Annals of Communications in Mathematics
Volume 9, Number 2 (2026), 5
DOI: 10.62072/acm.2026.09021
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

Research article

Distinguishing Numbers for Cartesian Powers and Wreath Products

Salihu Lawan Aliyu [1], **Nasiru Mohammed Mangga** [2]

[1,2]Department of Mathematics and Computer Science Kashim Ibrahim University Maiduguri, Nigeria.

[*]Corresponding Authors: Salihu Lawan Aliyu and Nasiru Mohammed Mangga. Email: salihu.lawana@kiu.edu.ng & nasiru.mangga@kiu.edu.ng

Received: 06 March 2026; Accepted: 01 May 2026; Published: 01 June 2026

Citation: S.L. Aliyu and N. M. Mangga, Distinguishing Numbers for Cartesian Powers and Wreath Products. Ann. Commun. Math. 9 **(2026)**, 5. https://doi.org/10.62072/acm.2026.09021

Abstract: In this paper we investigates distinguishing numbers of finite permutation group actions, with particular emphasis on Cartesian powers and wreath product constructions. For transitive actions, we establish that the distinguishing number of Cartesian powers grows at most linearly with the dimension, thereby improving the classical exponential bound. We further obtain exact values in several important cases. In particular, for regular actions, the distinguishing number is shown to be 2 for all Cartesian powers, while for primitive non-regular actions we prove that $D(G, X) = b(G) + 1$, where $b(G)$ denotes the base size. For wreath product actions, we derive general bounds expressed in terms of the component groups and demonstrate that these bounds are sharp for significant families, including symmetric groups. Overall, our results provide a clearer understanding of the interplay between distinguishing numbers, base size, and the structural properties of permutation groups.

Mathematics Subject Classification: 20B05; 05E18, 05C15, 20B15, 20E22
Keywords: Distinguishing number, permutation group action, wreath product, Cartesian power, symmetry breaking, Base size

1 INTRODUCTION

Symmetry is a central concept in mathematics, with deep connections to combinatorics, group theory, graph theory, and theoretical computer science. A fundamental problem in the study of symmetry is to determine how much information is required to destroy all nontrivial symmetries of a given structure. This idea is formalized through the notion of the *distinguishing number*, which provides a quantitative measure of symmetry breaking. The concept of the distinguishing number was introduced by Albertson and Collins [1] in the context of graph automorphisms. Given a graph, a labeling of its vertices is said to be distinguishing if no nontrivial automorphism preserves all labels. The minimum number of labels required for such a labeling is called the distinguishing number. This notion has since been extended to permutation group actions by Tymoczko [15], where it naturally arises as an invariant of a group action. Let G be a finite group acting faithfully on a finite set X, denoted by $G \curvearrowright X$. A labeling is a function

$$\phi : X \to \{1, 2, \dots, r\}$$

where $r \in \mathbb{N}$. The labeling ϕ is said to be *distinguishing* if the identity element is the only element $g \in G$ satisfying

$$\phi(g \cdot x) = \phi(x) \quad \text{for all } x \in X.$$

The smallest such r is called the *distinguishing number* of the action and is denoted by $D(G, X)$. A closely related invariant is the *base size* of the action. A subset $B \subseteq X$ is called a base if the pointwise stabilizer

$$\bigcap_{x \in B} G_x = \{1\}$$

where $G_x = \{g \in G : g \cdot x = x\}$ denotes the stabilizer of x. The minimum size of such a set is called the base size and is denoted by $b(G)$. It is well known that these invariants are related by the inequality

$$D(G, X) \leq b(G) + 1$$

highlighting the strong connection between symmetry breaking and structural properties of permutation groups. The study of distinguishing numbers has expanded significantly in recent years. Various authors have investigated distinguishing numbers for graphs, permutation groups, and their products (see [3, 6,7,11,13]). In particular, product constructions such as Cartesian powers and wreath products play an important role in both combinatorics and group theory, yet their distinguishing numbers remain only partially understood. Let $k \geq 1$ be an integer. The *k-fold Cartesian power* of X is defined by

$$X^k = X \times X \times \cdots \times X$$

where an element is written as $(x_1, x_2, \dots, x_k)$ with $x_i \in X$. The group G acts on X^k coordinatewise via

$$g \cdot (x_1, \dots, x_k) = (g \cdot x_1, \dots, g \cdot x_k).$$

Understanding how $D(G, X^k)$ behaves as k grows is a fundamental problem. Classical bounds show that $D(G, X^k) \leq (D(G, X))^k$, which is exponential in k, but this estimate is often far from optimal. Another important construction is the *wreath product*. For a positive integer m, let S_m denote the symmetric group on $\{1, \dots, m\}$. The wreath product is defined by

$$G \wr S_m = G^m \rtimes S_m,$$

where G^m is the direct product of m copies of G, and S_m acts by permuting coordinates. Elements of $G \wr S_m$ have the form

$$(g_1, g_2, \dots, g_m; \sigma)$$

with $g_i \in G$ and $\sigma \in S_m$. The natural product action on X^m is given by

$$(g_1, \dots, g_m; \sigma) \cdot (x_1, \dots, x_m) = (g_{\sigma^{-1}(1)} \cdot x_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(m)} \cdot x_{\sigma^{-1}(m)}).$$

Recent work has made progress in understanding these constructions. In particular, Aliyu et al. [2] investigated distinguishing labeling for Cartesian powers and wreath product actions, establishing structural bounds in terms of orbit structure, stabilizers, and base size. Their results provide general exponential bounds and highlight the role of group-theoretic parameters in controlling symmetry breaking. Motivated by these developments, the present paper refines and extends this line of research. We improve classical bounds for Cartesian powers by showing that, for transitive actions, the distinguishing number grows at most linearly with the dimension. We also determine exact values in key cases, including regular and primitive actions, and derive sharp bounds for wreath product actions in terms of the component groups.

2 PRELIMINARIES

In this section, we recall the basic definitions and results on permutation group actions, distinguishing numbers, and related constructions used throughout the paper. Standard references include [5,8].

2.1 Permutation Group Actions

Let G be a finite group acting on a finite set X. We write this action as

$$G \times X \to X, \quad (g, x) \mapsto g \cdot x.$$

The action is said to be *faithful* if

$$g \cdot x = x \text{ for all } x \in X \implies g = 1.$$

For each $x \in X$, the *stabilizer* of x is

$$G_x = \{g \in G : g \cdot x = x\}$$

and the *orbit* of x is

$$G \cdot x = \{g \cdot x : g \in G\}.$$

The action is called:

- *transitive* if there is a single orbit;

- *regular* if it is transitive and $|G_x| = 1$ for all $x \in X$;
- *primitive* if it preserves no nontrivial partition (block system) of X.

We refer to [5,8] for further background.

2.2 Distinguishing Number

Let G act faithfully on X. A labeling is a function

$$\varphi : X \to \{1, 2, \dots, r\}.$$

The labeling φ is said to be *distinguishing* if

$$\varphi(g \cdot x) = \varphi(x) \text{ for all } x \in X \implies g = 1.$$

The *distinguishing number* of the action is

$$D(G, X) = \min\{r : \text{there exists a distinguishing labeling } \varphi\}.$$

This concept was introduced for graphs by Albertson and Collins [**?**] and extended to group actions by Tymoczko [15]. See also [6,12] for further developments.

2.3 Base Size

A subset $B \subseteq X$ is called a *base* if its pointwise stabilizer is trivial:

$$\bigcap_{x \in B} G_x = \{1\}.$$

The *base size* of G is defined as

$$b(G) = \min\{|B| : B \text{ is a base}\}.$$

A fundamental inequality relating base size and distinguishing number is

$$D(G, X) \leq b(G) + 1,$$

see [3,5].

2.4 Cartesian Powers

For $k \geq 1$, the k-fold Cartesian power of X is

$$X^k = \underbrace{X \times \cdots \times X}_{k \text{ times}}.$$

An element of X^k is written as $x = (x_1, \dots, x_k)$. The group G acts componentwise:

$$g \cdot (x_1, \dots, x_k) = (g \cdot x_1, \dots, g \cdot x_k).$$

A general bound for distinguishing numbers is

$$D(G, X^k) \le (D(G, X))^k$$

obtained by labeling coordinates independently; see [9,10]. This bound is often not sharp.

2.5 Symmetric Groups

Let S_n denote the symmetric group on $\{1, \dots, n\}$. It is well known that

$$D(S_n, [n]) = \begin{cases} 1, & n = 1, \\ 2, & n = 2, \\ n, & n \ge 3, \end{cases}$$

see [15?].

2.6 Wreath Products

Let G act on X, and let S_m be the symmetric group on $\{1, \dots, m\}$. The wreath product is

$$G \wr S_m = G^m \rtimes S_m,$$

where S_m acts on G^m by permuting coordinates. Elements of $G \wr S_m$ are written as

$$(g_1, \dots, g_m; \sigma), \quad g_i \in G,\ \sigma \in S_m.$$

The product action on X^m is given by

$$(g_1, \dots, g_m; \sigma) \cdot (x_1, \dots, x_m) = \left(g_{\sigma^{-1}(1)} \cdot x_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(m)} \cdot x_{\sigma^{-1}(m)}\right).$$

This action combines:

- *internal symmetries* from G acting on each coordinate;
- *external symmetries* from permutations of coordinates by S_m.

See [5,8] for background on wreath products.

2.7 Basic Observations

We record several facts used later:

- If G acts regularly on X, then $g \cdot x \ne x$ for all $x \in X$ and all $g \ne 1$.

- If G acts primitively and non-regularly, then point stabilizers are maximal subgroups and strongly influence both $b(G)$ and $D(G, X)$ (see [3]).
- For Cartesian powers, distinguishing numbers may grow exponentially without structural assumptions, but can be significantly smaller in structured cases [10].
- For wreath product actions, one must simultaneously break internal and external symmetries, which governs the bounds for $D(G \wr S_m, X^m)$.

3 RESULTS

In this section, we present our main results on distinguishing numbers for permutation group actions. We establish improved bounds for Cartesian powers, determine exact values for regular and primitive actions, and derive sharp bounds for wreath product actions.

Lemma 1 (Diagonal Rigidity)**:** *Let G act on a finite set X, and let*

$$\varphi : X \to \{1, \dots, r\}$$

be a distinguishing labeling. Define $\Phi : X^k \to \{1, \dots, r + k - 1\}$ such that

$$\Phi(a, \dots, a) = \varphi(a) \quad \textit{for all } a \in X.$$

If $g \in G$ preserves Φ, then g preserves φ.

Proof: Let $g \in G$ satisfy $\Phi(g \cdot x) = \Phi(x)$ for all $x \in X^k$. For any $a \in X$, consider the diagonal tuple $x = (a, \dots, a)$. Then

$$\Phi(x) = \varphi(a), \quad \Phi(g \cdot x) = \Phi(g \cdot a, \dots, g \cdot a) = \varphi(g \cdot a).$$

Hence

$$\varphi(g \cdot a) = \varphi(a) \quad \text{for all } a \in X.$$

Thus g preserves φ, and since φ is distinguishing, $g = 1$. □

Lemma 2 (First-Deviation Detection)**:** *Define $\Phi : X^k \to \{1, \dots, r + k - 1\}$ by*

$$\Phi(x_1, \dots, x_k) = \begin{cases} \varphi(x_1), & \textit{if } x_2 = \cdots = x_k = x_1, \\ r + i - 1, & \textit{where } i = \min\{j \geq 2 : x_j \neq x_1\}. \end{cases}$$

If $g \in G$ preserves Φ, then for every $x \in X^k$, the index of first deviation is preserved under g.

Proof: Let $x = (x_1, \dots, x_k)$ and suppose the first deviation occurs at index i, i.e.

$$x_2 = \cdots = x_{i-1} = x_1, \quad x_i \neq x_1.$$

Then

$$\Phi(x) = r + i - 1.$$

Since g preserves Φ,

$$\Phi(g \cdot x) = r + i - 1.$$

Thus the first index $j \geq 2$ such that $(g \cdot x)_j \neq (g \cdot x)_1$ must also be i. Hence

$$g \cdot x_j = g \cdot x_1 \text{ for } j < i, \quad g \cdot x_i \neq g \cdot x_1.$$

Therefore, the position of first deviation is preserved. □

Lemma 3: *Let G act on a finite set X, and let $\varphi : X \to \{1, \dots, r\}$ be a distinguishing labeling. Suppose $\Phi : X^k \to \{1, \dots, r + k - 1\}$ is a labeling with the property that any element $g \in G$ that preserves Φ also preserves φ. Then Φ is a distinguishing labeling of X^k.*

Proof: Let $g \in G$ be such that

$$\Phi(g \cdot x) = \Phi(x) \quad \text{for all } x \in X^k.$$

By hypothesis, g preserves φ, that is,

$$\varphi(g \cdot x) = \varphi(x) \quad \text{for all } x \in X.$$

Since φ is distinguishing, it follows that $g = 1$. Therefore, the identity is the only element of G that preserves Φ, and hence Φ is distinguishing. □

Theorem 1: *(Bounds) Let G act transitively on a finite set X and suppose $D(G, X) = r$. Then for every integer $k \geq 1$,*

$$D(G, X^k) \leq r + (k - 1).$$

Proof: Let $\phi : X \to \{1, \dots, r\}$ be a distinguishing labeling for the action of G on X. We construct a labeling

$$\Phi : X^k \to \{1, 2, \dots, r + k - 1\}$$

as follows. For $x = (x_1, \dots, x_k) \in X^k$, define

$$\Phi(x) := \begin{cases} \phi(x_1), & \text{if } x_2 = x_3 = \cdots = x_k = x_1, \\ r + i - 1, & \text{if } i = \min\{j \geq 2 : x_j \neq x_1\}. \end{cases}$$

Thus, points whose coordinates are all equal inherit the distinguishing labeling from the first coordinate, while any deviation from the diagonal is detected by the smallest index at which a coordinate differs. We claim that Φ is distinguishing. Let $g \in G$ preserve Φ, i.e.

$$\Phi(g \cdot x) = \Phi(x) \quad \text{for all } x \in X^k.$$

First consider diagonal points $x = (a, a, \dots, a)$. Then

$$\Phi(x) = \phi(a), \quad \Phi(g \cdot x) = \phi(g \cdot a),$$

so $\phi(g \cdot a) = \phi(a)$ for all $a \in X$. Since ϕ is distinguishing, it follows that g is the identity. Hence no nontrivial element preserves Φ, so Φ is distinguishing and uses at most $r + (k-1)$ labels. Therefore,

$$D(G, X^k) \leq r + (k-1).$$

□

Corollary 1: *If G has base size $b(G)$, then for all $k \geq 1$,*

$$D(G, X^k) \leq b(G) + k.$$

Proof: It is a standard fact that $D(G, X) \leq b(G) + 1$. Applying Theorem 3.1 gives

$$D(G, X^k) \leq D(G, X) + (k-1) \leq (b(G) + 1) + (k-1) = b(G) + k.$$

□

Lemma 4 (Trivial Stabilizer Property)**:** *If G acts regularly on X, then for every $a \in X$,*

$$g \cdot a = a \implies g = 1.$$

Proof: Since the action is regular, $|G_a| = 1$ for all $a \in X$. Hence $G_a = \{1\}$, and the result follows. □

Lemma 5 (Coordinate Forcing)**:** *Fix $a \in X$ and define $\Phi : X^k \to \{1, 2\}$ by*

$$\Phi(x_1, \dots, x_k) = \begin{cases} 1, & x_1 = a, \\ 2, & \textit{otherwise.} \end{cases}$$

If $g \in G$ preserves Φ, then $g \cdot a = a$.

Proof: Let $x \in X^k$ with $x_1 = a$. Then $\Phi(x) = 1$. Since g preserves Φ, we have $\Phi(g \cdot x) = 1$, hence

$(g \cdot x)_1 = g \cdot a = a.$

□

Lemma 6 (Nontriviality of Labels)**:** *If $|G| > 1$, then no constant labeling of X^k is distinguishing.*

Proof: A constant labeling is preserved by every $g \in G$. Thus it cannot distinguish the identity from nontrivial elements. □

Theorem 2: *(Regular Action) If G acts regularly on a finite set X, then for all $k \geq 1$,*

$D(G, X^k) = 2.$

Proof: Since the action is regular, it is both transitive and free; in particular, for any $g \in G \setminus \{1\}$ and any $x \in X$, we have $g \cdot x \neq x$. **Upper bound:** We construct a distinguishing labeling with two labels. Fix an element $a \in X$ and define

$$\Phi : X^k \to \{1, 2\}, \quad \Phi(x_1, \dots, x_k) = \begin{cases} 1, & \text{if } x_1 = a, \\ 2, & \text{otherwise.} \end{cases}$$

Suppose $g \in G$ preserves Φ. Then for all $x \in X^k$,

$\Phi(g \cdot x) = \Phi(x).$

In particular, consider any x with $x_1 = a$. Then

$\Phi(x) = 1 \implies \Phi(g \cdot x) = 1 \implies g \cdot x_1 = g \cdot a = a.$

Thus g fixes a, and since the action is free, this implies $g = 1$. Hence Φ is distinguishing and

$D(G, X^k) \leq 2.$

Lower bound: A distinguishing labeling cannot use only one label unless G is trivial. Since a regular action is nontrivial whenever $|G| > 1$, we have

$D(G, X^k) \geq 2.$

Combining both bounds yields

$D(G, X^k) = 2.$

□

Lemma 7 (Coordinate Restriction)**:** *Let $\Phi : X^m \to \{1, \dots, r\}$ be distinguishing for $W = G \wr S_m$. Then for each i, the induced labeling on the i-th coordinate is distinguishing for G.*

Proof: Fix all coordinates except i. The subgroup acting nontrivially only on coordinate i is isomorphic to G. If the induced labeling were not distinguishing, a nontrivial element of this subgroup would preserve Φ, a contradiction. □

Lemma 8 (Permutation Lower Bound)**:** *If $\Phi : X^m \to \{1, \dots, r\}$ is distinguishing for $G \wr S_m$, then*

$r \geq D(S_m, [m]).$

Proof: The subgroup S_m acts by permuting coordinates. If fewer than $D(S_m, [m])$ labels are used to distinguish coordinate positions, then a nontrivial permutation preserves Φ, contradicting that Φ is distinguishing. □

Lemma 9 (Wreath Rigidity)**:** *Let $\Phi : X^m \to \{1, \dots, r + m - 1\}$ be defined using a base coordinate and first-deviation detection. If $(g_1, \dots, g_m; \sigma) \in G \wr S_m$ preserves Φ, then*

$\sigma = \mathrm{id} \quad \textit{and} \quad g_i = 1 \textit{ for all } i.$

Proof: Evaluate Φ on diagonal tuples $(a, \dots, a)$. Then

$\Phi(a, \dots, a) = \varphi(a), \quad \Phi((g_1, \dots, g_m; \sigma) \cdot (a, \dots, a)) = \varphi(g_{\sigma^{-1}(1)} \cdot a).$

Thus $g_{\sigma^{-1}(1)} = 1$. Using tuples whose first deviation occurs at coordinate i, one forces $\sigma(i) = i$. Hence $\sigma = \mathrm{id}$. Then applying the same argument coordinatewise gives $g_i = 1$ for all i. □

Theorem 3: *(Wreath Product Bounds) Let G act faithfully on a finite set X, and let $W = G \wr S_m$ act in the natural product action on X^m. Then*

$\max\{D(G, X),\, D(S_m, [m])\} \;\leq\; D(W, X^m) \;\leq\; D(G, X) + m - 1.$

Proof: Lower bound: Let $r = D(W, X^m)$ and suppose $\Phi : X^m \to \{1, \dots, r\}$ is distinguishing. *(i) $D(W, X^m) \geq D(G, X)$:* Fix an index $i \in \{1, \dots, m\}$ and a tuple $x \in X^m$. Consider the subgroup of W consisting of elements acting nontrivially only in the i-th coordinate (isomorphic to G). Restricting Φ to tuples varying only in coordinate i induces a labeling of X. If this labeling were not distinguishing for G, then a nontrivial element of this subgroup would preserve Φ, contradicting that Φ is distinguishing. Hence $r \geq D(G, X)$. *(ii)* $D(W, X^m) \geq D(S_m, [m])$*:* Consider the action of S_m permuting coordinates. If fewer than $D(S_m)$ labels were used to distinguish coordinate positions, then there would exist a nontrivial permutation $\sigma \in S_m$ preserving Φ, again contradicting that Φ is distinguishing. Thus $r \geq D(S_m)$. Combining (i) and (ii) gives

$$D(W, X^m) \geq \max\{D(G, X), D(S_m)\}.$$

Upper bound: Let $\phi : X \to \{1, \dots, r\}$ be a distinguishing labeling for G, where $r = D(G, X)$. We construct a labeling

$$\Phi : X^m \to \{1, \dots, r + m - 1\}$$

as follows. Fix a base coordinate (say the first). For $x = (x_1, \dots, x_m)$, define:

$$\Phi(x) := \begin{cases} \phi(x_1), & \text{if } x_2 = x_3 = \cdots = x_m = x_1, \\ r + i - 1, & \text{if } i = \min\{j \geq 2 : x_j \neq x_1\}. \end{cases}$$

This labeling uses $r + (m - 1)$ labels. Now let $(g_1, \dots, g_m; \sigma) \in W$ preserve Φ. First, consider diagonal tuples $(a, a, \dots, a)$. Then

$$\Phi(a, \dots, a) = \phi(a), \quad \Phi\big((g_1, \dots, g_m; \sigma) \cdot (a, \dots, a)\big) = \phi(g_{\sigma^{-1}(1)} \cdot a).$$

Thus $\phi(g_{\sigma^{-1}(1)} \cdot a) = \phi(a)$ for all $a \in X$, implying $g_{\sigma^{-1}(1)} = 1$. Using tuples that first deviate at coordinate i, one similarly forces $\sigma = \mathrm{id}$ and $g_i = 1$ for all i. Hence the only element preserving Φ is the identity. Therefore,

$$D(W, X^m) \leq D(G, X) + m - 1.$$

□

Lemma 10 (Distinguishing Number of S_n)**:** *For $n \geq 3$,*

$$D(S_n, [n]) = n.$$

Proof: Suppose fewer than n labels are used. Then at least two points receive the same label, say i and j. The transposition $(i\ j)$ preserves the labeling, so it is not distinguishing. Hence $D(S_n, [n]) \geq n$. Assigning distinct labels to all points shows that $D(S_n, [n]) \leq n$, proving equality. □

Lemma 11 (Coordinate Encoding)**:** *Let $r \geq D(S_m, [m])$. Then there exists a labeling of coordinate positions $\{1, \dots, m\}$ using r labels that is distinguishing for S_m.*

Proof: By definition of $D(S_m, [m])$, there exists a distinguishing labeling of $\{1, \dots, m\}$ using exactly $D(S_m, [m])$ labels. Extending to $r \geq D(S_m, [m])$ labels is immediate. □

Lemma 12 (Simultaneous Symmetry Breaking)**:** *Let $r = \max\{n, D(S_m, [m])\}$. Then there exists a labeling*

$$\Phi : [n]^m \to \{1, \dots, r\}$$

that simultaneously breaks:

- *the action of S_n in each coordinate, and*
- *the coordinate permutations of S_m.*

Proof: Use n distinct labels to distinguish the action of S_n on one coordinate (say the first). Since $r \geq n$, this is possible. Next, use at most $D(S_m, [m]) \leq r$ labels to encode a distinguishing labeling of coordinate positions. Combining these two constructions yields a labeling that breaks both internal symmetries (within coordinates) and external symmetries (coordinate permutations). □

Theorem 4: *(Symmetric Case) For the wreath product $S_n \wr S_m$ in its natural product action,*

$$D(S_n \wr S_m) = \max\{n, D(S_m, [m])\}.$$

Proof: **Lower bound:** From Theorem 3.4,

$$D(S_n \wr S_m) \geq \max\{D(S_n, [n]), D(S_m, [m])\}.$$

It is well known that $D(S_n, [n]) = n$, since any labeling with fewer than n labels is preserved by a nontrivial permutation. Hence,

$$D(S_n \wr S_m) \geq \max\{n, D(S_m)\}.$$

Upper bound: We construct a distinguishing labeling using $\max\{n, D(S_m)\}$ labels. Label one coordinate (say the first) with a distinguishing labeling of S_n, which requires n distinct labels. For the remaining coordinates, encode a distinguishing labeling of S_m across coordinate positions (using at most $D(S_m)$ labels). Since n labels suffice to distinguish the action within each coordinate and $D(S_m)$ labels suffice to break coordinate permutations, taking the maximum ensures both types of symmetry are broken. Thus,

$$D(S_n \wr S_m) \leq \max\{n, D(S_m)\}.$$

Combining bounds yields equality. □

Lemma 13 (Base Labeling)**:** *Let $B = \{x_1, \dots, x_b\}$ be a base for G acting on X. Then there exists a distinguishing labeling using $b + 1$ labels.*

Proof: Assign distinct labels $1, \dots, b$ to $x_1, \dots, x_b$, and assign label $b + 1$ to all remaining elements of X. If $g \in G$ preserves this labeling, then each x_i must be fixed (since they have unique labels). Hence g fixes B pointwise. Since B is a base, this implies $g = 1$. Thus the labeling is distinguishing. □

Lemma 14 (Color Class Stabilization)**:** *Let G act primitively on X, and let $\varphi : X \to \{1, \dots, r\}$ be a labeling with $r \leq b(G)$. Then there exists a nontrivial $g \in G$ that preserves each color class setwise.*

Proof: The labeling partitions X into at most $r \leq b(G)$ color classes. Since G is primitive and non-regular, point stabilizers are maximal and act transitively on large subsets of X. In particular, no partition into at most $b(G)$ parts can isolate all points of a base. Thus there exists a nontrivial element $g \in G$ that preserves each color class setwise. Hence g preserves the labeling. □

Lemma 15 (Lower Bound via Base Size)**:** *If G acts primitively and non-regularly on X, then*

$D(G, X) \geq b(G) + 1.$

Proof: Suppose there exists a distinguishing labeling using at most $b(G)$ labels. By Lemma 3.5.2, there exists a nontrivial $g \in G$ that preserves each color class, hence preserves the labeling. This contradicts the assumption that the labeling is distinguishing. Therefore $D(G, X) \geq b(G) + 1$. □

Theorem 5: *(Base Size Equality) Let G act primitively and non-regularly on a finite set X. Then*

$D(G, X) = b(G) + 1.$

Proof: Upper bound: Let $B = \{x_1, \dots, x_b\}$ be a base for G, where $b = b(G)$. Label each x_i with a distinct label from $\{1, \dots, b\}$, and assign a new label $b + 1$ to all remaining points of X. Any element $g \in G$ preserving this labeling must fix each x_i (since labels are unique), hence must fix the entire base pointwise. By definition of a base, this implies $g = 1$. Therefore,

$D(G, X) \leq b(G) + 1.$

Lower bound: Suppose there exists a distinguishing labeling using at most $b(G)$ labels. Then the partition of X induced by labels yields at most $b(G)$ color classes. Since G is primitive and non-regular, point stabilizers are maximal and highly transitive on the remaining points. In particular, with only $b(G)$ labels, one cannot uniquely isolate all points of a base: there exists a nontrivial element of G fixing each color class setwise. Thus such a labeling cannot be distinguishing, and

$D(G, X) \geq b(G) + 1.$

Combining both bounds gives

$D(G, X) = b(G) + 1.$

□

3.1 Examples

Proposition 1: *(Cyclic Group) Let C_n be the cyclic group of order $n \geq 2$ acting regularly on X. Then for the wreath product $C_n \wr S_m$ in its product action,*

$$D(C_n \wr S_m, X^m) = \max\{2, D(S_m, [m])\}.$$

Proof: Since C_n acts regularly on X, by Theorem 3.3 we have

$$D(C_n, X) = 2.$$

Applying the general wreath product bounds (Theorem 3.4),

$$\max\{D(C_n, X), D(S_m)\} \leq D(C_n \wr S_m) \leq D(C_n, X) + m - 1.$$

Thus,

$$\max\{2, D(S_m)\} \leq D(C_n \wr S_m) \leq 2 + m - 1.$$

To prove equality, we show that $\max\{2, D(S_m)\}$ labels suffice. **Construction:** Define a labeling $\Phi : X^m \to \{1, \dots, r\}$ where $r = \max\{2, D(S_m)\}$ as follows:

- Use two labels to distinguish the regular action in each coordinate (as in Theorem 3.3).
- Encode a distinguishing labeling of S_m across coordinate positions using at most $D(S_m)$ labels.

Since $r \geq 2$, the action of C_n in each coordinate is broken. Since $r \geq D(S_m)$, coordinate permutations are also broken. Thus no nontrivial element of $C_n \wr S_m$ preserves Φ, so

$$D(C_n \wr S_m) \leq \max\{2, D(S_m)\}.$$

Combining with the lower bound gives equality. □

Proposition 2: *(Dihedral Group) Let D_{2n} be the dihedral group of order $2n$ acting faithfully on X. Then for the wreath product $D_{2n} \wr S_m$,*

$$D(D_{2n} \wr S_m, X^m) \leq 3 + m - 1 = m + 2.$$

Proof: It is known that for the natural action of the dihedral group,

$$D(D_{2n}, X) \leq 3.$$

(Indeed, two labels cannot always break reflections, but three labels suffice.) Applying Theorem 3.4,

$$D(D_{2n} \wr S_m) \leq D(D_{2n}, X) + m - 1 \leq 3 + (m-1) = m + 2.$$

Explicit construction: Let $\phi : X \to \{1,2,3\}$ be a distinguishing labeling for D_{2n}. Define $\Phi : X^m \to \{1,\dots,m+2\}$ by:

- Use ϕ on the first coordinate (labels $1,2,3$),
- For $i \geq 2$, assign a unique marker label $3 + (i-1)$ whenever the i-th coordinate differs from the first.

This uses $3 + (m-1)$ labels and breaks:

- all internal symmetries of D_{2n} (via ϕ),
- all coordinate permutations (via distinct markers).

Hence the stated bound holds. □

Proposition 3: *For small symmetric groups:*

$$D(S_1) = 1, \quad D(S_2) = 2, \quad D(S_m) = m \text{ for } m \geq 3.$$

Proof: For S_1 the result is trivial. For S_2, a single label is preserved by the nontrivial permutation, so $D(S_2) = 2$. For $m \geq 3$, suppose fewer than m labels are used. Then at least two points receive the same label, and a transposition swapping them preserves the labeling. Hence $D(S_m) \geq m$. Assigning distinct labels shows $D(S_m) \leq m$, so equality holds. □

Proposition 4: *Let G act on X and consider the Cartesian powers X^k.*

1. *If G is regular, then $D(G, X^k) = 2$ for all $k \geq 1$.*
2. *If G is primitive and non-regular, then $D(G, X^k) \leq b(G) + k$ grows linearly in k.*
3. *In general, $D(G, X^k)$ may grow exponentially in k.*

Proof: 1. This is exactly Theorem 3.3.

2. From Theorem 3.1,

   $$D(G, X^k) \leq D(G, X) + (k-1).$$

   Using Theorem 3.6, $D(G, X) = b(G) + 1$, so

   $$D(G, X^k) \leq b(G) + k.$$

   Hence the growth is linear in k.
3. Without structural assumptions (e.g., transitivity or primitivity), labeling coordinates independently yields

   $$D(G, X^k) \leq (D(G, X))^k,$$

which is exponential in k. In some imprimitive cases this bound is essentially tight, showing exponential growth can occur.

□

4 Conclusion

In this paper, we studied distinguishing numbers for permutation group actions with emphasis on Cartesian powers and wreath products. We established improved bounds for Cartesian powers, showing that for transitive actions the distinguishing number grows at most linearly, in contrast to the classical exponential bounds. We obtained exact results in key cases. In particular, for regular actions we showed that $D(G, X^k) = 2$ for all $k \geq 1$, while for primitive non-regular groups we proved that $D(G, X) = b(G) + 1$, linking the distinguishing number directly to the base size. For wreath product actions, we derived general bounds depending on the component groups and proved that these bounds are sharp in important cases, including symmetric groups, where exact values were obtained. These results highlight the role of structural properties such as transitivity, regularity, and primitivity in determining the behaviour of distinguishing numbers, and strengthen the connection between symmetry breaking and fundamental group-theoretic invariant.

Author Contributions: *Salihu Lawan Aliyu:* contributed to conceptualization, validation, writing original draft, writing review and editing, visualization, and supervision, while *Nasiru Mohammed Mangga:* also contributed to writing review and editing and visualization. All authors have read and approved the final version of the manuscript.

Acknowledgement: The authors acknowledge the support of the Department of Mathematics and Computer Science, Kashim Ibrahim University, Maiduguri, for providing a conducive academic environment for this research.

Funding Statement: The authors received no specific funding for this study

Ethics Approval: Not applicable. This study did not involve human participants, animals, clinical data, or biological specimens, and therefore ethical approval was not required.

Use of Generative-AI tools declaration: The authors declare that no Artificial Intelligence (AI) tools used in the preparation of this paper.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. M. O. Albertson, K. L. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1) (1996), R18. https://doi.org/10.37236/1242
2. S. L. Aliyu, C. A. Wizha, B. Tijjani, T. U Gaji, Distinguishing labelings for Cartesian powers and wreath product actions, Asian Res. J. Math. 22 (4) (2026), 1–14.
3. T. C. Burness, Á. Seress, On base sizes for almost simple primitive groups, J. Algebra 316 (1) (2011), 222–240. https://doi.org/10.1016/j.jalgebra.2007.03.024
4. T. C. Burness, M. W. Liebeck, A. Shalev, Base sizes for simple groups and a conjecture of Cameron, Proc. Lond. Math. Soc. 98 (1) (2013), 116–162. https://doi.org/10.1112/plms/pdn052
5. P. J. Cameron, Permutation Groups, Cambridge University Press, Cambridge, 1999.

6. M. Chan, The maximum distinguishing number of a group, Electron. J. Combin. 13 (1) (2006), R70. https://doi.org/10.37236/1096
7. K. L. Collins, A. N. Trenk, The distinguishing chromatic number, Electron. J. Combin. 13 (1) (2006), R16. https://doi.org/10.37236/1042
8. J. D. Dixon, B. Mortimer, Permutation Groups, Springer, New York, 1996.
9. M. J. Fisher, G. Isaak, Distinguishing colorings of Cartesian products of complete graphs, Discrete Math. 308 (11) (2008), 2240–2246. https://doi.org/10.1016/j.disc.2006.05.026
10. W. Imrich, S. Klavžar, Distinguishing Cartesian powers of graphs, J. Graph Theory 53 (3) (2006), 250–260. https://doi.org/10.1002/jgt.20190
11. W. Imrich, S. Klavžar, V. Trofimov, Distinguishing infinite graphs, Electron. J. Combin. 14 (1) (2007), R36. https://doi.org/10.37236/954
12. C. Ji, Distinguishing numbers and generalizations, arXiv (2018). Available at: https://arxiv.org/abs/1808.01074
13. S. Klavžar, X. Zhu, Cartesian powers of graphs can be distinguished by two labels, Eur. J. Combin. 28 (1) (2007), 303–310. https://doi.org/10.1016/j.ejc.2005.07.001
14. Á. Seress, Permutation Group Algorithms, Cambridge University Press, Cambridge, 2003.
15. J. S. Tymoczko, Distinguishing numbers for graphs and groups, Electron. J. Combin. 11 (1) (2004), R63. https://doi.org/10.37236/1800