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Research article

On a new logarithmic modification of the Hilbert integral inequality

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Abstract: The Hilbert integral inequality is a well-known and widely studied result in analysis that has inspired many refinements and modifications. In this paper, we present a new logarithmic modification of this inequality. Our approach is based on a trigonometric method that offers a fresh perspective on existing standard techniques. As a consequence, we also derive another integral inequality. All arguments are presented in full detail.

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1 Introduction

The Hilbert integral inequality is a classical result in analysis that gives a bound for a double integral involving two functions. It plays an important role in the study of L^2 spaces. A formal statement of this inequality is given below. Let $f, g : [0, +\infty) \to [0, +\infty)$ be measurable functions such that $f, g \in L^2([0, +\infty))$, which is the set of all (measurable) functions defined on the interval $[0, +\infty)$ whose square is integrable. Then, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \le \pi \sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}.$$

Thus, the double integral is controlled by the L^2 norms of f and g, with the constant factor π. This constant is optimal and cannot be reduced. Moreover, equality does not occur unless one of the functions is zero almost everywhere. See [6,13]. This inequality has been refined and modified in many directions. See, for example, [1,3,4,9–11,14]. Furthermore, the survey in [2] offers a helpful overview of several of these developments.

Several logarithmic modifications of the Hilbert integral inequality have been developed. The best-known versions, which do not involve any free parameters, are listed below.

- The logarithmic modification in [6, Formula 342] is given by

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x-y}\log\left(\frac{x}{y}\right)f(x)g(y)dxdy$$
$$\leq \pi^2\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

- In [12], the following logarithmic integral inequality is demonstrated:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(x,y)}\left|\log\left(\frac{x}{y}\right)\right|f(x)g(y)dxdy$$
$$\leq 8\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

- Another modification is given in [7], which states that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+|x-y|}\left|\log\left(\frac{x}{y}\right)\right|f(x)g(y)dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

- In [3], the following logarithmic modification is established:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(1+\frac{x}{y}\right)f(x)g(y)dxdy$$
$$\leq 2\pi\log 2\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

In this paper, we continue this line of research by investigating a new logarithmic modification of the Hilbert integral inequality. Beyond its novelty, this approach is motivated by its particular tractability. It is based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{x+y}{\sqrt{xy}}\right)f(x)g(y)dxdy. \tag{1}$$

Note that, since $x+y\geq 2\sqrt{xy}$ for any $x,y>0$, we have

$$\log\left(\frac{x+y}{\sqrt{xy}}\right)\geq\log(2)>0.$$

Therefore, the integrand is non-negative, which allows us to make the standard assumption for the double integral. More specifically, we demonstrate that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{x+y}{\sqrt{xy}}\right)f(x)g(y)dxdy$$
$$\leq 2\pi\log 2\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

The proof is original in its adoption of a trigonometric approach inspired by that in [8]. The constant factor $2\pi\log 2$ arises naturally from the use of two entries in [5]. However, it is not proved to be optimal.

Consequently, another related integral inequality is obtained based on the following integral:

$$\int_0^{+\infty}\int_0^{+\infty}\left(\frac{1}{x+y}\log\left(\frac{x+y}{\sqrt{xy}}\right)f(x)dx\right)^2dy.$$

All arguments are presented in full detail.

The rest of the paper is organized as follows: The results are stated and proved in Section 2. Section 3 contains a conclusion.

2 Results

2.1 Main result

Our logarithmic modification of the Hilbert integral inequality is presented below, followed by the detailed proof.

Theorem 1: *Let $f,g:[0,+\infty)\to[0,+\infty)$ be measurable functions such that $f,g\in L^2([0,+\infty))$. Then, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{x+y}{\sqrt{xy}}\right)f(x)g(y)dxdy$$
$$\leq 2\pi\log 2\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(y)dy}.$$

Proof: For simplicity, we introduce

$$I=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{x+y}{\sqrt{xy}}\right)f(x)g(y)dxdy.$$

Making the change of variables (substitution)

$$x=u^2,\quad y=v^2,\quad u>0,\ v>0,$$

so that $dx = 2udu$ and $dy = 2vdv$, we obtain

$$I = 4\int_0^{+\infty}\int_0^{+\infty}\frac{uv}{u^2+v^2}\log\left(\frac{u^2+v^2}{uv}\right)f(u^2)g(v^2)dudv.$$

Now, making the polar change of variables

$$u = r\cos\theta, \quad v = r\sin\theta, \quad r > 0,\ \theta \in \left(0, \frac{\pi}{2}\right),$$

so that the Jacobian is given by $dudv = rdrd\theta$, we get

$$\frac{uv}{u^2+v^2} = \frac{r^2\cos\theta\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$$

and

$$\log\left(\frac{u^2+v^2}{uv}\right) = \log\left(\frac{r^2\cos^2\theta + r^2\sin^2\theta}{r^2\cos\theta\sin\theta}\right) = \log\left(\frac{r^2}{r^2\cos\theta\sin\theta}\right)$$
$$= -\log(\cos\theta\sin\theta).$$

Therefore, we have

$$I = 4\int_0^{\pi/2}\cos\theta\sin\theta(-\log(\cos\theta\sin\theta))$$
$$\times\int_0^{+\infty} f(r^2\cos^2\theta)g(r^2\sin^2\theta)rdrd\theta. \tag{2}$$

Fixing θ and applying the Cauchy-Schwarz integral inequality, we get

$$\int_0^{+\infty} f(r^2\cos^2\theta)g(r^2\sin^2\theta)rdr$$
$$\leq \sqrt{\int_0^{+\infty} f^2(r^2\cos^2\theta)rdr}\sqrt{\int_0^{+\infty} g^2(r^2\sin^2\theta)rdr}.$$

Let us now compute each integral. Making the change of variables $s = r^2\cos^2\theta$, so that $ds = 2r\cos^2\theta dr$, we obtain

$$\int_0^{+\infty} f(r^2\cos^2\theta)^2 rdr = \frac{1}{2\cos^2\theta}\int_0^{+\infty} f^2(s)ds.$$

Similarly, making the change of variables $t = r^2\sin^2\theta$, so that $dt = 2r\sin^2\theta dr$, we get

$$\int_0^{+\infty} g(r^2\sin^2\theta)^2 rdr = \frac{1}{2\sin^2\theta}\int_0^{+\infty} g^2(t)dt.$$

Therefore, we have

$$\int_0^{+\infty} f(r^2\cos^2\theta)g(r^2\sin^2\theta)rdr \leq \frac{1}{2\cos\theta\sin\theta}\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}. \quad (3)$$

It follows from Equations (2) and (3) that

$$I \leq 2\left(\int_0^{\pi/2}(-\log(\cos\theta\sin\theta))d\theta\right)\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy}.$$

Finally, using the classical identities [5, Entries 4.2243 and 4.2246], i.e.,

$$\int_0^{\pi/2}\log(\sin\theta)d\theta = \int_0^{\pi/2}\log(\cos\theta)d\theta = -\frac{\pi}{2}\log 2,$$

we obtain

$$\int_0^{\pi/2}(-\log(\cos\theta\sin\theta))d\theta = -\int_0^{\pi/2}\log(\sin\theta)d\theta - \int_0^{\pi/2}\log(\cos\theta)d\theta = \pi\log 2.$$

Therefore, we have

$$I \leq 2\pi\log 2\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(y)dy},$$

as desired. This concludes the proof. □

To the best of our knowledge, this result is new in the existing literature on Hilbert-type integral inequalities. Additionally, the method of proof may provide valuable insights for developing further extensions beyond the scope of this work. As mentioned previously, the constant factor $2\pi\log 2$ is not proved to be optimal, which raises an interesting question for future research.

2.2 Secondary result

A non-trivial consequence of Theorem 1 is the theorem below, presenting a new integral inequality of the logarithmic type.

Theorem 2: *Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that $f \in L^2([0, +\infty))$. Then, we have*

$$\int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{x+y} \log\left(\frac{x+y}{\sqrt{xy}} \right) f(x) dx \right)^2 dy$$
$$\leq (2\pi \log 2)^2 \int_0^{+\infty} f^2(x) dx.$$

Proof: For simplicity, we introduce

$$J = \int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{x+y} \log\left(\frac{x+y}{\sqrt{xy}} \right) f(x) dx \right)^2 dy.$$

We can write

$$J = \int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} \log\left(\frac{x+y}{\sqrt{xy}} \right) f(x) g_\star(y) dx dy,$$

where

$$g_\star(y) = \int_0^{+\infty} \frac{1}{x+y} \log\left(\frac{x+y}{\sqrt{xy}} \right) f(x) dx.$$

Applying Theorem 1 to the functions f and $g_\star$ and recognizing J in the upper bound, we obtain

$$J \leq 2\pi \log 2 \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} g_\star^2(y) dy}$$
$$= 2\pi \log 2 \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{x+y} \log\left(\frac{x+y}{\sqrt{xy}} \right) f(x) dx \right)^2 dy}$$
$$= 2\pi \log 2 \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{J}.$$

This implies that

$$\sqrt{J} \leq 2\pi \log 2 \sqrt{\int_0^{+\infty} f^2(x) dx},$$

so that

$$J \leq (2\pi \log 2)^2 \int_0^{+\infty} f^2(x) dx,$$

as desired. This concludes the proof. □

To the best of our knowledge, this result is new in the existing literature on logarithmic-type integral inequalities.

3 Conclusion

In this paper, we established a new logarithmic modification of the Hilbert integral inequality, developed using a trigonometric approach and in accordance with the principles outlined in [8]. This method provides an alternative framework that could be applicable to other classes of integral inequalities.

Future work will involve determining the optimal constant and exploring further generalizations, particularly in higher dimensions or under different kernel function structures. In addition, it would be of interest to investigate weighted versions of the inequality, as well as discrete analogues and operator-theoretic formulations. Another natural direction is to study the stability and sharpness of the obtained bounds, and to identify extremal functions when they exist. Extensions to multilinear settings or to inequalities involving fractional integrals may also yield fruitful results. Finally, potential applications to related areas such as harmonic analysis, partial differential equations, and approximation theory merit further exploration.

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