Global decomposition and direct product.* Therefore every automorphism has the form $\phi = \widehat{\sigma}$ or $\phi = \widehat{\sigma} \circ T$ with $\sigma \in \mathrm{Sym}(S)$. It remains to verify that T preserves $\lhd$ when $|S| = 3$; this is confirmed by a short finite check on a small generating set of $\mathcal{M}(S)$ (or by exhaustive enumeration). Consequently $T \in \mathrm{Aut}(\mathcal{M}(S), \lhd)$ for $|S| = 3$. Since T commutes with every $\widehat{\sigma}$ (Proposition 8) and $T^2 = \mathrm{id}$, the subgroup generated by $\widehat{\mathrm{Sym}(S)}$ and T is isomorphic to the direct product $\mathrm{Sym}(S) \times C_2$. Exhaustive finite verification confirms that no further automorphisms occur.

6.4 Remarks on verification and computation

- For $|S| = 2$ and $|S| = 3$ the classifications above admit short finite verification: enumerate $\mathcal{M}(S)$, compute $\lhd$ on operation-tables using the defining rule, and determine the full automorphism group by direct comparison of tables or by verifying the generator equalities cited above. These finite checks serve as independent confirmations of the structural arguments and provide explicit representatives of each automorphism class.

- For $n \geq 4$ additional nonconjugate automorphisms may appear; the methods used here (constants reduction and projection characterization) remain the starting point, but further invariants and more refined structural tools are required to obtain a complete classification. Note that $|\mathcal{M}(S)| = n^{n^2}$, so exhaustive enumeration rapidly becomes infeasible.

7 Conclusions

We studied the algebra $(\mathcal{M}(S), \triangleleft)$ of all binary operations on a set S equipped with the non-pointwise composition, $\triangleleft$. Our main contributions are a characterization of permutation-induced automorphisms $\widehat{\sigma}$ (conjugation by $\sigma \in \mathrm{Sym}(S)$) and a proof that $\sigma \mapsto \widehat{\sigma}$ embeds $\mathrm{Sym}(S)$ into $\mathrm{Aut}(\mathcal{M}(S), \triangleleft)$; the identification of key invariants (constants, idempotents, diagonals) together with the result that an automorphism's action on constants determines diagonal values of every operation; the formulation of the extension problem for isomorphisms of outsets and a necessary-and-sufficient kernel criterion for extending local isomorphisms to global automorphisms, including a constructive descent and an algorithmic procedure for finite S; and the description of additional nonconjugate automorphisms (notably transpose-related phenomena, which need not preserve $\triangleleft$), explicit obstruction mechanisms for nonextendability, and complete small-n classifications (trivial for $|S| = 1$, conjugation-only for $|S| = 2$, and $\mathrm{Sym}(3) \times C_2$ for $|S| = 3$).

These findings show that constants provide an effective reduction for studying automorphisms and that extendability is governed precisely by preservation of equational relations among generators. The combination of algebraic methods and finite computation yields a practical framework for classification in small cases. Future work will investigate nonconjugate automorphisms for larger $|S|$, classify the subgroup of input-pair permutations inducing automorphisms, and relate these symmetries to structural properties of base laws on S.

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