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Research article

On Total Product Cordial Labeling of Some Snake Graphs

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Abstract: A total product cordial labeling of a graph G is a function $f : V \to \{0, 1\}$. For each xy, assign the label $f(x)f(y)$, f is called total product cordial labeling of G if it satisfies the condition that $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the set of vertices and edges which are labeled with $i = 0, 1$, respectively. A graph with a total product cordial labeling defined on it is called a total product cordial graph. In this paper, we determined the total product cordial labeling of the snake graphs T_n, $A(T_n)$, $D(T_n)$, $DA(T_n)$, Q_n, $A(Q_n)$, $D(Q_n)$, and $DA(Q_n)$.

Mathematics Subject Classification: 54C05, 54C08, 54C10
Keywords: Graph Labeling; Total Product Cordial Labeling; Snake Graphs

1 Introduction

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called vertex labeling (or an edge labeling). It was first introduced in 1967 in the classic paper of $\beta-$valuations by Rosa [5] which leads to the foundation of various graph labeling methods. In 2004, M. Sundaram, R. Ponraj and S. Somasundaram introduced the notion of product cordial labeling. A product cordial labeling of a graph G with vertex set V is a function f from V to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1 [8]. Building on this concept, the same authors introduced a new labeling approach called total product cordial labeling. This method considers both vertices and edges. A graph is said to be total product cordial if a function f from V to $\{0, 1\}$, and for each edge uv assigned with the label $f(u)f(v)$, the number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1. A graph with a total product cordial labeling is called a *total product cordial graph* [1]. Pedrano and Rulete [3,4] determined that some cartesian product graphs and the Generalized petersen graph admits a total product cordial labeling. They also showed that the corona of some graphs admits total product cordial labeling. There exist several variations of cordial labeling, each differing in structure and conditions while preserving the balance properties that define the concept. See [2–4,7,10**?** ,11] for some of the variants of cordial labeling. Throughout this paper, we determined the total product cordial labeling of the snake graphs T_n, $A(T_n)$, $D(T_n)$, $DA(T_n)$, Q_n, $A(Q_n)$, $D(Q_n)$, and $DA(Q_n)$.

2 Preliminaries

Definition 1: *[4] For a simple graph $G = (V, E)$ and a function $f : V \to \{0, 1\}$ assign the label $f(x)f(y)$ for each edge xy. This function f is called total product cordial labeling if $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and edges labeled with $i = 0, 1$. A graph with a total product cordial labeling defined on it is called total product cordial.*

The following are results of total product cordial labeling on some graphs.

Theorem 1 ([?])**:** *C_n is total product cordial if $n \neq 4$.*

Corollary 1 ([?])**:** *The ladder $L_m = P_m \times P_2$ $(m \neq 2)$ is total product cordial.*

Theorem 2 ([4])**:** *The crown graph $P_m \circ P_n$ is total product cordial graph for all $m, n \geq 1$.*

3 TOTAL PRODUCT CORDIAL GRAPHS

This section present the results of total product cordial labeling on some snake graphs.

Theorem 3: The Triangular Snake Graph T_n is total product cordial for all $n \geq 2$.

Proof: Let $V(T_n) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq n-1\}$ and $E(T_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n-1\} \cup \{v_{i+1} u_i | 1 \leq i \leq n-1\}$. Hence, the order and size of T_n is $2n-1$ and $3n-3$, respectively. To show that T_n is total product cordial, we consider the following cases:

Case 1: n is even, $n \geq 2$.

Subcase 1.1: $n = 2$.

Observe that $T_2 \cong C_3$. By Theorem 1, T_2 is a total product cordial graph.

Subcase 1.2: n is even, $n \geq 4$.

Let $f : V(T_n) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases} \qquad f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-2}{2} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n - 1$ and $v_f(1) = n$. Now, the edges of T_n with labels zero are the following:

$$f(v_i v_{i+1}) = 0, \qquad 1 \le i \le \frac{n}{2};$$
$$f(v_i u_i) = 0, \qquad 1 \le i \le \frac{n}{2};$$
$$f(v_{i+1} u_i) = 0, \qquad 1 \le i \le \frac{n-2}{2}.$$

As a result of these edge labels, we have $e_f(0) = \frac{3n-2}{2}$ and consequently, $e_f(1) = \frac{3n-4}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{2n-2+3n-2-2n-3n+4}{2}\right| = 0$. Thus, T_n is total product cordial when n is even, $n \ge 2$.
Case 2: n is odd, $n \ge 3$.
Let $f : V(T_n) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases} \qquad f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n - 1$ and $v_f(1) = n$. Now, the edges of T_n with labels zero are the following:

$$f(v_i v_{i+1}) = 0, \qquad 1 \le i \le \frac{n-1}{2};$$
$$f(v_i u_i) = 0, \qquad 1 \le i \le \frac{n-1}{2};$$
$$f(v_{i+1} u_i) = 0 \qquad 1 \le i \le \frac{n-1}{2}.$$

As a result of these edge labels, we have $e_f(0) = \frac{3n-3}{2}$ and consequently, $e_f(1) = \frac{3n-3}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{2n-2+3n-3-2n-3n+3}{2}\right| = 1$. Thus, T_n is total product cordial for all $n \ge 3$, n is odd. Considering all the cases presented above, we have shown that T_n is total product cordial for all $n \ge 2$. □

Theorem 4: The Alternate Triangular Snake Graph $A(T_n)$ is total product cordial for all $n \ge 2$.

Proof: To prove the theorem, we consider the following cases:
Case 1: n is even, $n \ge 2$.
Let $V(A(T_n)) = \{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le \frac{n}{2}\}$ and $E(A(T_n)) = \{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{v_{2i-1} u_i | 1 \le i \le \frac{n}{2}\} \cup \{v_{2i} u_i | 1 \le i \le \frac{n}{2}\}$. Hence, the order and size of $A(T_n)$ is $\frac{3n}{2}$ and $2n - 1$, respectively.
Subcase 1.1: $n = 2$.
Observe that $A(T_2) \cong C_3$. By Theorem. 2.1, $A(T_2)$ is a total product cordial graph.
Subcase 1.2: $n \equiv 0 \pmod 4$, $n \ge 4$.

Let $f : V(A(T_n)) \to \{0,1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{2} \\ 1, & otherwise \end{cases} \qquad f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n}{4}$ and $v_f(1) = \frac{3n}{4}$. Now, the edges of $A(T_n)$ with labels zero are the following:

$$f(v_i v_{i+1}) = 0, \qquad 1 \le i \le \frac{n}{2};$$
$$f(v_{2i-1} u_i) = 0, \qquad 1 \le i \le \frac{n}{4};$$
$$f(v_{2i} u_i) = 0 \qquad 1 \le i \le \frac{n}{4}.$$

As a result of these edge labels, we have $e_f(0) = n$ and consequently, $e_f(1) = n - 1$.

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n+4n-3n-4n+4}{4}\right| = 1$. Thus, $A(T_n)$ is total product cordial for all $n \equiv 0 \pmod 4$, $n \ge 4$.

Subcase 1.3: $n \equiv 2 \pmod 4$, $n \ge 6$.

Let $f : V(A(T_n)) \to \{0,1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{2} \\ 1, & otherwise \end{cases} \qquad f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n-2}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-2}{4}$ and $v_f(1) = \frac{3n+2}{4}$. Now, the edges of $A(T_n)$ with labels zero are the following:

$$f(v_i v_{i+1}) = 0, \qquad 1 \le i \le \frac{n}{2};$$
$$f(v_{2i-1} u_i) = 0, \qquad 1 \le i \le \frac{n+2}{4};$$
$$f(v_{2i} u_i) = 0, \qquad 1 \le i \le \frac{n-2}{4}.$$

As a result of these edge labels, we have $e_f(0) = n$ and consequently, $e_f(1) = n - 1$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-2+4n-3n-2-4n+4}{4}\right| = 0$. Thus, $A(T_n)$ is total product cordial for all $n \equiv 2 \pmod 4$, $n \ge 6$. By Subcases 1.1, 1.2 and 1.3, $A(T_n)$ is a total product cordial graph when n is even, $n \ge 2$.

Case 2: n is odd, $n \ge 3$.

Let $V(A(T_n)) = \{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le \frac{n-1}{2}\}$ and $E(A(T_n)) = \{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{v_{2i-1} u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i} u_i | 1 \le i \le \frac{n-1}{2}\}$. Hence, the order and size of $A(T_n)$ is $\frac{3n-1}{2}$ and $2n-2$, respectively.

Subcase 2.1: $n = 3$.

Let $f : V(A(T_3)) \to \{0,1\}$ be the function defined by:

$$f(v_1) = f(v_3) = f(u_1) = 1 \quad \text{and} \quad f(v_2) = 0.$$

Then the edge labels would be:

$$f(v_1v_2) = 0; \quad f(v_2v_3) = 0; \quad f(v_1u_1) = 1; \quad f(v_2u_2) = 0.$$

Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |1 + 3 - 3 - 1| = 0$. Thus, $A(T_3)$ is a total product cordial graph.
Subcase 2.2: $n \equiv 1 \pmod 4$, $n \geq 5$.
Let $f : V(A(T_n)) \rightarrow \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases} \qquad f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-3}{4}$ and $v_f(1) = \frac{3n+1}{4}$. Now, the edges of $A(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, & 1 \leq i \leq \frac{n-1}{2}; \\ f(v_{2i-1}u_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i}u_i) &= 0 & 1 \leq i \leq \frac{n-1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = n - 1$ and consequently, $e_f(1) = n - 1$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-3+4n-4-3n-1-4n+4}{4}\right| = 1$. Thus, $A(T_n)$ is total product cordial for all $n \equiv 1 \pmod 4$, $n \geq 5$.
Subcase 2.3: $n \equiv 3 \pmod 4$, $n \geq 7$.
Let $f : V(A(T_n)) \rightarrow \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+1}{2} \\ 1, & otherwise \end{cases} \qquad f(u_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+1}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-5}{4}$ and $v_f(1) = \frac{3n+3}{4}$. Now, the edges of $A(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, & 1 \leq i \leq \frac{n+1}{2}; \\ f(v_{2i-1}u_i) &= 0, & 1 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i}u_i) &= 0 & 1 \leq i \leq \frac{n+1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = n$ and consequently, $e_f(1) = n - 2$. Hence,$|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-5+4n-3n-3-4n+8}{4}\right| = 0$. Thus, $A(T_n)$ is total product cordial for all $n \equiv 3 \pmod 4$, $n \geq 7$. By

Subcases 2.1, 2.2, and 2.3, $A(T_n)$ is a total product cordial graph when n is odd, $n \geq 3$. Considering all cases presented above, we have shown that $A(T_n)$ is total product cordial for all $n \geq 2$. □

Theorem 5: The Double Triangular Snake Graph $D(T_n)$ is total product cordial for all $n \geq 2$.

Proof: Let $V(D(T_n)) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq n-1\} \cup \{w_i | 1 \leq i \leq n-1\}$ and $E(D(T_n)) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n-1\} \cup \{v_{i+1} u_i | 1 \leq i \leq n-1\} \cup \{v_i w_i | 1 \leq i \leq n-1\} \cup \{v_{i+1} w_i | 1 \leq i \leq n-1\}$. Hence, the order and size of $D(T_n)$ is $3n-2$ and $5n-5$, respectively.
To show that $D(T_n)$ is total product cordial, we consider the following cases:
Case 1: n is even, $n \geq 2$.
Subcase 1.1: $n = 2$.

Observe that $D(T_2) \cong P_1 \circ P_3$. By Theorem. 2.9, $D(T_2)$ is total product cordial graph.
Subcase 1.2: n is even, $n \geq 4$.
Let $f : V(D(T_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-2}{2} \\ 1, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-2}{2} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-4}{2}$ and $v_f(1) = \frac{3n}{2}$. Now, the edges of $D(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, \qquad 1 \leq i \leq \frac{n}{2}; \\ f(v_i u_i) &= 0, \qquad 1 \leq i \leq \frac{n}{2}; \\ f(v_{i+1} u_i) &= 0, \qquad 1 \leq i \leq \frac{n-2}{2}; \\ f(v_i w_i) &= 0, \qquad 1 \leq i \leq \frac{n}{2}; \\ f(v_{i+1} w_i) &= 0, \qquad 1 \leq i \leq \frac{n-2}{2}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{5n-4}{2}$ and consequently, $e_f(1) = \frac{5n-6}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-4+5n-4-3n-5n+6}{2}\right| = 1$. Thus, the $D(T_n)$ is total product cordial for all $n \geq 4$, n is even. By Subcase 1.1 and Subcase 1.2, $D(T_n)$ is total product cordial when n is even, $n \geq 2$.
Case 2: n is odd, $n \geq 3$.
Let $f : V(D(T_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-3}{2}$ and $v_f(1) = \frac{3n-1}{2}$. Now, the edges of $D(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) = 0, &\qquad 1 \leq i \leq \frac{n-1}{2}; \\ f(v_i u_i) = 0, &\qquad 1 \leq i \leq \frac{n-1}{2}; \\ f(v_{i+1} u_i) = 0, &\qquad 1 \leq i \leq \frac{n-1}{2}; \\ f(v_i w_i) = 0, &\qquad 1 \leq i \leq \frac{n-1}{2}; \\ f(v_{i+1} w_i) = 0, &\qquad 1 \leq i \leq \frac{n-1}{2}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{5n-5}{2}$ and consequently, $e_f(1) = \frac{5n-5}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-3+5n-5-3n+1-5n+5}{2}\right| = 1$. Thus, $D(T_n)$ is total product cordial for all $n \geq 3$, n is odd. Considering all the cases presented above, we have shown that $D(T_n)$ is total product cordial for all $n \geq 2$. □

Theorem 6: The Double Alternate Triangular Snake Graph $DA(T_n)$ is total product cordial for all $n \geq 2$.

Proof: To prove the theorem, we consider the following cases:
Case 1: n is even, $n \geq 2$.

Let $V(DA(T_n)) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq \frac{n}{2}\} \cup \{w_i | 1 \leq i \leq \frac{n}{2}\}$ and $E(DA(T_n)) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_{2i-1} u_i | 1 \leq i \leq \frac{n}{2}\} \cup \{v_{2i} u_i | 1 \leq i \leq \frac{n}{2}\} \cup \{v_{2i-1} w_i | 1 \leq i \leq \frac{n}{2}\} \cup \{v_{2i} w_i | 1 \leq i \leq \frac{n}{2}\}$. Hence, its order and size is $2n$ and $3n-1$, respectively.

Subcase 1.1: $n = 2$.

Observe that $DA(T_2) \cong P_1 \circ P_3$. By Theorem. 2.9, $DA(T_2)$ is a total product cordial graph.

Subcase 1.2: $n \equiv 0 \pmod 4$, $n \geq 4$.

Let $f : V(DA(T_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n$ and $v_f(1) = n$. Now, the edges of $DA(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, \qquad 1 \leq i \leq \frac{n}{2}; \\ f(v_{2i-1} u_i) &= 0, \qquad 1 \leq i \leq \frac{n}{4}; \\ f(v_{2i} u_i) &= 0, \qquad 1 \leq i \leq \frac{n}{4}; \\ f(v_{2i-1} w_i) &= 0, \qquad 1 \leq i \leq \frac{n}{4}; \\ f(v_{2i} w_i) &= 0, \qquad 1 \leq i \leq \frac{n}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{3n}{2}$ and consequently, $e_f(1) = \frac{3n-2}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{2n+3n-2n-3n+2}{2}\right| = 1$. Thus, $DA(T_n)$ is total product cordial for all $n \equiv 0 \pmod 4$, $n \geq 4$.

Subcase 1.3: $n \equiv 2 \pmod 4$, $n \geq 6$.

Let $f : V(DA(T_n)) \rightarrow \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-2}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-2}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n-1$ and $v_f(1) = n+1$. Now, the edges of $DA(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, & 1 \le i \le \frac{n}{2}; \\ f(v_{2i-1} u_i) &= 0, & 1 \le i \le \frac{n+2}{4}; \\ f(v_{2i} u_i) &= 0, & 1 \le i \le \frac{n-2}{4}; \\ f(v_{2i-1} w_i) &= 0, & 1 \le i \le \frac{n+2}{4}; \\ f(v_{2i} w_i) &= 0, & 1 \le i \le \frac{n-2}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{3n}{2}$ and consequently, $e_f(1) = \frac{3n-2}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{2n-2+3n-2n-2-3n+2}{2}\right| = 1$. Thus, the $DA(T_n)$ is total product cordial for all $n \equiv 2 \pmod 4$, $n \ge 6$. By Subcases 1.1, 1.2, and 1.3, $DA(T_n)$ is total product cordial when n is even, $n \ge 2$.

Case 2: n is odd, $n \ge 3$.

Let $V(DA(T_n)) = \{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{w_i | 1 \le i \le \frac{n-1}{2}\}$ and $E(DA(T_n)) = \{v_i v_{i+1} | 1 \le i \le n-1\} \cup \{v_{2i-1} u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i} u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i-1} w_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i} w_i | 1 \le i \le \frac{n-1}{2}\}$. Hence, its order and size is $2n-1$ and $3n-3$, respectively.

Subcase 2.1: $n = 3$.

Let $f : V(DA(T_3)) \to \{0,1\}$ be the function defined by:

$$f(v_1) = f(v_3) = f(u_1) = f(w_1) = 1 \quad \text{and} \quad f(v_2) = 0.$$

Then the edge labels would be:

$$\begin{aligned} f(v_1 v_2) &= 0; \\ f(v_2 v_3) &= 0; \\ f(v_1 u_1) &= 1; \\ f(v_2 u_1) &= 0; \\ f(v_1 w_1) &= 1; \\ f(v_2 w_1) &= 0. \end{aligned}$$

Observe that $v_f(0) = 1$, $v_f(1) = 4$, $e_f(0) = 4$, and $e_f(1) = 2$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |1+4-4-2| = |-1| = 1$. Thus, $DA(T_3)$ is a total product cordial graph.

Subcase 2.2: $n \equiv 1 \pmod 4$, $n \ge 5$.

Let $f : V(DA(T_n)) \to \{0,1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n - 1$ and $v_f(1) = n$. Now, the edges of $DA(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, \qquad 1 \leq i \leq \frac{n-1}{2}; \\ f(v_{2i-1} u_i) &= 0, \qquad 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i} u_i) &= 0, \qquad 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i-1} w_i) &= 0, \qquad 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i} w_i) &= 0, \qquad 1 \leq i \leq \frac{n-1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{3n-3}{2}$ and $e_f(1) = \frac{3n-3}{2}$. Hence, $|v_f(0)+e_f(0)-v_f(1)-e_f(1)| = \left|\frac{2n-2+3n-3-2n-3n+3}{2}\right| = 1$. Thus, $DA(T_n)$ is total product cordial for all $n \equiv 1 \pmod 4$, $n \geq 5$.

Subcase 2.3: $n \equiv 3 \pmod 4$, $n \geq 7$.

Let $f : V(DA(T_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+1}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+1}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+1}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n - 2$ and $v_f(1) = n + 1$. Now, the edges of $DA(T_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, \qquad 1 \leq i \leq \frac{n+1}{2}; \\ f(v_{2i-1} u_i) &= 0, \qquad 1 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i} u_i) &= 0, \qquad 1 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i-1} w_i) &= 0, \qquad 1 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i} w_i) &= 0, \qquad 1 \leq i \leq \frac{n+1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{3n-1}{2}$ and consequently, $e_f(1) = \frac{3n-5}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{2n-2+3n-1-2n-2-3n+5}{2}\right| = 1$. Thus, $DA(T_n)$ is total product cordial for all $n \equiv 3 \pmod 4$, $n \geq 7$. By Subcases 2.1, 2.2, and 2.3, $DA(T_n)$ is a total product cordial graph when n is odd, $n \geq 3$. Considering all the cases presented above, we have shown that $DA(T_n)$ is total product cordial for all $n \geq 2$. □

Theorem 7: The Quadrilateral Snake Graph Q_n is total product cordial for all $n \geq 3$.

Proof: Let $V(Q_n) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq n-1\} \cup \{w_i | 1 \leq i \leq n-1\}\}$ and $E(Q_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n-1\} \cup \{v_{i+1} w_i | 1 \leq i \leq n-1\} \cup \{u_i w_i | 1 \leq i \leq n-1\}$. Hence, the order and size of Q_n is $3n-2$ and $4n-4$, respectively.
To show that Q_n is total product cordial, we consider the following cases:
Case 1: n is odd, $n \geq 3$.
Let $f : V(Q_n) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-3}{2}$ and $v_f(1) = \frac{3n-1}{2}$. Now, edges of Q_n with labels zero are the following:

$$f(v_i v_{i+1}) = 0, \quad 1 \leq i \leq \tfrac{n-1}{2}; \qquad f(v_i u_i) = 0, \quad 1 \leq i \leq \tfrac{n-1}{2};$$
$$f(v_{i+1} w_i) = 0, \quad 1 \leq i \leq \tfrac{n-1}{2}; \qquad f(u_i w_i) = 0, \quad 1 \leq i \leq \tfrac{n-1}{2}.$$

As a result of these edge labels, we have $e_f(0) = 2n-2$ and consequently, $e_f(1) = 2n-2$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-3+4n-4-3n+1-4n+4}{2}\right| = 1$. Thus, Q_n is total product cordial for all $n \geq 3$, n is odd.
Case 2: n is even, $n \geq 4$.
Let $f : V(Q_n) \to \{0, 1\}$ be the function defined by:

$$f(u_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = \frac{3n-4}{2}$ and $v_f(1) = \frac{3n}{2}$. Now, the edges of Q_n with labels zero are the following:

$$f(v_iv_{i+1}) = 0, \quad 1 \le i \le \tfrac{n}{2} \qquad f(v_iu_i) = 0, \quad 1 \le i \le \tfrac{n}{2}$$
$$f(v_{i+1}w_i) = 0, \quad 1 \le i \le \tfrac{n}{2} \qquad f(u_iw_i) = 0, \quad 2 \le i \le \tfrac{n}{2}$$

As a result of these edge labels, we have $e_f(0) = 2n - 1$ and consequently, $e_f(1) = 2n - 3$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-4+4n-2-3n-4n+6}{2}\right| = 0$. Thus, Q_n is total product cordial for all $n \ge 4$, n is even. Considering all the cases presented above, we have shown that Q_n is total product cordial for all $n \ge 3$. □

Theorem 8: The Alternate Quadrilateral Snake Graph $A(Q_n)$ is total product cordial for all $n \ge 3$.

Proof: To prove the theorem, we consider the following cases:
Case 1: n is odd, $n \ge 3$.

Let $V(A(Q_n)) = \{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{w_i | 1 \le i \le \frac{n-1}{2}\}$ and $E(A(Q_n)) = \{v_iv_{i+1} | 1 \le i \le n-1\} \cup \{v_{2i-1}u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i}w_i | 1 \le i \le \frac{n-1}{2}\} \cup \{u_iw_i | 1 \le i \le \frac{n-1}{2}\}$. Thus, the order and size of $A(Q_n)$ is $2n - 1$ and $\frac{5n-5}{2}$, respectively.
Subcase 1.1: $n = 3$.
Let $f : V(A(Q_3)) \to \{0, 1\}$ be the function defined by:

$$f(v_1) = f(v_2) = f(v_3) = 1 \text{ and } f(u_1) = f(w_1) = 0.$$

Then the induced edge labeling would be

$$f(v_1v_2) = 1; \quad f(v_2v_3) = 1; \quad f(v_1u_1) = 0; \quad f(v_2w_1) = 0; \quad f(u_1w_1) = 0.$$

Observe that $v_f(0) = 2$, $v_f(1) = 3$, $e_f(0) = 3$, and $e_f(1) = 2$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |2 + 3 - 3 + 2| = 0$. Thus, $A(Q_3)$ is total product cordial.
Subcase 1.2: $n \equiv 1 \pmod 4$, $n \ge 5$.
Let $f : V(A(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n-1$ and $v_f(1) = n$. Now, the edges of $A(Q_n)$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, \qquad 1 \le i \le \frac{n-1}{2}; \\ f(v_{2i-1}u_i) &= 0, \qquad 1 \le i \le \frac{n-1}{4}; \\ f(v_{2i}w_i) &= 0, \qquad 1 \le i \le \frac{n-1}{4}; \\ f(u_iw_i) &= 0, \qquad 1 \le i \le \frac{n-1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{5n-5}{4}$ and consequently, $e_f(1) = \frac{5n-5}{4}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |\frac{4n-4+5n-5-4n-5n+5}{4}| = 1$. Thus, the $A(Q_n)$ is total product cordial for all $n \equiv 1 \pmod 4$, $n \ge 5$.

Subcase 1.3: $n \equiv 3 \pmod 4$, $n \ge 7$.

Let $f : V(A(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 1, & 1 \le i \le \frac{n+1}{2} \\ 0, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 1, & 2 \le i \le \frac{n+1}{4} \\ 0, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 1, & 1 \le i \le \frac{n+1}{4} \\ 0, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n-1$ and $v_f(1) = n$. Now, the edges of $A(Q_n)$ with labels one are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, \qquad 1 \le i \le \frac{n-1}{2}; \\ f(v_{2i-1}u_i) &= 0, \qquad 1 \le i \le \frac{n+1}{4}; \\ f(v_{2i}w_i) &= 0, \qquad 1 \le i \le \frac{n+1}{4}; \\ f(u_iw_i) &= 0, \qquad 1 \le i \le \frac{n+1}{4}. \end{aligned}$$

As a result these edge labels, we have $e_f(1) = \frac{5n-7}{4}$ and consequently, $e_f(0) = \frac{5n-3}{4}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |\frac{4n-4+5n-3-4n-5n+7}{4}| = 0$. Thus, the $A(Q_n)$ is total product cordial for all $n \equiv 3 \pmod 4$, $n \ge 7$. By Subcases 1.1, 1.2, and 1.3, $A(Q_n)$ is total product cordial when n is odd, $n \ge 3$.

Case 2: n is even, $n \ge 4$.

Let $V(A(Q_n)) = \{v_i|1 \le i \le n\} \cup \{u_i|1 \le i \le \frac{n}{2}\} \cup \{w_i|1 \le i \le \frac{n}{2}\}$ and $E(A(Q_n)) = \{v_iv_{i+1}|1 \le i \le n-1\} \cup \{v_{2i-1}u_i|1 \le i \le \frac{n}{2}\} \cup \{v_{2i}w_i|1 \le i \le \frac{n}{2}\} \cup \{u_iw_i|1 \le i \le \frac{n}{2}\}$. Thus, the order and size of $A(Q_n)$ is $2n$ and $\frac{5n-2}{2}$, respectively.

Subcase 2.1: $n \equiv 0 \pmod 4$, $n \ge 4$.

Let $f : V(A(Q_n)) \to \{0,1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n$ and $v_f(1) = n$. Now, the edges of $A(Q_n)$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, & 1 \le i \le \frac{n}{2}; \\ f(v_{2i-1}u_i) &= 0, & 1 \le i \le \frac{n}{4}; \\ f(v_{2i}w_i) &= 0, & 1 \le i \le \frac{n}{4}; \\ f(u_iw_i) &= 0, & 1 \le i \le \frac{n}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{5n}{4}$ and consequently, $e_f(1) = \frac{5n-4}{4}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{4n+5n-4n-5n+4}{4}\right| = 1$. Thus, the $A(Q_n)$ is total product cordial for all $n \equiv 0 \pmod 4$, $n \ge 4$.

Subcase 2.2: $n \equiv 2 \pmod 4$, $n \ge 6$.

Let $f : V(A(Q_n)) \to \{0,1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 2 \le i \le \frac{n+2}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 2 \le i \le \frac{n+2}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 2 \le i \le \frac{n+2}{4} \\ 1, & otherwise \end{cases}$$

From the established labeling, we have $v_f(0) = n-1$ and $v_f(1) = n+1$. Now, the edges of $A(Q_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, \qquad 1 \leq i \leq \frac{n+2}{2}; \\ f(v_{2i-1} u_i) &= 0, \qquad 1 \leq i \leq \frac{n+2}{4}; \\ f(v_{2i} w_i) &= 0, \qquad 1 \leq i \leq \frac{n}{4}; \\ f(u_i w_i) &= 0, \qquad 1 \leq i \leq \frac{n}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{5n+2}{4}$ and consequently, $e_f(1) = \frac{5n-6}{4}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{4n-4+5n+2-4n-4-5n+6}{4}\right| = 0$. Thus, $A(Q_n)$ is total product cordial for all $n \equiv 2 \pmod 4$, $n \geq 6$. By Subcases 2.1 and 2.2, $A(Q_n)$ is total product cordial when $n \geq 4$, n is even. Considering all the cases presented above, we have shown that $A(Q_n)$ is total product cordial for all $n \geq 3$. □

Theorem 9: The Double Quadrilateral Snake Graph $D(Q_n)$ is total product cordial for all $n \geq 2$.

Proof: Let $V(D(Q_n)) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq n-1\} \cup \{w_i | 1 \leq i \leq n-1\} \cup \{x_i | 1 \leq i \leq n-1\} \cup \{y_i | 1 \leq i \leq n-1\}$ and $E(D(Q_n)) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i | 1 \leq i \leq n-1\} \cup \{v_{i+1} x_i | 1 \leq i \leq n-1\} \cup \{u_i x_i | 1 \leq i \leq n-1\} \cup \{v_i w_i | 1 \leq i \leq n-1\} \cup \{v_{i+1} y_i | 1 \leq i \leq n-1\} \cup \{w_i y_i | 1 \leq i \leq n-1\}$. Thus, the order and size of $D(Q_n)$ is $5n-4$ and $7n-7$, respectively.
To show that $D(Q_n)$ is total product cordial, we consider the following cases:
Case 1: n is even, $n \geq 2$.
Subcase 1.1: $n = 2$.
Observe that $D(Q_2) \cong L_3$. By Corollary 1, $D(Q_2)$ is total product cordial.
Subcase 1.2: n is even, $n \geq 4$.
Let $f : V(D(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$
$$f(x_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$
$$f(y_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n}{2} \\ 1, & otherwise \end{cases}$$

From the labeling specified above, we have $v_f(0) = \frac{5n-8}{2}$ and $v_f(1) = \frac{5n}{2}$. Now, the edges of $D(Q_n)$ with labels zero are the following:

$$\begin{aligned}
f(v_iv_{i+1}) &= 0, \qquad 1 \le i \le \frac{n}{2};\\
f(v_iu_i) &= 0, \qquad 1 \le i \le \frac{n}{2};\\
f(v_{i+1}x_i) &= 0, \qquad 1 \le i \le \frac{n}{2};\\
f(u_ix_i) &= 0, \qquad 2 \le i \le \frac{n}{2};\\
f(v_iw_i) &= 0. \qquad 1 \le i \le \frac{n}{2};\\
f(v_{i+1}y_i) &= 0, \qquad 1 \le i \le \frac{n}{2};\\
f(w_iy_i) &= 0, \qquad 2 \le i \le \frac{n}{2}.
\end{aligned}$$

As a result of these edge labels, we have $e_f(0) = \frac{7n-4}{2}$ and consequently, $e_f(1) = \frac{7n-10}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{5n-8+7n-4-5n-7n+10}{2}\right| = 1$. Thus, $D(Q_n)$ is total product cordial for all $n \ge 4$, n is even. By Subcases 1.1 and 1.2, $D(Q_n)$ is total product cordial when n is even, $n \ge 2$.
Case 2: n is odd, $n \ge 3$.
Let $f : V(D(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(x_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$
$$f(y_i) = \begin{cases} 0, & 1 \le i \le \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

From the labeling specified above, we have $v_f(0) = \frac{5n-5}{2}$ and $v_f(1) = \frac{5n-3}{2}$. Now, the edges of $D(Q_n)$ with labels zero are the following:

$$
\begin{aligned}
f(v_iv_{i+1}) &= 0, \qquad 1 \le i \le \frac{n-1}{2};\\
f(v_iu_i) &= 0, \qquad 1 \le i \le \frac{n-1}{2};\\
f(v_{i+1}x_i) &= 0, \qquad 1 \le i \le \frac{n-1}{2};\\
f(u_ix_i) &= 0, \qquad 2 \le i \le \frac{n-1}{2};\\
f(v_iw_i) &= 0. \qquad 1 \le i \le \frac{n-1}{2};\\
f(v_{i+1}y_i) &= 0, \qquad 1 \le i \le \frac{n-1}{2};\\
f(w_iy_i) &= 0, \qquad 2 \le i \le \frac{n-1}{2}.
\end{aligned}
$$

As a result of these edge labels, we have $e_f(0) = \frac{7n-7}{2}$ and consequently, $e_f(1) = \frac{7n-7}{2}$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{5n-5+7n-7-5n+3-7n+7}{2}\right| = 1$. Thus, $D(Q_n)$ is total product cordial for all $n \ge 3$, n is odd. Considering all the cases presented above, we have shown that $D(Q_n)$ is total product cordial for all $n \ge 2$. □

Theorem 10: The Double Alternate Quadrilateral Snake Graph $DA(Q_n)$ is total product cordial for all $n \ge 2$.

Proof: To prove the theorem, we consider the following cases:
Case 1: n is even, $n \ge 2$.

Let $V(DA(Q_n)) = \{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le \frac{n}{2}\} \cup \{w_i | 1 \le i \le \frac{n}{2}\} \cup \{x_i | 1 \le i \le \frac{n}{2}\} \cup \{y_i | 1 \le i \le \frac{n}{2}\}$ and $E(DA(Q_n)) = \{v_iv_{i+1} | 1 \le i \le n-1\} \cup \{v_{2i-1}u_i | 1 \le i \le \frac{n}{2}\} \cup \{v_{2i}x_i | 1 \le i \le \frac{n}{2}\} \cup \{u_ix_i | 1 \le i \le \frac{n}{2}\} \cup \{v_{2i-1}w_i | 1 \le i \le \frac{n}{2}\} \cup \{v_{2i}y_i | 1 \le i \le \frac{n}{2}\} \cup \{w_iy_i | 1 \le i \le \frac{n}{2}\}$. Thus, the order and size of $DA(Q_n)$ is $3n$ and $4n-1$, respectively.

Subcase 1.1: $n = 2$.

Observe that $DA(Q_2) \cong L_3$. By Corollary 2.2, $D(Q_2)$ is total product cordial.

Subcase 1.2: $n \equiv 0 \pmod 4$, $n \ge 4$.

Let $f : V(DA(Q_n)) \to \{0, 1\}$ be the function defined by:

$$
f(v_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{2} \\ 1, & otherwise \end{cases}
$$

$$
f(u_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{4} \\ 1, & otherwise \end{cases}
$$

$$
f(x_i) = \begin{cases} 0, & 1 \le i \le \frac{n}{4} \\ 1, & otherwise \end{cases}
$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{4} \\ 1, & otherwise \end{cases}$$
$$f(y_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n}{4} \\ 1, & otherwise \end{cases}$$

From the labeling specified above, we have $v_f(0) = \frac{3n}{2}$ and $v_f(1) = \frac{3n}{2}$. Now, the edges of $DA(Q_n)$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, & 1 \leq i \leq \frac{n}{2}; \\ f(v_{2i-1}u_i) &= 0, & 1 \leq i \leq \frac{n}{4}; \\ f(v_{2i}x_i) &= 0, & 1 \leq i \leq \frac{n}{4}; \\ f(u_ix_i) &= 0, & 1 \leq i \leq \frac{n}{4}; \\ f(v_{2i-1}w_i) &= 0, & 1 \leq i \leq \frac{n}{4}; \\ f(v_{2i}y_i) &= 0, & 1 \leq i \leq \frac{n}{4}; \\ f(w_iy_i) &= 0, & 1 \leq i \leq \frac{n}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = 2n$ and consequently, $e_f(1) = 2n - 1$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n+4n-3n-4n+2}{2}\right| = 1$. Thus, $DA(Q_n)$ is total product cordial for all $n \equiv 0 \pmod 4$, $n \geq 4$.

Subcase 1.3 $n \equiv 2 \pmod 4$, $n \geq 6$.

Let $f : V(DA(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 3 \leq i \leq \frac{n+2}{2} \\ 1, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n+2}{4} \\ 1, & otherwise \end{cases}$$
$$f(x_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+2}{4} \\ 1, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n+2}{4} \\ 1, & otherwise \end{cases}$$
$$f(y_i) = \begin{cases} 0, & 2 \leq i \leq \frac{n+2}{4} \\ 1, & otherwise \end{cases}$$

From the labeling specified above, we have $v_f(0) = \frac{3n-2}{2}$ and $v_f(1) = \frac{3n+2}{2}$. Now, the edges of $DA(Q_n)$ with labels zero are the following:

$$\begin{aligned} f(v_iv_{i+1}) &= 0, && 1 \le i \le \frac{n+2}{2};\\ f(v_{2i-1}u_i) &= 0, && 1 \le i \le \frac{n+2}{4};\\ f(v_{2i}x_i) &= 0, && 1 \le i \le \frac{n+2}{4};\\ f(u_ix_i) &= 0, && 1 \le i \le \frac{n+2}{4};\\ f(v_{2i-1}w_i) &= 0, && 1 \le i \le \frac{n+2}{4};\\ f(v_{2i}y_i) &= 0, && 1 \le i \le \frac{n+2}{4};\\ f(w_iy_i) &= 0, && 1 \le i \le \frac{n+2}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = 2n+1$ and consequently, $e_f(1) = 2n-2$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-2+4n+2-3n-2-4n+4}{2}\right| = 0$. Thus, $DA(Q_n)$ is total product cordial for all $n \equiv 2 \pmod 4$, $n \ge 6$. By Subcases 1.1, 1.2, and 1.3, it follows that $DA(T_n)$ is total product cordial when n is even, $n \ge 2$.

Case 2: n is odd, $n \ge 3$.

Let $V(DA(Q_n)) = \{v_i | 1 \le i \le n\} \cup \{u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{w_i | 1 \le i \le \frac{n-1}{2}\} \cup \{x_i | 1 \le i \le \frac{n-1}{2}\} \cup \{y_i | 1 \le i \le \frac{n-1}{2}\}$ and $E(DA(Q_n)) = \{v_iv_{i+1} | 1 \le i \le n-1\} \cup \{v_{2i-1}u_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i}x_i | 1 \le i \le \frac{n-1}{2}\} \cup \{u_ix_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i-1}w_i | 1 \le i \le \frac{n-1}{2}\} \cup \{v_{2i}y_i | 1 \le i \le \frac{n-1}{2}\} \cup \{w_iy_i | 1 \le i \le \frac{n-1}{2}\}$. Thus, the order and size of $DA(Q_n)$ is $3n-2$ and $4n-4$, respectively.

Subcase 2.1: $n = 3$.

Let $f : V(DA(Q_3)) \to \{0, 1\}$ be the function defined by:

$$f(v_1) = f(v_2) = f(x_1) = f(y_1) = 1 \text{ and } f(v_3) = f(u_1) = f(w_1) = 0.$$

Now, the induced edge labeling would be:

$$\begin{aligned} f(v_1v_2) &= 1;\\ f(v_2v_3) &= 0;\\ f(v_1u_1) &= 0;\\ f(v_1w_1) &= 0;\\ f(v_2x_1) &= 1;\\ f(v_2y_1) &= 1;\\ f(u_1x_1) &= 0;\\ f(w_1y_1) &= 0. \end{aligned}$$

Observe that $v_f(0) = 3$, $v_f(1) = 4$, $e_f(0) = 5$, and $e_f(1) = 3$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = |3 + 5 - 4 - 3| = |-1| = 1$. Thus, $DA(Q_3)$ is a total product cordial graph.

Subcase 2.2: $n \equiv 1 \pmod 4$, $n \geq 5$.

Let $f : V(DA(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{2} \\ 1, & otherwise \end{cases}$$

$$f(u_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

$$f(x_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

$$f(w_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

$$f(y_i) = \begin{cases} 0, & 1 \leq i \leq \frac{n-1}{4} \\ 1, & otherwise \end{cases}$$

From the labeling specified above, we have $v_f(0) = \frac{3n-3}{2}$ and $v_f(1) = \frac{3n-1}{2}$. Now, the edges of $DA(Q_n)$ with labels zero are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, & 1 \leq i \leq \frac{n-1}{2}; \\ f(v_{2i-1} u_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i} x_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}; \\ f(u_i x_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i-1} w_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}; \\ f(v_{2i} y_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}; \\ f(w_i y_i) &= 0, & 1 \leq i \leq \frac{n-1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(0) = 2n - 2$. and consequently, $e_f(1) = 2n - 2$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-3+4n-4-3n+1-4n+4}{2}\right| = 1$. Thus, the $DA(Q_n)$ is total product cordial for all $n \equiv 1 \pmod 4$, $n \geq 5$.

Subcase 2.3: $n \equiv 3 \pmod 4$, $n \geq 7$.

Let $f : V(DA(Q_n)) \to \{0, 1\}$ be the function defined by:

$$f(v_i) = \begin{cases} 1, & 1 \leq i \leq \frac{n+1}{2} \\ 0, & otherwise \end{cases}$$
$$f(u_i) = \begin{cases} 1, & 2 \leq i \leq \frac{n+1}{4} \\ 0, & otherwise \end{cases}$$
$$f(x_i) = \begin{cases} 1, & 1 \leq i \leq \frac{n+1}{4} \\ 0, & otherwise \end{cases}$$
$$f(w_i) = \begin{cases} 1, & 2 \leq i \leq \frac{n+1}{4} \\ 0, & otherwise \end{cases}$$
$$f(y_i) = \begin{cases} 1, & 1 \leq i \leq \frac{n+1}{4} \\ 0, & otherwise \end{cases}$$

From the labeling specified above, we have $v_f(0) = \frac{3n-3}{2}$ and $v_f(1) = \frac{3n-1}{2}$. Now, the edges of $DA(Q_n)$ with labels one are the following:

$$\begin{aligned} f(v_i v_{i+1}) &= 0, & 1 \leq i \leq \frac{n-1}{2}; \\ f(v_{2i-1} u_i) &= 0, & 2 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i} x_i) &= 0, & 1 \leq i \leq \frac{n+1}{4}; \\ f(u_i x_i) &= 0, & 2 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i-1} w_i) &= 0, & 2 \leq i \leq \frac{n+1}{4}; \\ f(v_{2i} y_i) &= 0, & 1 \leq i \leq \frac{n+1}{4}; \\ f(w_i y_i) &= 0, & 2 \leq i \leq \frac{n+1}{4}. \end{aligned}$$

As a result of these edge labels, we have $e_f(1) = 2n - 3$ and consequently, $e_f(0) = 2n - 1$. Hence, $|v_f(0) + e_f(0) - v_f(1) - e_f(1)| = \left|\frac{3n-3+4n-2-3n+1-4n+6}{2}\right| = 1$. Thus, $DA(Q_n)$ is total product cordial for all $n \equiv 3 \pmod 4$, $n \geq 7$. By Subcases 2.1, 2.2, and 2.3, it follows that $DA(Q_n)$ is a total product cordial graph when $n \geq 3$, n is odd. Considering all the cases presented above, we have shown that $DA(Q_n)$ is total product cordial for all $n \geq 2$. □

4 Conclusion

In conclusion, the study on total product cordial labeling of snake graphs demonstrates that this class of graphs admits structured and systematic labeling schemes that satisfy the cordiality condition. By carefully assigning binary labels to vertices and edges, the induced product labeling achieves a near-balanced distribution, highlighting the compatibility of snake graph structures with total product cordial labeling.

These results not only extend existing work on cordial-type labelings but also suggest that other path-related and recursively constructed graphs may exhibit similar properties, opening opportunities for further exploration and generalization in graph labeling theory.

For future work, researchers may consider extending total product cordial labeling to other families of graphs such as triangular snakes, ladder graphs, caterpillar graphs, and generalized snake graphs. Further investigations may also focus on determining necessary and sufficient conditions for graphs to admit total product cordial labelings, as well as studying the behavior of this labeling under various graph operations including union, join, corona, and graph products. In addition, exploring algorithmic approaches and computational methods for constructing total product cordial labelings in larger and more complex graph classes may provide deeper insights and broader applications in graph theory and combinatorial mathematics.

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