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Research article

On Connected Secure Domination Polynomial of Some Graphs

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Abstract: A set $S \subseteq V(G)$ of the connected graph $G = (V(G), E(G))$ is said to be a connected secure dominating set if S is a dominating set, S is a secure set, and $\langle S \rangle_G$ is a connected graph. The connected secure domination polynomial of G is $D_s^c(G, x) = \sum_{i=\gamma_s^c(G)}^{n} d_s^c(G, i)x^i$ where $\gamma_s^c(G) = \min\{|S| : S \text{ is a connected secure dominating set of } G\}$ and $d_s^c(G, i)$ is the number of connected secure dominating sets with cardinality i. In this paper, we will determine the connected secure domination polynomial of path graph P_n, cycle graph C_n, complete graph K_n, star graph $K_{1,n}$, and corona graph $G \circ K_1$.

Mathematics Subject Classification: 05C30, 05C31, 05C76
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1 Introduction

All graphs considered in this paper are finite, simple, and connected. A graph $G = (V, E)$ has a *vertex set* $V(G)$ and an *edge set* $E = E(G)$, and the cardinalities $|V|$ and $|E|$ are called the *order* and *size* of G, respectively. If $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u : uv \in E(G)\}$, and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. For $S \subseteq V(G)$, the *open neighborhood* of S is $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is $N[S] = S \cup N(S)$. If $S \subseteq V(G)$, then the *induced subgraph* $\langle S \rangle_G$ is the graph obtained from G such that $V(\langle S \rangle_G) = S$ and $uv \in E(\langle S \rangle_G)$ if and only if $uv \in E(G)$. A vertex $v \in V(G)$ is called a *cut vertex* if $\langle V(G) - \{v\} \rangle_G$ has more components than G. In addition, for a set $S \subseteq V(G)$, S is called a *k-vertex cut* set if $|S| = k$, $\langle V(G) - S \rangle_G$ has more components than G, and $\langle V(G) - X \rangle_G$ is connected for every $X \subset S$. For more graph-theoretic notions, refer to [3].

The concept of secure sets was first introduced by Brigham et al. [2] in an attempt to develop a more realistic notion of alliances in graphs. For a set $S \subseteq V(G)$ with $S = \{s_1, s_2, \ldots, s_k\}$, the *attack* on S is defined as the collection of pairwise disjoint sets $A = \{A_1, A_2, \ldots, A_k\}$ such that $A_i \subseteq N[s_i] - S$ for $i = 1, 2, \ldots, k$. The *defense* of S is a collection of pairwise disjoint sets $D = \{D_1, D_2, \ldots, D_k\}$ for which $D_i \subseteq N[s_i] \cap S$. An attack

A is *defendable* if there exists a defense D such that $|D_i| \geq |A_i|$ for each $i = 1, 2, \dots, k$. The set S is *secure* if every attack on S is defendable.

Another important topic in graph theory is domination. The concept of dominating sets in graphs dates back to the 1950s and 1960s, originally studied in the context of network control and facility location problems. Since then, domination has become a central topic in graph theory. A set $S \subseteq V(G)$ is called a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. Equivalently, $N[S] = V(G)$. This notion of dominating sets has inspired several extensions, including vertex-edge dominating sets [7], pendant dominating sets [5], and connected total dominating set [4]. Secure dominating sets were first studied by Johnson and Jones in [6], combining the concepts of secure sets and dominating sets. A set $S \subseteq V(G)$ is a *secure dominating set* if it is both a dominating set and a secure set. If, in addition, the induced subgraph $\langle S \rangle_G$ is connected, then S is called a *connected secure dominating set.*

The concept of dominating sets was later extended to the domination polynomial, which was introduced by Alikhani and Peng [1]. The idea is to study the number of dominating sets of a graph with respect to their cardinalities in a single polynomial. This notion has led to several variants of the domination polynomial, including the connected domination polynomial [8].

In this paper, we extend the study of connected secure dominating sets by introducing a new variation of the domination polynomial, called the connected secure domination polynomial. For a simple connected graph G of order n, the *connected secure domination polynomial* is defined as

$$D_s^c(G, x) = \sum_{i=\gamma_s^c(G)}^{n} d_s^c(G, i) x^i,$$

where $\gamma_s^c(G) = \min\{|S| : S \text{ is a connected secure dominating set of } G\}$ is the connected secure domination number of G, and $d_s^c(G, i)$ denotes the number of connected secure dominating sets of cardinality i.

2 Preliminaries

Definition 1: A *path graph* of order n, denoted by P_n, is a graph with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.

Theorem 1 ([6])**:** Let P_n be a path graph of order n. Then $\gamma_s^c(P_n) = n - 2$ for $n \geq 4$.

Definition 2: A *cycle graph* of order n, denoted by C_n, is a graph with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_1v_n\}$.

Theorem 2 ([6])**:** Let C_n be a cycle graph of order n. Then $\gamma_s^c(C_n) = n - 2$ for $n \geq 4$.

Definition 3: A graph is said to be a *complete graph* if every two distinct vertices of the graph are adjacent. The complete graph of order n is denoted by K_n.

Definition 4: The *complete multipartite graph*, denoted by $K_{n_1,n_2,\ldots,n_k}$, is a graph whose vertex set can be partitioned into $k \geq 2$ subsets $V_1, V_2, \ldots, V_k$ (also called partite sets) with $|V_i| = n_i$ for $i = 1, 2, \ldots, k$ such that $uv \in E(K_{n_1,n_2,\ldots,n_k})$ if $u \in V_i$ and $v \in V_j$, where $1 \leq i, j \leq k$ and $i \neq j$. The graph $K_{1,n}$ is called a star graph.

Definition 5: Let G be a graph of order n and H be a graph. Then the *corona graph* $G \circ H$ is a graph obtained by taking one copy of G and n copies of H, and then joining ith vertex of G to every vertex of ith copy of H.

The number of connected dominating sets of G with cardinality i is denoted $d_c(G, i)$.

Theorem 3 ([8])**:** Let G be a connected graph with k cut vertices and r 2-vertex cut sets. Then

i. $d_c(G, n) = 1$;

ii. $d_c(G, n-1) = n - k$;

iii. $d_c(G, n-2) = \binom{n-k}{2} - r$.

Theorem 4 ([8])**:** Let G be a connected graph of order n. Then $d_c(G \circ K_1, i) = \binom{n}{i-n}$ for all i, $n \leq i \leq 2n$.

Theorem 5 ([2])**:** Let G be a graph with vertex set $V(G)$. Then the set $S \subseteq V(G)$ is secure if and only if $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$.

Theorem 6 ([6])**:** For any connected graph G of order n, $\gamma_s^c(G) \geq \frac{n}{2}$.

Theorem 7 ([6])**:** Let G be a graph of order n. Suppose that $S \subseteq V(G)$, S is a dominating set, each vertex in S is adjacent to at most one vertex in $V(G) - S$, and $|S| = \left\lceil \frac{n}{2} \right\rceil$. Then S is a secure dominating set.

Theorem 8 ([6])**:** Let $K_{n_1,n_2,\ldots,n_m}$ be a complete multipartite graph. Then $\gamma_s^c(K_{n_1,n_2,\ldots,n_m}) = \left\lceil \frac{n_1+n_2+\ldots+n_m}{2} \right\rceil$ for $m \geq 2$.

3 Results and Discussions

Remark 1: For any positive integer n, $\left\lceil \frac{n}{2} \right\rceil \geq n - \left\lceil \frac{n}{2} \right\rceil$.

Lemma 1: Let G be a connected graph of order n. Suppose that $S \subseteq V(G)$ is a dominating set with $|S| \geq \left\lceil \frac{n}{2} \right\rceil$. If $X \subseteq S$ is a dominating set of G, then $|N[X] \cap S| \geq |N[X] - S|$.

Proof: Since S and X are dominating sets, we have $N[S] = V(G)$ and $N[X] = V(G)$, and as a result $N[S] = N[X]$. Note that $N[S] \cap S = S$. So, $|N[X] \cap S| = |N[S] \cap S| = |S| \geq \left\lceil \frac{n}{2} \right\rceil$, and $|N[X] - S| = |V(G) - S| \leq n - \left\lceil \frac{n}{2} \right\rceil$. By Remark 1, it follows that $|N[X] \cap S| \geq |N[X] - S|$. □

Lemma 2: Let G be a nontrivial connected graph and suppose that $S \subseteq V(G)$. If X is a subset of S such that no vertex in X is adjacent to any vertex in $V(G) - S$, then

i. $N[X] \subseteq S$;

ii. $|N[X] \cap S| > |N[X] - S|$; and

iii. If Y is a subset of S that contains a vertex which is adjacent to a vertex in $V(G) - S$ and $|N[Y] \cap S| \geq |N[Y] - S|$, then $|N[X \cup Y] \cap S| \geq |N[X \cup Y] - S|$.

Proof: (i): Assume to the contrary that $N[X] \nsubseteq S$.

Case 1. Let $S \subset N[X]$. Then there exists a vertex $v \in N[X]$ such that $v \notin S$. As a result, $v \in X$ or $v \in N(X)$, and $v \in V(G) - S$. If $v \in X$, then $v \in S$ since $X \subseteq S$. This is not possible because $v \notin S$. On the other hand, if $v \in N(X)$, then there exists a vertex $u \in X$ that is adjacent to v. However, $v \in V(G) - S$, which contradicts the assumption on X.

Case 2. Let $N[X] \cap S \neq \emptyset$ and $S \nsubseteq N[X]$. The argument is the same as in Case 1.

Case 3. Let $N[X] \cap S = \emptyset$. Notice that $N[X] \cap S = (N(X) \cup X) \cap S = (N(X) \cap S) \cup (X \cap S)$. Since $X \subseteq S$ and $X \neq \emptyset$, we have $X \cap S \neq \emptyset$. This means that $N[X] \cap S \neq \emptyset$, a contradiction.

Therefore, $N[X] \subseteq S$.

(ii): Because $N[X] \subseteq S$, we have $N[X] - S = \emptyset$, that is, $|N[X] - S| = 0$. Also, since $X \neq \emptyset$, it follows that $|N[X] \cap S| \geq 1$. Thus, $|N[X] \cap S| > |N[X] - S|$.

(iii): Observe that $N[X \cup Y] \cap S = (N[X] \cup N[Y]) \cap S = (N[X] \cap S) \cup (N[Y] \cap S)$. So,

$$\begin{aligned} |N[X \cup Y] \cap S| &= |N[X] \cap S| + |N[Y] \cap S| - |(N[X] \cap S) \cap (N[Y] \cap S)| \\ &= |N[X] \cap S| + |N[Y] \cap S| - |N[X] \cap S \cap N[Y]|. \end{aligned}$$

Since $N[X] \subseteq S$ by (i), it follows that

$$|N[X \cup Y] \cap S| = |N[X] \cap S| + |N[Y] \cap S| - |N[X] \cap N[Y]|.$$

Because $N[X] \cap N[Y] \subseteq N[X]$, we have $|N[X]| \geq |N[X] \cap N[Y]|$, that is, $|N[X]| - |N[X] \cap N[Y]| \geq 0$. Hence, $|N[X \cup Y] \cap S| \geq |N[Y] \cap S|$. In addition, notice that

$$\begin{aligned} N[X \cup Y] - S &= (N[X] \cup N[Y]) \cap S^c \\ &= (N[X] \cap S^c) \cup (N[Y] \cap S^c) \\ &= (N[X] - S) \cup (N[Y] - S). \end{aligned}$$

Note that $N[X] - S = \emptyset$. So, $N[X \cup Y] - S = N[Y] - S$, and consequently, $|N[X \cup Y] - S| = |N[Y] - S|$. Since $|N[Y] \cap S| \geq |N[Y] - S|$ by assumption, we have $|N[X \cup Y] \cap S| \geq |N[X \cup Y] - S|$. □

Lemma 3: Let G be a connected graph of order n. Suppose that k is the number of cut vertices and r is the number of 2-vertex cut sets of G. Then

i. $d_s^c(G, n) = 1$;

ii. $d_s^c(G, n-1) = n - k$; and

iii. $d_s^c(G, n-2) = \binom{n-k}{2} - r$ for $n \geq 4$.

Proof: (i) is clear.

(ii): By Theorem 3 (ii), G has $n-k$ connected dominating sets with cardinality $n-1$. Let S be a connected dominating set of G with cardinality $n-1$. Then, there must be one vertex of G that is not in S, say v. Note that v must not be a cut vertex, because if v is a cut vertex, then $\langle S\rangle_G$ is disconnected. Let $X \subseteq S$. If no vertex in X is adjacent to v, then by Lemma 2, $|N[X] \cap S| > |N[X] - S|$. Also, if there is a vertex in X adjacent to v, then $N[X] - S = \{v\}$, that is, $|N[X] - S| = 1$. Because $X \neq \emptyset$, $|N[X] \cap S| \geq 1$. Thus, $|N[X] \cap S| \geq |N[X] - S|$. Therefore, S is a secure set by Theorem 5. So, S is a connected secure dominating set. Clearly, $d_s^c(G, n-1) = n-k$.

(iii): According to Theorem 3 (iii), G has $\binom{n-k}{2} - r$ connected dominating sets with cardinality $n-2$. Let S be a connected dominating set of G with cardinality $n-2$. This implies that there are two vertices u and v such that $u, v \notin S$. Note that u and v are not cut vertices and $\{u, v\}$ is not a 2-vertex cut set. Otherwise, $\langle S\rangle_G$ is disconnected or S is not a dominating set. Let $X \subseteq S$. Because $n \geq 4$, we have $|N[X] \cap S| \geq 2$. In addition, there are only four possible sets that describe $N[X] - S$, namely, $\emptyset, \{u\}, \{v\}, \{u, v\}$. So, $0 \leq |N[X] - S| \leq 2$. As a consequence, $|N[X] \cap S| \geq |N[X] - S|$. By Theorem 5, S is a secure set. Therefore, S is a connected secure dominating set. Consequently, $d_s^c(G, n-2) = \binom{n-k}{2} - r$. □

Theorem 9: Let P_n be a path graph of order $n \geq 4$. Then

$$d_s^c(P_n, i) = \begin{cases} 1 & \text{if } i = n \\ 2 & \text{if } i = n-1 \\ 1 & \text{if } i = n-2 \\ 0 & \text{if } i \leq n-3. \end{cases}$$

Thus, $D_s^c(P_n, x) = x^n + 2x^{n-1} + x^{n-2}$.

Proof: Case 1. Clearly, $d_s^c(P_n, n) = 1$ by Lemma 3 (i).

Case 2. Since P_n has $n-2$ cut vertices, by Lemma 3 (ii), $d_s^c(P_n, n-1) = n - (n-2) = 2$.

Case 3. Notice that P_n has no 2-vertex cut set but has $n-2$ cut vertices. By Lemma 3 (iii), $d_s^c(P_n, n-2) = \binom{n-(n-2)}{2} - 0 = 1$.

Case 4. Because $\gamma_s^c(P_n) = n-2$ by Theorem 1, $d_s^c(P_n, i) = 0$ if $i \leq n-3$.

Immediately, $D_s^c(P_n, x) = x^n + 2x^{n-1} + x^{n-2}$. □

Theorem 10: Let C_n be a cycle graph of order $n \geq 4$. Then

$$d_s^c(C_n, i) = \begin{cases} 1 & \text{if } i = n \\ n & \text{if } i = n-1 \\ n & \text{if } i = n-2 \\ 0 & \text{if } i \leq n-3. \end{cases}$$

Thus, $D_s^c(C_n, x) = x^n + nx^{n-1} + nx^{n-2}$.

Proof: Case 1. It is obvious that $d_s^c(C_n, n) = 1$ by Lemma 3 (i).

Case 2. Because C_n has no cut vertex, by Lemma 3 (ii), $d_s^c(C_n, n-1) = n$.

Case 3. Note that $V(C_n)$ has $\binom{n}{2}$ subsets with cardinality 2. Among these subsets, n are not 2-vertex cut sets (those subsets whose elements are adjacent to each other). So, the number of 2-vertex cut sets of C_n is $\binom{n}{2} - n$. Also, note that C_n has no cut vertex. Thus, by Lemma 3 (iii), $d_s^c(C_n, n-2) = \binom{n-0}{2} - \left[\binom{n}{2} - n\right] = n$.

Case 4. Since $\gamma_s^c(C_n) = n-2$ by Theorem 2, $d_s^c(C_n, i) = 0$ whenever $i \leq n-3$.

Consequently, $D_s^c(C_n, x) = x^n + nx^{n-1} + nx^{n-2}$. □

Theorem 11: Let K_n be a complete graph of order $n \geq 4$. Then

$$d_s^c(K_n, i) = \begin{cases} \binom{n}{i} & \text{if } \lceil \frac{n}{2} \rceil \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $D_s^c(K_n, x) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \binom{n}{i} x^i$.

Proof: Notice that $K_{m_1, m_2, \ldots, m_n} \cong K_n$ if $m_i = 1$ for all $i = 1, 2, \ldots, n$. By Theorem 8, $\gamma_s^c(K_n) = \lceil \frac{n}{2} \rceil$. It is clear that $d_s^c(K_n, i) = 0$ for $i < \lceil \frac{n}{2} \rceil$. Observe that every subset of $V(K_n)$ is a connected dominating set. Suppose that S is a connected dominating set of K_n with $\lceil \frac{n}{2} \rceil \leq |S| \leq n$. Let $X \subseteq S$. It is clear that X is a dominating set. By Lemma 1, $|N[X] \cap S| \geq |N[X] - S|$. Hence, S is a secure set by Theorem 5 and so S is a connected secure dominating set.

For counting: By Lemma 3 (i), $d_s^c(K_n, n) = 1 = \binom{n}{n}$. Since K_n has no cut vertex and 2-vertex cut set, by Lemma 3 (ii) and (iii), $d_s^c(K_n, n-1) = n = \binom{n}{n-1}$ and $d_s^c(K_n, n-2) = \binom{n}{2} = \binom{n}{n-2}$. Now, there are $\binom{n}{n-4}$ ways to obtain all connected secure dominating sets with cardinality $n-4$. Similarly, there are $\binom{n}{n-5}$ ways to obtain all connected secure dominating sets with cardinality $n-5$, and so on, until there are $\binom{n}{\lceil n/2 \rceil}$ ways to obtain all connected secure dominating sets with cardinality $\lceil \frac{n}{2} \rceil$. Therefore, $d_s^c(K_n, i) = \binom{n}{i}$ for $\lceil \frac{n}{2} \rceil \leq i \leq n$. Consequently, $D_s^c(K_n, x) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \binom{n}{i} x^i$. □

Theorem 12: Let $K_{1,n}$ be a star graph of order $n+1$ with $n \geq 3$. Then

$$d_s^c(K_{1,n}, i) = \begin{cases} 1 & \text{if } i = n+1 \\ \binom{n}{i-1} & \text{if } \lceil \frac{n+1}{2} \rceil \leq i \leq n \\ 0 & \text{if } i < \lceil \frac{n+1}{2} \rceil. \end{cases}$$

Therefore, $D_s^c(K_{1,n}, x) = x^{n+1} + \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} \binom{n}{i-1} x^i$.

Proof: Observe that $\gamma_s^c(K_{1,n}) = \lceil \frac{n+1}{2} \rceil$ according to Theorem 8. Thus, $d_s^c(K_{1,n}, i) = 0$ for $i < \lceil \frac{n+1}{2} \rceil$.

Case 1. By Lemma 3 (i), $d_s^c(K_{1,n}, n+1) = 1$.

Case 2. Let v be the central vertex of $K_{1,n}$. Suppose that $S \subseteq V(K_{1,n})$ with $\lceil \frac{n+1}{2} \rceil \leq |S| \leq n$. Note that $v \in S$ because if $v \notin S$, $\langle S \rangle_{K_{1,n}}$ is disconnected. So, S is a connected dominating set. Now, let $X \subseteq S$. If $v \in X$, then X is a dominating set and as a result $|N[X] \cap S| \geq |N[X] - S|$ by Lemma 1. Also, if $v \notin X$, then by Lemma 2 (ii), $|N[X] \cap S| > |N[X] - S|$. Hence, by Theorem 5, S is a secure set and it follows that S is a connected secure dominating set.

For counting: Because $K_{1,n}$ has one cut vertex and no 2-vertex cut set, by Lemma 3 (ii) and (iii), $d_s^c(K_{1,n}, n) = n + 1 - 1 = \binom{n}{n-1}$ and $d_s^c(K_{1,n}, n-1) = \binom{n+1-1}{2} = \binom{n}{n-2} = \binom{n}{(n-1)-1}$. Now, there are $\binom{n}{(n-2)-1}$ ways to obtain all connected secure dominating sets with cardinality $n-2$. In addition, there are $\binom{n}{(n-3)-1}$ ways to obtain all connected secure dominating sets with cardinality $n-3$, and so on, until there are $\binom{n}{\lceil (n+1)/2 \rceil - 1}$ ways to obtain all connected secure dominating sets with cardinality $\lceil \frac{n+1}{2} \rceil$. Consequently, $d_s^c(K_{1,n}, i) = \binom{n}{i-1}$ for $\lceil \frac{n+1}{2} \rceil \leq i \leq n$.

Therefore, $D_s^c(K_{1,n}, x) = x^{n+1} + \sum_{i=\lceil \frac{n+1}{2} \rceil}^{n} \binom{n}{i-1} x^i$. □

Theorem 13: Let G be a connected graph of order $n \geq 2$. Then

$$d_s^c(G \circ K_1, i) = \begin{cases} \binom{n}{i-n} & \text{if } n \leq i \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $D_s^c(G \circ K_1, x) = x^n (x+1)^n$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $V(K_1^i) = \{v_1^i\}$ is the vertex set of the ith copy of K_1, whose vertex is adjacent to the ith vertex of G. It is clear that $V(G)$ is a secure dominating set by Theorem 7. Because G is connected, $V(G)$ is a connected secure dominating set. Note that $\gamma_s^c(G \circ K_1) \geq \frac{2n}{2} = n$ by Theorem 6. Hence, $\gamma_s^c(G \circ K_1) = n$. As a consequence, $d_s^c(G \circ K_1, i) = 0$ for $i < n$. Now, according to Theorem 4, there are $\binom{n}{i-n}$ connected dominating sets with cardinality i, $n \leq i \leq 2n$. Let S be a connected dominating set of $G \circ K_1$ with $n \leq |S| \leq 2n$. Note that $V(G) \subseteq S$ because if at least one vertex in $V(G)$ is not in S, $\langle S \rangle_{G \circ K_1}$ is disconnected.

Case 1. Let $X \subseteq S$ for which no vertex in X is adjacent to a vertex in $V(G \circ K_1) - S$. By Lemma 2 (ii), $|N[X] \cap S| \geq |N[X] - S|$.

Case 2. Let $Y \subseteq S$ with $Y = \{v_{a_1}, v_{a_2}, \ldots, v_{a_p}\}$ where $1 \leq a_j \leq n$ for $j = 1, 2, \ldots, p$. Now,

$$N[Y] \cap S = (N(Y) \cup Y) \cap S = (N(Y) \cap S) \cup (Y \cap S) = (N(Y) \cap S) \cup Y$$

since $Y \subseteq S$. Thus, $Y \subseteq N[Y] \cap S$ and as a result $|N[Y] \cap S| \geq |Y| = p$. In addition, observe that $N[Y] - S = \{v_1^{a_1}, v_1^{a_2}, \ldots, v_1^{a_p}\}$. Thus, $|N[Y] - S| = p$. This means that $|N[Y] \cap S| \geq |N[Y] - S|$.

Case 3. Consider the set $X \cup Y \subseteq S$ where X and Y are constructed in Cases 1 and 2, respectively. It is immediate that $|N[X \cup Y] \cap S| \geq |N[X \cup Y] - S|$ by Lemma 2 (iii).

Therefore, S is a secure set by Theorem 5 and it follows that S is a connected secure dominating set. Thus, $d_s^c(G \circ K_1, i) = \binom{n}{i-n}$ for $n \leq i \leq 2n$. Consequently, $D_s^c(G \circ K_1, x) = x^n (x+1)^n$. □

4 Conclusion

This paper introduced the connected secure domination polynomial and determined its explicit algebraic forms for paths, cycles, complete graphs, stars, and corona graphs. These structural results establish a strong mathematical foundation for evaluating how secure alliances behave under explicit connectivity constraints. Moving forward, a natural extension of this research is to investigate the connected secure domination polynomial for complex operations like cartesian, strong, or lexicographic graph products. Additionally, generalizing the current corona graph result from a single vertex copy to arbitrary graph families presents a highly promising open problem. Finally, future investigations could explore the analytical properties of the polynomial, including its roots, coefficient unimodality, and graph uniqueness characterizations.

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References

1. S. Alikhani and Y. H. Peng, Introduction to domination polynomial of a graph, *Ars Combinatoria 114* **(2009)**, 257–266.
2. R. Brigham, R. Dutton and S. Hedetniemi, Security in graphs, *Discrete Applied Mathematics 155* **(2007)**, 1708–1714. https://doi.org/10.1016/j.dam.2007.03.009
3. G. Chartrand and P. Zhang. Chromatic Number Theory, 1st Ed., Taylor and Francis Group: New York, USA, **(2009)**.
4. G. Entero and A. C. Pedrano, On connected total domination polynomial of some lexicographic product graphs, *Advances and Applications in Discrete Mathematics 27(1)* **(2021)**, 147–155. https://doi.org/10.17654/DM027010147
5. C. R. Giganto and A. C. Pedrano, On pendant domination polynomial in the corona of some graphs, *Annals of Communications in Mathematics 8(4)* **(2025)**, 442–450. https://doi.org/10.62072/acm.2025.080402
6. P. Johnson and C. Jones, Secure-dominating sets in graphs, In: V. R. Kulli (Ed.), *Advances in Domination Theory II*, Vishwa International Publications, India, **(2013)**, 1–9.

7. I. Majaba and R. F. Rulete, Total vertex-edge domination number of some graphs, *Communications in Mathematics and Applications 16(4)* **(2025)**, 1067–1075. https://doi.org/10.26713/cma.v16i4.3332
8. D. A. Mojdeh and A. S. Emadi, Connected domination polynomial of graphs, *Fasciculi Mathematici 60(1)* **(2018)**, 103–121. https://doi.org/10.1515/fascmath-2018-0007