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Research article

Infinitely many Concrete Multiple-Composite and Amplified Fuzzy ordinary and Fractional Neural Network Approximations

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Abstract: Here we investigate further the univariate fuzzy ordinary and fractional quantitative approximation of fuzzy real valued functions on a compact interval. This is done by quasi-interpolation sigmoid multicomposite activation functions based infinitely many specific and amplified multicomposite fuzzy neural network operators. These approximations are derived by establishing fuzzy multicomposite Jackson type inequalities involving the fuzzy moduli of continuity of the function, or of the right and left Caputo fuzzy fractional derivatives of the involved function. The approximations are fuzzy pointwise and fuzzy uniform. The related feed-forward fuzzy multicomposite neural networks are with one hidden layer. We study in particular the fuzzy integer derivative and just fuzzy continuous cases. Our fuzzy fractional multicomposite approximation result using higher order fuzzy differentiation converges better than in the multicomposite fuzzy just continuous case. All these approximations are generated by 7 specific and basic activation functions.

Mathematics Subject Classification: [2010]: 26A33, 26E50, 41A17, 41A25, 41A30, 41A36, 47S40.
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1 Introduction

From AI and computer science we have the following: In essence, composing activation functions in neural networks offers the advantage of potentially tailoring the network's ability to learn and model complex, non-linear relationships in data. Here's a breakdown of the potential benefits:

1. Enhanced Capacity for Complex Modeling:

- Diversification of Non-linearity: Different activation functions have different characteristics. For example, ReLU introduces sparsity, while Sigmoid squashes values into a range. By composing them, the network potentially can learn a wider variety of non-linear transformations and capture more intricate patterns in the data.

2. Improved Training Dynamics:

- Mitigating Gradient Problems: Activation functions influence gradient flow during training. Using different activation functions can potentially help address issues like vanishing or exploding gradients, which hinder learning in deep networks.
- Faster Convergence: Certain activation functions, like ReLU, can accelerate the convergence of the training process compared to others like Sigmoid or Tanh. Combining different functions can potentially lead to faster training and competitive performance.

3. Enhanced Generalization and Robustness:

- Better Generalization: By learning richer representations of the data through diverse activation functions, the network's ability to generalize well to unseen data improves, reducing the risk of overfitting.
- Increased Robustness: Networks with carefully chosen activation functions can handle variations in input data more effectively, adapting to noise , missing data, or unexpected perturbations.

4. Adaptation to Input Characteristics:

- Handling Diverse Data: Different activation functions can be suited to different data characteristics. For instance, tanh can be useful when dealing with data containing both positive and negative values.

5. Potential for Architectural Interpretability:

- Insight into Learning: By using distinct activation functions, different parts of the network might become responsible for capturing specific features, which can potentially offer insights into how the model learns.

In summary, composing activation functions potentially allows for a more flexible and powerful neural network capable of:

- Learning more complex patterns.
- Faster and more stable training.
- Better generalization to new data.
- Greater adaptability to diverse data.

Warning: While composing activation functions can offer benefits, it is important to choose them judiciously and with consideration for the specific problem at hand, as some combinations might not be beneficial or could even lead to unwanted behaviors like exploding gradients. Empirical testing and validation are crucial when exploring different activation function compositions.

The author in [1] and [2], see chapters 2-5, was the first to derive quantitative neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He studied there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support.

The author inspired by [23], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [10], [13] - [22], by treating both the univariate and multivariate cases.

Continuation of the author's works ([17], [18], [19, Ch. 20], [22, Ch. 23]) is this article where 7 basic specific and typical activation functions generate infinitely many multicomposite and amplified Neural Network Operators at fuzzy setting that are taken at the fractional and ordinary levels resulting into higher rates of approximation. We involve the fuzzy ordinary derivatives and the right and left Caputo fuzzy fractional derivatives of the fuzzy function under approximation and we establish tight fuzzy Jackson type inequalities. An extensive background is given on fuzziness, fractional calculus and multicomposite neural networks, all needed to present our work.

Our fuzzy feed-forward multicomposite neural networks (FFNNs) are with one hidden layer. About neural networks in general study [29], [32], [33].

2 Fuzzy Fractional Mathematical Analysis Basics

(see also [19], pp. 432-444)

We need the following basic background

Definition 1: *(see [36]) Let* $\mu : \mathbb{R} \to [0,1]$ *with the following properties:*

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R}; \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}, \forall\, x, y \in \mathbb{R}, \forall\, \lambda \in [0,1]$ *(μ is called a convex fuzzy subset).*

(iii) μ *is upper semicontinuous on* $\mathbb{R}$*, i.e.* $\forall\, x_0 \in \mathbb{R}$ *and* $\forall\, \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) The set $\overline{supp(\mu)}$ *is compact in* $\mathbb{R}$ *(where* $supp(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, *for any* $x_0 \in \mathbb{R}$, *where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ *and* $\mu \in \mathbb{R}_{\mathcal{F}}$ *define*

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ *is a closed and bounded interval on* $\mathbb{R}$ *([28]).*

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r,\ [\lambda \odot u]^r = \lambda [u]^r,\ \forall\, r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and

$\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [36]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ by

$$D(u,v) := \sup_{r\in[0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right];\ u, v \in \mathbb{R}_\mathcal{F}.$$

We have that D is a metric on $\mathbb{R}_\mathcal{F}$.

Then $(\mathbb{R}_\mathcal{F}, D)$ is a complete metric space, see [36], [37].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_\mathcal{F}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u,\ \ \forall\, u \in \mathbb{R}_\mathcal{F}.$$

Denote

$$D^*(f,g) = \sup_{x\in X\subseteq\mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_\mathcal{F}$.

We mention

Definition 2: *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_\mathcal{F}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y\in X,\ |x-y|\leq\delta} D(f(x), f(y)),\ \ \delta > 0.$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta) = \omega_1(g,\delta)_X = \sup_{x,y\in X,\ |x-y|\leq\delta} |g(x) - g(y)|.$$

We define by $C_{\mathcal{F}}^{U}(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 1: *([5]) Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_1^{(\mathcal{F})}(f, \delta)_X < \infty$, for any $\delta > 0$.*

By [9], p. 129 we have that $C_{\mathcal{F}}^{U}([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 2: *([5]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f, \delta)_X = \omega_1^{(\mathcal{F})}(f, 0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$, where X is a compact interval.

Proposition 3: *([5]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 1: *It is clear by Propositions 2, 3, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_\pm^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_\pm^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).*

Proposition 4: *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f, \delta)_X$, $\omega_1\left(f_-^{(r)}, \delta\right)_X$, $\omega_1\left(f_+^{(r)}, \delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathcal{F})}(f, \delta)_X = \sup_{r \in [0,1]} \max\left\{ \omega_1\left(f_-^{(r)}, \delta\right)_X, \omega_1\left(f_+^{(r)}, \delta\right)_X \right\}.$$

Proof: Similar to Proposition 14.15, p. 246 of [9]. □

We need

Remark 1: *([3]). Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, \ldots, m \in \mathbb{N}$. Suppose that*

$$\sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, \ldots, m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max\left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)}\right) \leq \sum_{i=1}^{m} \sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{1}$$

We need

Definition 3: *Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 4: *([35]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \ \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^N(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C_{\mathcal{F}}^N([a,b])$, $[a,b] \subset \mathbb{R}$.

We mention

Theorem 2: *([30]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$\left(f' \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \ \ \forall \, r \in [0,1].$$

Remark 2: *([4]) Let $f \in C_{\mathcal{F}}^N(\mathbb{R})$, $N \geq 1$. Then by Theorem 2 we obtain*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 3: *([4]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2 we have $f_{\pm}^{(r)} \in C^{N}(\mathbb{R})$, for any $r \in [0,1]$.*

Items 11-13 are valid also on $[a,b]$.

By [9], p. 131, if $f \in C_{\mathcal{F}}([a,b])$, then f is a fuzzy bounded function.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [36].

Definition 5: *([27], p. 644) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have*

$$D\left(\sum_{P}^{*}(v-u) \odot f(\xi), I\right) < \varepsilon.$$

We write

$$I := (FR)\int_{a}^{b} f(x)dx. \tag{3}$$

We mention

Theorem 4: *([28]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_{a}^{b} f(x)dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR)\int_{a}^{b} f(x)dx\right]^{r} = \left[\int_{a}^{b} (f)_{-}^{(r)}(x)dx, \int_{a}^{b} (f)_{+}^{(r)}(x)dx\right],$$

$\forall\, r \in [0,1]$.

For the definition of general fuzzy integral we follow [31] next.

Definition 6: *Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [31].

Theorem 5: *([31]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \le r \le 1 \right\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\ r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are measurable.

Following [31], given that for each $r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \le r \le 1 \right\} \tag{4}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 7: *([31]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \le r \le 1 \right\}$$

is integrable if for each $r \in [0,1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \le r \le 1 \right\}.$$

By [31], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right),\ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 6: *([31]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG) d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F,G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F,G) d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}} d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.
Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall\, r \in [0,1].$$

We need

Definition 8: *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [24], pp. 49-52, [26], [34]) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) dt, \tag{6}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.
Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a,b]$.
We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Lemma 1: *([8]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_{\infty}([a,b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 9: *(see also [6], [25], [26]) Let $f \in AC^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$. The right Caputo fractional derivative of order $\beta > 0$ is given by*

$$D_{b-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_x^b (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta) d\zeta, \tag{7}$$

$\forall\, x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice that $D_{b-}^{\beta} f \in L_1([a,b])$ and $D_{b-}^{\beta} f$ exists a.e. on $[a,b]$.

Lemma 2: *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$. Then $D_{b-}^{\beta} f(b) = 0$.*

Convention 7. We assume that

$$D_{*x_0}^{\beta} f(x) = 0, \text{ for } x < x_0, \tag{8}$$

and

$$D_{x_0-}^{\beta} f(x) = 0, \text{ for } x > x_0, \tag{9}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 5: *([8]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous on $[a,b]$.*

Also we have

Proposition 6: *([8]) Let $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$. Then $D_{b-}^{\beta} f(x)$ is continuous on $[a,b]$.*

We further mention

Proposition 7: *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ and*

$$D_{*x_0}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_{x_0}^{x} (x-t)^{m-\beta-1} f^{(m)}(t)\,dt, \tag{10}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\beta} f(x)$ is continuous in x_0.

Proposition 8: *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ and*

$$D_{x_0-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_{x}^{x_0} (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta)\,d\zeta, \tag{11}$$

for all $x, x_0 \in [a,b] : x \leq x_0$.

Then $D_{x_0-}^{\beta} f(x)$ is continuous in x_0.

We need

Proposition 9: *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\,dt, \text{ for } x \geq x_0, \tag{12}$$

and $L(x, x_0) = 0$, *for* $x < x_0$.

Then L *is jointly continuous in* (x, x_0) *on* $[a, b]^2$.

We mention

Proposition 10: *([8]) Let* $g \in C([a, b])$, $0 < c < 1$, $x, x_0 \in [a, b]$. *Define*

$$K(x, x_0) = \int_{x_0}^{x} (\zeta - x)^{c-1} g(\zeta) d\zeta, \text{ for } x \leq x_0, \tag{13}$$

and $K(x, x_0) = 0$, *for* $x > x_0$.

Then $K(x, x_0)$ *is jointly continuous from* $[a, b]^2$ *into* $\mathbb{R}$.

Based on Propositions 9, 10 we derive

Corollary 1: *([8]) Let* $f \in C^m([a, b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a, b]$. *Then* $D^{\beta}_{*x_0} f(x)$, $D^{\beta}_{x_0 -} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $[a, b]^2$ *into* $\mathbb{R}$.

We need

Theorem 8: *([8]) Let* $f : [a, b]^2 \to \mathbb{R}$ *be jointly continous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta)_{[x,b]}, \tag{14}$$

$\delta > 0$, $x \in [a, b]$.

Then G *is continuous in* $x \in [a, b]$.

Also it holds

Theorem 9: *([8]) Let* $f : [a, b]^2 \to \mathbb{R}$ *be jointly continous. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta)_{[a,x]}, \tag{15}$$

$x \in [a, b]$, *is continuous in* $x \in [a, b]$, $\delta > 0$.

So that for $f \in C^m([a, b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a, b]$, we have that $\omega_1\left(D^{\beta}_{*x} f, h\right)_{[x,b]}$, $\omega_1\left(D^{\beta}_{x-} f, h\right)_{[a,x]}$ are continuous functions in $x \in [a, b]$, $h > 0$ is fixed.

We make

Remark 3: *([8]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then we have*

$$|D_{*a}^{\nu} f(x)| \leq \frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)}(x-a)^{n-\nu}, \ \forall\, x \in [a,b]. \tag{16}$$

Thus we observe

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y)| \tag{17}$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)}(x-a)^{n-\nu} + \frac{\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)}(y-a)^{n-\nu} \right)$$

$$\leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \tag{18}$$

Consequently

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) \leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \tag{19}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\beta} f, \delta\right) \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\beta+1)}(b-a)^{m-\beta}. \tag{20}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\beta} f, \delta\right)_{[x_0,b]} \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\beta+1)}(b-a)^{m-\beta}, \tag{21}$$

and

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{x_0-}^{\beta} f, \delta\right)_{[a,x_0]} \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\beta+1)}(b-a)^{m-\beta}. \tag{22}$$

By Proposition 15.114, p. 388 of [7], we get here that $D_{*x_0}^{\beta} f \in C([x_0,b])$, and by [12] we obtain that $D_{x_0-}^{\beta} f \in C([a,x_0])$.

We need

Definition 10: *([11]) Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous on $[a,b] \subset \mathbb{R}$), $\nu > 0$.*
We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^\nu f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)dt, \quad x \in [a,b], \tag{23}$$

$J_a^0 f := f.$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^\nu f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)dt, \quad x \in [a,b], \tag{24}$$

$I_{b-}^0 f := f.$

We mention

Definition 11: *([11]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall\ \epsilon > 0, \exists\ \delta > 0$ for every finite, pairwise disjoint, family*

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \ \ \text{with} \ \ \sum_{k=1}^n (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \epsilon. \tag{25}$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.
If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 11: *([11]) $f \in AC_{\mathcal{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b]), \forall\ r \in [0,1]$ (absolutely equicontinuous).*

We need

Definition 12: *([11]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.*
Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu \mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)dt \tag{26}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)dt \right) | 0 \le r \le 1 \right\} =$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)dt \right) | 0 \le r \le 1 \right\}. \tag{27}$$

So, we get

$$\left[D_{*a}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)dt \right) \right], \quad 0 \le r \le 1. \tag{28}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [7], [24].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{29}$$

$\forall\, x \in [a,b]$, *in short*

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1]. \tag{30}$$

We need

Lemma 3: *([11]) $D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 13: *([11]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.*

Let $f \in C^n_{\mathcal{F}}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$D^{\nu\mathcal{F}}_{b-} f(x) := \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)dt$$

$$= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)dt \right) | 0 \le r \le 1 \right\} \tag{31}$$

$$= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)dt \right) | 0 \le r \le 1 \right\}.$$

We get

$$\left[D^{\nu\mathcal{F}}_{b-} f(x)\right]^r = \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)dt \right) \right], \quad 0 \le r \le 1.$$

That is

$$\left(D^{\nu\mathcal{F}}_{b-} f(x)\right)_\pm^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_\pm^{(r)}\right)^{(n)}(t)dt = \left(D^\nu_{b-}\left(f_\pm^{(r)}\right)\right)(x),$$

see [6].

I.e. we get that

$$\left(D^{\nu\mathcal{F}}_{b-} f(x)\right)_\pm^{(r)} = \left(D^\nu_{b-}\left(f_\pm^{(r)}\right)\right)(x), \tag{32}$$

$\forall\, x \in [a,b]$, *in short*

$$\left(D^{\nu\mathcal{F}}_{b-} f\right)_\pm^{(r)} = D^\nu_{b-}\left(f_\pm^{(r)}\right), \;\; \forall\, r \in [0,1]. \tag{33}$$

Clearly,

$$D^\nu_{b-}\left(f_-^{(r)}\right) \le D^\nu_{b-}\left(f_+^{(r)}\right), \;\; \forall\, r \in [0,1].$$

We need

Lemma 4: *([11]) $D_{b-}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a, b]$.*

3 Foundations of Multicomposite Neural Networks

We need

Definition 14: *([22], p. 530) The general sigmoid activation functions we deal with here are described as follows $h : \mathbb{R} \to [-1, 1]$, strictly increasing $h(0) = 0$, $h(-x) = -h(x)$, $x \in \mathbb{R}$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also h is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.*

More precisely we will deal with the following sigmoid activation functions which possess all of the above properties: the algebraic activation function,

$$g_1(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}}, \ m \in \mathbb{N}, x \in \mathbb{R}, \tag{34}$$

(see [21], pp. 2-3);

the normalized and parametrized arctangent

$$g_2(x) = \frac{2}{\pi} \arctan\left(\frac{x}{2}\lambda x\right) = \frac{2}{\pi} \int_0^{\frac{\pi\lambda x}{2}} \frac{dz}{1 + z^2}, \quad x \in \mathbb{R}, \lambda > 0, \tag{35}$$

(see [22], p.p. 156-157);

the normalized and parametrized Gudermannian function

$$g_3(x) = \frac{2}{\pi} gd(\lambda x) = \frac{4}{\pi} \arctan\left(\tanh\left(\frac{\lambda x}{2}\right)\right)$$

$$= \frac{2}{\pi} \int_0^{\lambda x} \frac{dt}{\cosh t} = \frac{4}{\pi} \int_0^{\lambda x} \frac{dt}{e^t + e^{-t}}, \quad x \in \mathbb{R}, \lambda > 0, \tag{36}$$

where

$$gd(x) := \int_0^x \frac{dt}{\cosh t} = 2 \arctan\left(\tanh\left(\frac{x}{2}\right)\right), \ \ \forall\, x \in \mathbb{R},$$

is the Gudermannian function (see [22], p. 198-199);

the parametrized error function

$$g_4(x) = erf \lambda x = \frac{2}{\sqrt{\pi}} \int_0^{\lambda x} e^{-t^2} dt, \quad x \in \mathbb{R}, \lambda > 0, \tag{37}$$

(see [22], pp. 226-227);

the parametrized hyperbolic tangent function

$$g_5(x) = \tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}} = \frac{e^{2\lambda x} - 1}{e^{2\lambda x} + 1}$$

$$= \frac{1 - e^{-2\lambda x}}{1 + e^{-2\lambda x}}, \quad x > 0, \lambda > 0, \tag{38}$$

(see [22], p. 72);

for small $0 < \lambda < 1$, $\tanh \lambda x$ is expected to behave better than $ReLu$ and $Leaky\ ReLu$ activation functions;

the parametrized hyperbolic tangent like activation function,

$$g_6(x) = \frac{A^{\lambda x} - A^{-\lambda x}}{A^{\lambda x} + A^{-\lambda x}}, \quad A > 1, \lambda > 0, x \in \mathbb{R} \tag{39}$$

(see [22], pp. 262-263);

finally, the β-parametrized half hyperbolic tangent function

$$g_7(x) = \frac{1 - e^{-\beta x}}{1 + e^{-\beta x}}, \quad \beta > 0, \forall\, x \in \mathbb{R}, \tag{40}$$

(see [22], p. 509-510).

So, we denote them by $g_i(x)$, $x \in \mathbb{R}$, $i = 1, 2, ..., 7$.

Here we deal with the specific sequences of activation functions $\left(h_i^{(q)}\right)$, $q \in \mathbb{N}$, $i \in \mathbb{N}$, such that all $h_i^{(q)} \in G := \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$. That is $h_i^{(q)}$ for different $i \in \mathbb{N}$ could be identical, $q \in \mathbb{N}$.

Therefore in the sequences of activation functions $h_1^{(q)}, h_2^{(q)}, ...$, we allow random repetitions. Thus any two random $h_i^{(q)}$ activation functions are not necessarily different, but all belong to G. Call $H^{(q)} = \left\{ h_1^{(q)}, h_2^{(q)}, ... \right\}$, $q \in \mathbb{N}$, and by $\Omega := \left\{ H^{(q)},\ \text{all } q \in \mathbb{N} \right\}$. For simplicity next we can consider such a sequence $\{h_1, h_2, ...\}$.

Notice here $0 < h_i(1) \leq 1$, $i = 1, 2, ...$ Any composition $h_1 \circ h_2 \circ h_3 \circ ... \circ h_\lambda$ is meant to be $h_1|_{[-1,1]} \circ h_2|_{[-1,1]} \circ h_3|_{[-1,1]} \circ ... \circ h_{\lambda-1}|_{[-1,1]} \circ h_\lambda$, $\lambda \in \mathbb{N}$, and for convenience, we denote it by $G_\lambda := h_1 \circ h_2 \circ h_3 \circ ... \circ h_\lambda$. We have for any $\lambda \in \mathbb{N} : 0 < h_\lambda(1) \leq 1$, hence $0 < h_{\lambda-1}(h_\lambda(1)) \leq h_{\lambda-1}(1) \leq 1$, and $0 < h_{\lambda-2}(h_{\lambda-1}(h_\lambda(1))) \leq h_{\lambda-2}(h_{\lambda-1}(1)) \leq h_{\lambda-2}(1) \leq 1$.

Inductively we derive that $0 < G_\lambda(1) \leq 1$, $\forall\, \lambda \in \mathbb{N}$.

Clearly, it is $G_\lambda(0) = 0$ and G_λ is strictly increasing over $\mathbb{R}$. Furthermore it holds

$$G_\lambda(-x) = h_1(h_2(h_3(...(h_{\lambda-1}(h_\lambda(-x)))))) = h_1(h_2(h_3(...(h_{\lambda-1}(-h_\lambda(x))))))$$

$$= ... = -h_1(h_2(h_3(...(h_{\lambda-1}(h_\lambda(x)))))) = -G_\lambda(x), \quad x \in \mathbb{R}.$$

Clearly it holds $G_{\lambda}^{(2)} \in C(\mathbb{R})$.

We notice that

$$G_{\lambda}(+\infty) = h_1(h_2(h_3(...(h_{\lambda-1}(h_{\lambda}(+\infty)))))) =$$

$$h_1(h_2(h_3(...(h_{\lambda-1}(1))))) = G_{\lambda-1}(1), \tag{41}$$

and

$$G_{\lambda}(-\infty) = h_1(h_2(h_3(...(h_{\lambda-1}(h_{\lambda}(-\infty)))))) = h_1(h_2(h_3(...(h_{\lambda-1}(-1)))))$$

$$= -h_1(h_2(h_3(...(h_{\lambda-1}(1))))) = -G_{\lambda-1}(1). \tag{42}$$

Consequently, it holds

$$-G_{\lambda-1}(1) \leq G_{\lambda}(x) \leq G_{\lambda-1}(1), \quad \forall\, x \in \mathbb{R}. \tag{43}$$

Thus, $y = \pm G_{\lambda-1}(1)$ are asymptotes of $G_{\lambda}(x)$, any $\lambda \in \mathbb{N}$.

Next we act over $(-\infty, 0]$: let $\lambda, \mu \geq 0$: $\lambda + \mu = 1$. Then by convexity of h_2 there we have

$$h_2(\lambda x + \mu y) \leq \lambda h_2(x) + \mu h_2(y), \;\forall\, x, y \in (-\infty, 0];$$

and

$$h_1(h_2(\lambda x + \mu y)) \leq h_1(\lambda h_2(x) + \mu h_2(y)) \leq$$

$$\lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x, \;y \in (-\infty, 0]. \tag{44.}$$

So that $h_1 \circ h_2$ is convex over $(-\infty, 0]$.

Now we work on $[0, +\infty)$: let $\lambda, \mu \geq 0$: $\lambda + \mu = 1$. Then by concavity of h_2 there we have

$$h_2(\lambda x + \mu y) \geq \lambda h_2(x) + \mu h_2(y), \;\forall\, x, y \in [0, +\infty);$$

and

$$h_1(h_2(\lambda x + \mu y)) \geq h_1(\lambda h_2(x) + \mu h_2(y)) \geq$$

$$\lambda(h_1 \circ h_2)(x) + \mu(h_1 \circ h_2)(y), \quad \forall x, \ y \in [0, +\infty). \tag{45}$$

Thus, $h_1 \circ h_2$ is concave over $[0, +\infty)$.

Therefore $G_2 = h_1 \circ h_2$ is a general sigmoid activation function with asymptotes $y = \pm h_1(1)$, and fulfilling the rest of conditions of Definition 14.

Arguing as above $h_2 \circ h_3 : \mathbb{R} \to [-1, 1]$, fulfills Definition 14, and $h_1 \circ h_2 \circ h_3$ does the same with asymptotes $h = \pm h_1(h_2(1))$.

Inductively, we prove that G_λ fulfills Definition 14 with asymptotes $y = \pm G_{\lambda-1}(1)$.

We have established the following:

Theorem 10: *Let $q, \lambda \in \mathbb{N}$. Then $G_\lambda^{(q)} := h_1^{(q)} \circ h_2^{(q)} \circ h_3^{(q)} \circ ... \circ h_\lambda^{(q)}$ fulfill all the properties of Definition 14 with asymptotes $y = \pm G_{\lambda-1}^{(q)}(1)$. That is $G_\lambda^{(q)}$ are multi-composite general sigmoid activation functions from $\mathbb{R} \to [-1, 1]$.*

Corollary 2: *$\frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}$ fulfill Definition 14 with asymptotes $y = \pm 1$.*

We call

$$\widetilde{G}_\lambda^{(q)} := \frac{G_\lambda^{(q)}}{G_{\lambda-1}^{(q)}(1)}. \tag{46}$$

Remark 4: *Next we consider the functions*

$$T_\lambda^{(q)}(x) := \frac{1}{4}\Big(\widetilde{G}_\lambda^{(q)}(x+1) - \widetilde{G}_\lambda^{(q)}(x-1)\Big) > 0, \ \forall\, x \in \mathbb{R}, \ q, \lambda \in \mathbb{N}. \tag{47}$$

We observe that

$$T_\lambda^{(q)}(-x) = \frac{1}{4}\Big(\widetilde{G}_\lambda^{(q)}(-x+1) - \widetilde{G}_\lambda^{(q)}(-x-1)\Big) =$$

$$\frac{1}{4}\Big(\widetilde{G}_\lambda^{(q)}(-(x-1)) - \widetilde{G}_\lambda^{(q)}(-(x+1))\Big) = \frac{1}{4}\Big(-\widetilde{G}_\lambda^{(q)}(x-1) + \widetilde{G}_\lambda^{(q)}(x+1)\Big) = \tag{48}$$

$$\frac{1}{4}\Big(\widetilde{G}_\lambda^{(q)}(x+1) - \widetilde{G}_\lambda^{(q)}(x-1)\Big) = T_\lambda^{(q)}(x).$$

That is $T_\lambda^{(q)}$ are even functions,

$$T_\lambda^{(q)}(-x) = T_\lambda^{(q)}(x), \ \ \forall\, x \in \mathbb{R}, \ q, \lambda \in \mathbb{N}. \tag{49}$$

We see that

$$T_{\lambda}^{(q)}(0) = \frac{\widetilde{G}_{\lambda}^{(q)}(1)}{2}. \tag{50}$$

Let $x > 1$, we have that

$$T_{\lambda}^{(q)\prime}(x) = \frac{1}{4}\Big(\widetilde{G}_{\lambda}^{(q)\prime}(x+1) - \widetilde{G}_{\lambda}^{(q)\prime}(x-1) \Big) < 0,$$

by $\widetilde{G}_{\lambda}^{(q)\prime}$ being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1-x > 0$ and $0 < 1-x < 1+x$. It holds $\widetilde{G}_{\lambda}^{(q)\prime}(x-1) = \widetilde{G}_{\lambda}^{(q)\prime}(1-x) > \widetilde{G}_{\lambda}^{(q)\prime}(x+1)$, so that again $T_{\lambda}^{(q)\prime}(x) < 0$. Consequently $T_{\lambda}^{(q)}$ is strictly decreasing on $(0, +\infty)$.

Clearly, $T_{\lambda}^{(q)}$ is strictly increasing on $(-\infty, 0)$, and $T_{\lambda}^{(q)\prime}(0) = 0$.

Observe that

$$\lim_{x \to +\infty} T_{\lambda}^{(q)}(x) = \frac{1}{4}\Big(\widetilde{G}_{\lambda}^{(q)}(+\infty) - \widetilde{G}_{\lambda}^{(q)}(+\infty) \Big) = 0, \tag{51}$$

and

$$\lim_{x \to -\infty} T_{\lambda}^{(q)}(x) = \frac{1}{4}\Big(\widetilde{G}_{\lambda}^{(q)}(-\infty) - \widetilde{G}_{\lambda}^{(q)}(-\infty) \Big) = 0, \quad q, \lambda \in \mathbb{N}. \tag{52}$$

That is the x-axis is the horizontal asymptote on $T_{\lambda}^{(q)}$.

As a result $T_{\lambda}^{(q)}$ is a bell shaped symmetric function with maximum

$$T_{\lambda}^{(q)}(0) = \frac{\widetilde{G}_{\lambda}^{(q)}(1)}{2}. \tag{53}$$

We need

Theorem 11: *It holds*

$$\sum_{i=-\infty}^{\infty} T_{\lambda}^{(q)}(x - i) = 1, \ \forall\, x \in \mathbb{R}; \quad q, \lambda \in \mathbb{N}. \tag{54}$$

Proof: As similar to [22], p. 286 is omitted. □

Theorem 12: *We have that*

$$\int_{-\infty}^{\infty} T_{\lambda}^{(q)}(x)dx = 1, \quad q, \lambda \in \mathbb{N}. \tag{55}$$

Proof: As similar to [22], p. 287, it is omitted. □

So that $T_{\lambda}^{(q)}(x)$ can serve as a density function in general.

We need

Theorem 13: *Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. Then*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k|\geq n^{1-\alpha} \end{cases}}^{\infty} T_{\lambda}^{(q)}(nx-k) < \left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right), \tag{56}$$

and

$$\lim_{n\to+\infty}\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)=0. \tag{57}$$

Proof: As in [22], pp. 531-532, it is omitted. □

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We also need

Theorem 14: *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} T_{\lambda}^{(q)}(nx-k)} < \frac{1}{T_{\lambda}^{(q)}(1)}, \quad \forall\, x\in[a,b]; \quad q,\lambda\in\mathbb{N}. \tag{58}$$

Proof: As similar to [22], p. 289 is omitted. □

Remark 5: *We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} T_{\lambda}^{(q)}(nx-k)\neq 1; \quad q,\lambda\in\mathbb{N}, \tag{59}$$

for at least some $x\in[a,b]$.

See [22], p. 290, same reasoning.

Note 15: *For large enough n we always obtain $\lceil na\rceil\leq\lfloor nb\rfloor$. Also $a\leq\frac{k}{n}\leq b$, iff $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. In general it holds (by (21))*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} T_{\lambda}^{(q)}(nx-k)\leq 1. \tag{60}$$

4 Deterministic Multicomposite Neural Network Approximation

We need to define.

Definition 15: *Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the multicomposite linear neural network operator*

$$ {}_{\lambda}^{(q)}A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) T_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} T_{\lambda}^{(q)}(nx-k)}, \quad x \in [a,b],\ q,\lambda \in \mathbb{N}. \tag{61} $$

Notice that ${}_{\lambda}^{(q)}A_n(1) = 1$, *$q, \lambda \in \mathbb{N}$.*

The first modulus of continuity is defined by

$$ \omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{62} $$

In this work $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$; where $\|\cdot\|_{\infty}$ is the supremum norm.

Clearly here ${}_{\lambda}^{(q)}A_n(f,x) \in C([a,b])$. We present results for the pointwise and uniform convergence of ${}_{\lambda}^{(q)}A_n(f,x)$ to $f(x)$ with rates.

We first give

Theorem 16: *(see also [22, Ch. 21]) Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$; $q, \lambda \in \mathbb{N}$. Then*

i)

$$ \left| {}_{\lambda}^{(q)}A_n(f,x) - f(x) \right| \leq \frac{1}{T_{\lambda}^{(q)}(1)} \left[\omega_1\left(f, \frac{1}{n^{\alpha}}\right) + 2\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right) \|f\|_{\infty} \right] =: \rho_{\lambda_n}^{(q)}, \tag{63} $$

and

ii)

$$ \left\| {}_{\lambda}^{(q)}A_n(f) - f \right\|_{\infty} \leq \rho_{\lambda_n}^{(q)}. \tag{64} $$

We notice $\lim_{n\to\infty} {}_{\lambda}^{(q)}A_n(f) = f$, *pointwise and uniformly.*

The speed of convergence is $\max\left(\frac{1}{n^{\alpha}}, \left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)\right)$.

In the next we discuss high order neural network approximation by using the smoothness of f.

Theorem 17: *(see also [22, Ch. 21]) Let $f \in C^N([a,b])$, $q, \lambda, n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then* *i)*

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right|\leq\frac{1}{T_{\lambda}^{(q)}(1)}\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left[\frac{1}{n^{\alpha j}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)(b-a)^j\right]+\right. \tag{65}$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!}\right]\right\}$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$, for some $x_0\in[a,b]$, it holds

$$\left|{}_{\lambda}^{(q)}A_n(f,x_0)-f(x_0)\right|\leq\frac{1}{T_{\lambda}^{(q)}(1)}$$

$$\left\{\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!}\right\}, \tag{66}$$

and

iii)

$$\left\|{}_{\lambda}^{(q)}A_n(f)-f\right\|_{\infty}\leq\frac{1}{T_{\lambda}^{(q)}(1)}\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\alpha j}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)(b-a)^j\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!}\right]\right\}. \tag{67}$$

Again we obtain $\lim_{n\to\infty}{}_{\lambda}^{(q)}A_n(f)=f$, pointwise and uniformly.

We present the following fractional approximation result by multicomposite neural networks.

Theorem 18: *(see also [22, Ch. 21]) Let $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C^N([a,b])$, $0<\beta<1$, $x\in[a,b]$, $\lambda,q,n\in\mathbb{N}:\ n^{1-\beta}>2$. Then*

i)

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\,{}_{\lambda}^{(q)}A_n\left((\cdot-x)^j\right)(x)-f(x)\right|\leq$$

$$\frac{\left(T_{\lambda}^{(q)}(1)\right)^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\left(\|D_{x-}^{\alpha}f\|_{\infty,[a,x]}(x-a)^{\alpha}+\|D_{*x}^{\alpha}f\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{68}$$

ii) if $f^{(j)}(x)=0$, *for* $j=1,\ldots,N-1$, *we have*

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right|\leq\frac{\left(T_{\lambda}^{(q)}(1)\right)^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\left(\|D_{x-}^{\alpha}f\|_{\infty,[a,x]}(x-a)^{\alpha}+\|D_{*x}^{\alpha}f\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{69}$$

iii)

$$\left|{}_{\lambda}^{(q)}A_n(f,x)-f(x)\right|\leq\left(T_{\lambda}^{(q)}(1)\right)^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\|f^{(j)}(x)\|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^{j}\left(1-\widetilde{G}_{\lambda}^{(q)}h\left(n^{1-\beta}-2\right)\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\left(\|D_{x-}^{\alpha}f\|_{\infty,[a,x]}(x-a)^{\alpha}+\|D_{*x}^{\alpha}f\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\}, \tag{70}$$

$\forall\, x\in[a,b]$,

and

iv)

$$\left\|{}_{\lambda}^{(q)}A_n f-f\right\|_{\infty}\leq\left(T_{\lambda}^{(q)}(1)\right)^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\|f^{(j)}\|_\infty}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^\alpha f,\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^\alpha f,\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)(b-a)^\alpha\left(\sup_{x\in[a,b]}\|D_{x-}^\alpha f\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\|D_{*x}^\alpha f\|_{\infty,[x,b]}\right)\right\}\right\}. \tag{71}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally pointwise and uniform convergence with rates of ${}_\lambda^{(q)}A_n\to I$, the unit operator, as $n\to\infty$.

5 Main Results: Approximation by general Fuzzy Multicomposite Neural Network Operators

Let $f\in C_{\mathcal{F}}([a,b])$ (fuzzy continuous functions on $[a,b]\subset\mathbb{R}$), $q,\lambda,n\in\mathbb{N}$. We define the following Fuzzy Multicomposite Quasi-Interpolation Neural Network operator

$${}_\lambda^{(q)}A_n^{\mathcal{F}}(f,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{k}{n}\right)\odot\frac{T_\lambda^{(q)}(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}T_\lambda^{(q)}(nx-k)}, \tag{72}$$

$\forall\ x\in[a,b]$, see also (61).

The fuzzy sum in (72) is finite.

Let $r\in[0,1]$, we observe that

$$\left[{}_\lambda^{(q)}A_n^{\mathcal{F}}(f,x)\right]^r=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f\left(\frac{k}{n}\right)\right]^r\left(\frac{T_\lambda^{(q)}(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}T_\lambda^{(q)}(nx-k)}\right)=$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left[f_-^{(r)}\left(\frac{k}{n}\right),f_+^{(r)}\left(\frac{k}{n}\right)\right]\left(\frac{T_\lambda^{(q)}(nx-k)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}T_\lambda^{(q)}(nx-k)}\right)=$$

$$\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f_{-}^{(r)}\left(\frac{k}{n}\right)\left(\frac{T_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} T_{\lambda}^{(q)}(nx-k)}\right), \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f_{+}^{(r)}\left(\frac{k}{n}\right)\left(\frac{T_{\lambda}^{(q)}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} T_{\lambda}^{(q)}(nx-k)}\right)\right] \tag{73}$$

$$= \left[{}_{\lambda}^{(r)}A_n\left(f_{-}^{(q)}, x\right),\ {}_{\lambda}^{(r)}A_n\left(f_{+}^{(q)}, x\right)\right].$$

We have proved that

$$\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f, x)\right)_{\pm}^{(r)} = {}_{\lambda}^{(q)}A_n\left(f_{\pm}^{(r)}, x\right), \tag{74}$$

respectively, $\forall\ r \in [0, 1]$, $\forall\ x \in [a, b]$, all $q, \lambda \in \mathbb{N}$.

Therefore we get

$$D\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f, x), f(x)\right) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|{}_{\lambda}^{(q)}A_n\left(f_{-}^{(r)}, x\right) - f_{-}^{(r)}(x)\right|, \left|{}_{\lambda}^{(q)}A_n\left(f_{+}^{(r)}, x\right) - f_{+}^{(r)}(x)\right|\right\}, \tag{75}$$

$\forall\ x \in [a, b]$; $q, \lambda \in \mathbb{N}$.

We present our first fuzzy multicomposite neural network approximation result.

Theorem 19: *Let $f \in C_{\mathcal{F}}([a, b])$, $0 < \alpha < 1$, $x \in [a, b]$, $q, \lambda, n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

1)

$$D\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f, x), f(x)\right) \leq \frac{1}{T_{\lambda}^{(q)}(1)}\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^{\alpha}}\right) + 2\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)D^{*}(f, \widetilde{o})\right] =: K_{\lambda_n}^{(q)}, \tag{76}$$

and

2)

$$D^{*}\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f), f\right) \leq K_{\lambda_n}^{(q)}. \tag{77}$$

We notice that $\lim_{n\to\infty}\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\lim_{n\to\infty} {}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f) \overset{D^{*}}{\to} f$, *fuzzy pointwise and uniformly.*

Proof: We have that $f_{\pm}^{(r)} \in C([a,b])$, $\forall\, r \in [0,1]$. Hence by (63), we obtain

$$\left| {}_{\lambda}^{(q)}A_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \frac{1}{T_{\lambda}^{(q)}(1)}\left[\omega_1\left(f_{\pm}^{(r)}, \frac{1}{n^{\alpha}}\right) + 2\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)\left\|f_{\pm}^{(r)}\right\|_{\infty}\right] \tag{78}$$

(by Proposition 4 and $\left\|f_{\pm}^{(r)}\right\|_{\infty} \leq D^*(f, \widetilde{o})$)

$$\leq \frac{1}{T_{\lambda}^{(q)}(1)}\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^{\alpha}}\right) + 2\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)D^*(f, \widetilde{o})\right]. \tag{79}$$

Taking into account (75) the theorem is proved. □

We also give the fuzzy differentiation result.

Theorem 20: *Let $f \in C_{\mathcal{F}}^{N}([a,b])$, $N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$, $q, \lambda, n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

1)

$$D\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f, x), f(x)\right) \leq \frac{1}{T_{\lambda}^{(q)}(1)}$$

$$\left\{\sum_{j_*=1}^{N} \frac{D\left(f^{(j_*)}(x), \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + \left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)(b-a)^{j_*}\right] + \right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N} N!} + 2\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{80}$$

2) assume further that $f^{(j_*)}(x_0) = \widetilde{o}$, $j_* = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$D\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f, x_0), f(x_0)\right) \leq$$

$$\frac{1}{T_{\lambda}^{(q)}(1)}\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N} N!} + 2\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha} - 2\right)\right)D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right], \tag{81}$$

notice here the extremely high rate of convergence $n^{-(N+1)\alpha}$,

3)

$$D^*\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f), f\right) \leq \frac{1}{T_{\lambda}^{(q)}(1)}$$

$$\left\{\sum_{j_*=1}^{N}\frac{D^*\left(f^{(j_*)},\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)D^*\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}. \tag{82}$$

Proof: Since $f\in C_{\mathcal{F}}^{N}([a,b])$, $N\geq 1$, we have that $f_{\pm}^{(r)}\in C^{N}([a,b])$, $\forall\ r\in[0,1]$. Using (65), we get

$$\left|{}_{\lambda}^{(q)}A_n\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\frac{1}{T_{\lambda}^{(q)}(1)} \tag{83}$$

$$\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1\left(\left(f_{\pm}^{(r)}\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\left\|\left(f_{\pm}^{(r)}\right)^{(N)}\right\|_{\infty}\frac{(b-a)^N}{N!}\right]\right\} \tag{84}$$

(by Remark 2)

$$=\frac{1}{T_{\lambda}^{(q)}(1)}\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f^{(j_*)}\right)_{\pm}^{(r)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1\left(\left(f^{(N)}\right)_{\pm}^{(r)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}\frac{(b-a)^N}{N!}\right]\right\}\leq$$

$$\frac{1}{T_{\lambda}^{(q)}(1)}\left\{\sum_{j_*=1}^{N}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)(b-a)^{j_*}\right]+\right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+2\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\alpha}-2\right)\right)D^*\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{85}$$

by Proposition 4, $\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}\leq D^*\left(f^{(N)},\widetilde{o}\right)$ and apply (75).

The theorem is proved. □

Next we present the fuzzy fractional differentiation result.

Theorem 21: *Let $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C_{\mathcal{F}}^{N}([a,b])$, $0<\beta<1$, $x\in[a,b]$, $q,\lambda,n\in\mathbb{N}$, $n^{1-\beta}>2$. Then* *i)*

$$D\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f,x), f(x)\right) \le \frac{1}{T_{\lambda}^{(q)}(1)}$$

$$\left\{\sum_{j_*=1}^{N-1} \frac{D\left(f^{(j_*)}(x), \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\beta j_*}} + \left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta} - 2\right)\right)(b-a)^{j_*}\right] + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}} + \right. \tag{86}$$

$$\left.\left.\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta} - 2\right)\right)\left[D^*\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]}(x-a)^{\alpha} + D^*\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\right\}\right\},$$

ii) if $f^{(j_*)}(x_0) = 0$, $j_* = 1, ..., N-1$, *for some* $x_0 \in [a,b]$, *we have*

$$D\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f,x_0), f(x_0)\right) \le$$

$$\frac{\left(T_{\lambda}^{(q)}(1)\right)^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x_0-}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[a,x_0]} + \omega_1^{(\mathcal{F})}\left(\left(D_{*x_0}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[x_0,b]}\right]}{n^{\alpha\beta}} + \right. \tag{87}$$

$$\left.\left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta} - 2\right)\right)\left[D^*\left(\left(D_{x_0-}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x_0]}(x_0-a)^{\alpha} + D^*\left(\left(D_{*x_0}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[x_0,b]}(b-x_0)^{\alpha}\right]\right\},$$

when $\alpha > 1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$,

and

iii)

$$D^*\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f), f\right) \le$$

$$\frac{1}{T_{\lambda}^{(q)}(1)}\left\{\sum_{j_*=1}^{N-1} \frac{D^*\left(f^{(j_*)}, \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\beta j_*}} + \left(1 - \widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta} - 2\right)\right)(b-a)^{j_*}\right] + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}} + \right.$$

$$\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)(b-a)^{\alpha}\left[\sup_{x\in[a,b]} D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]} D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\Bigg\}\Bigg\}, \tag{88}$$

above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally the fuzzy pointwise and uniform convergence with rates of ${}_{\lambda}^{(q)}A_{n}^{\mathcal{F}}\to I$ *the unit operator, as* $n\to\infty$.

Proof: Here $f_{\pm}^{(r)}\in C^{N}([a,b])$, $\forall\, r\in[0,1]$, and $D_{x-}^{\alpha\mathcal{F}}f$, $D_{*x}^{\alpha\mathcal{F}}f$ are fuzzy continuous over $[a,b]$, $\forall\, x\in[a,b]$, so that $\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\in C([a,b])$, $\forall\, r\in[0,1]$, $\forall\, x\in[a,b]$.

We observe by (68), $\forall\, x\in[a,b]$, that (respectively in $\pm$)

$$\left|{}_{\lambda}^{(q)}A_{n}\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\frac{1}{T_{\lambda}^{(q)}(1)}$$

$$\left\{\sum_{j_{*}=1}^{N-1}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_{*})}(x)\right|}{j_{*}!}\left\{\frac{1}{n^{\beta j_{*}}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)(b-a)^{j_{*}}\right\}+\right. \tag{89}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_{1}\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\left(\left\|D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}=$$

(by Remark 2, (30), (33))

$$\frac{1}{T_{\lambda}^{(q)}(1)}\left\{\sum_{j_{*}=1}^{N-1}\frac{\left|\left(f^{(j_{*})}(x)\right)_{\pm}^{(r)}\right|}{j_{*}!}\left\{\frac{1}{n^{\beta j_{*}}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)(b-a)^{j_{*}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_{1}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_{1}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right. \tag{90}$$

$$\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\left(\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}\leq$$

$$\frac{1}{T_{\lambda}^{(r)}(1)}\left\{\sum_{j_*=1}^{N-1}\frac{D\left(f^{(j_*)}(x),\widetilde{o}\right)}{j_*!}\left\{\frac{1}{n^{\beta j_*}}+\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)(b-a)^{j_*}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{91}$$

$$\left(1-\widetilde{G}_{\lambda}^{(q)}\left(n^{1-\beta}-2\right)\right)\left[D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}(x-a)^{\alpha}+D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\Bigg\}\Bigg\},$$

along with (75) proving all inequalities of theorem.

Here we notice that

$$\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{(-1)^N}{\Gamma(N-\alpha)}\int_t^x(s-t)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)ds,$$

where $a\leq t\leq x$.

Hence

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma(N-\alpha)}\int_t^x(s-t)^{N-\alpha-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds$$

$$\leq\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

So we have

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

all $a\leq t\leq x$.

And it holds

$$\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{92}$$

that is

$$D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]} D^*\Big(\big(D_{x-}^{\alpha\mathcal{F}} f\big), \widetilde{o}\Big)_{[a,x]} \le \frac{D^*\big(f^{(N)}, \widetilde{o}\big)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha} < \infty. \tag{93}$$

Similarly we have

$$\Big(D_{*x}^{\alpha\mathcal{F}} f\Big)_{\pm}^{(r)}(t) = \Big(D_{*x}^{\alpha}\Big(f_{\pm}^{(r)}\Big)\Big)(t)$$

$$= \frac{1}{\Gamma(N-\alpha)}\int_x^t (t-s)^{N-\alpha-1}\Big(f_{\pm}^{(r)}\Big)^{(N)}(s)ds,$$

where $x \le t \le b$.

Hence

$$\left|\Big(D_{*x}^{\alpha\mathcal{F}} f\Big)_{\pm}^{(r)}(t)\right| \le \frac{1}{\Gamma(N-\alpha)}\int_x^t (t-s)^{N-\alpha-1}\left|\Big(f^{(N)}\Big)_{\pm}^{(r)}(s)\right|ds \le$$

$$\frac{\left\|\Big(f^{(N)}\Big)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha} \le \frac{D^*\big(f^{(N)}, \widetilde{o}\big)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

$x \le t \le b$.

So we have

$$\left\|\Big(D_{*x}^{\alpha\mathcal{F}} f\Big)_{\pm}^{(r)}\right\|_{\infty,[x,b]} \le \frac{D^*\big(f^{(N)}, \widetilde{o}\big)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{94}$$

that is

$$D^*\Big(\big(D_{*x}^{\alpha\mathcal{F}} f\big), \widetilde{o}\Big)_{[x,b]} \le \frac{D^*\big(f^{(N)}, \widetilde{o}\big)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]} D^*\Big(\big(D_{*x}^{\alpha\mathcal{F}} f\big), \widetilde{o}\Big)_{[x,b]} \le \frac{D^*\big(f^{(N)}, \widetilde{o}\big)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha} < +\infty. \tag{95}$$

Furthermore we notice

$$\omega_1^{(\mathcal{F})}\left(\big(D_{x-}^{\alpha\mathcal{F}} f\big), \frac{1}{n^\beta}\right)_{[a,x]} = \sup_{\substack{s,t\in[a,x]\\ |s-t|\le\frac{1}{n^\beta}}} D\Big(\big(D_{x-}^{\alpha\mathcal{F}} f\big)(s), \big(D_{x-}^{\alpha\mathcal{F}} f\big)(t)\Big) \le$$

$$\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^\beta}}}\left\{D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\widetilde{o}\right)+D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t),\widetilde{o}\right)\right\}\leq 2D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}$$

$$\leq\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

Therefore it holds

$$\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x]}\leq\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}<+\infty. \tag{96}$$

Similarly we observe

$$\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}=\sup_{\substack{s,t\in[x,b]\\|s-t|\leq\frac{1}{n^\beta}}}D\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right)(s),\left(D_{*x}^{\alpha\mathcal{F}}f\right)(t)\right)\leq$$

$$2D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

Consequently it holds

$$\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\leq\frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}<+\infty. \tag{97}$$

So everything in the statements of the theorem makes sense.

The proof of the theorem is now completed. □

Corollary 3: *(to Theorem 21, $N=1$ case) Let* $0<\alpha,\beta<1$, $f\in C_{\mathcal{F}}^1([a,b])$, $\lambda,q,n\in\mathbb{N}$, $n^{1-\beta}>2$. *Then*

$$D^*\left({}_{\lambda}^{(q)}A_n^{\mathcal{F}}(f),f\right)\leq$$

$$\frac{\left(T_\lambda^{(q)}(1)\right)^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right. \tag{98}$$

$$\left.\left(1-\widetilde{G}_\lambda^{(q)}\left(n^{1-\beta}-2\right)\right)(b-a)^\alpha\left[\sup_{x\in[a,b]}D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}.$$

Proof: By (88). □

Finally we specialize to $\alpha = \frac{1}{2}$.

Corollary 4: *(to Theorem 21) Let* $0 < \beta < 1$, $f \in C^1_{\mathcal{F}}([a,b])$, $\lambda, q, n \in \mathbb{N}$, $n^{1-\beta} > 2$. *Then*

$$D^*\left({}^{(r)}_{\lambda}A^{\mathcal{F}}_n(f), f\right) \leq$$

$$\frac{2\left(T^{(q)}_{\lambda}(1)\right)^{-1}}{\sqrt{\pi}}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D^{\frac{1}{2}\mathcal{F}}_{x-}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D^{\frac{1}{2}\mathcal{F}}_{*x}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}}+\right. \tag{99}$$

$$\left.\left(1-\widetilde{G}^{(q)}_{\lambda}\left(n^{1-\beta}-2\right)\right)\sqrt{b-a}\left[\sup_{x\in[a,b]}D^*\left(\left(D^{\frac{1}{2}\mathcal{F}}_{x-}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^*\left(\left(D^{\frac{1}{2}\mathcal{F}}_{*x}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}.$$

Proof: By (98). □

6 Conclusion

In this paper, we developed fuzzy multicomposite neural network operators generated by seven basic activation functions for the approximation of fuzzy-valued functions. Fuzzy pointwise and uniform approximation results were established through Jackson-type inequalities involving fuzzy moduli of continuity and fuzzy fractional derivatives. The obtained results show that the use of fuzzy fractional derivatives provides improved approximation rates compared with the purely continuous fuzzy case. Future work may extend these techniques to multivariate settings, deep neural network architectures, and other classes of fuzzy fractional operators.

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