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Research article

Interior hop Roman dominating function in graphs

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Abstract: Let $G = (V(G), E(G))$ be a simple non-complete graph and let $\xi : V(G) \to \{0, 1, 2\}$ be an HRDF on G. For each $j \in \{0, 1, 2\}$, let $V_j = \{x \in V(G) : \xi(x) = j\}$. Then $\xi = (V_0, V_1, V_2)$. A function ξ is an interior hop Roman dominating function (InHRDF) on G if for each $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 2$, and either $V_1 = V(G)$ or for every $w \in V_2$, w is an interior vertex of G. The weight of InHRDF ξ is denoted by $\omega_G^{InhR}(\xi)$ and is defined as $\omega_G^{InhR}(\xi) = \sum_{u \in V(G)} \xi(u) = |V_1| + 2|V_2|$. The minimum weight of an InHRDF ξ on G, denoted and defined by $\gamma_{InhR}(G) = min\{\omega_G^{InhR}(\xi) : \xi \; is \; an$ InHRDF $on \; G\}$, is called the *interior hop Roman domination number*. Every InHRDF ξ on G satisfying the condition $\omega_G^{InhR}(\xi) = \gamma_{InhR}(G)$ is called a γ_{InhR}-function on G. In this paper, we investigate a new restricted parameter of a hop Roman dominating function in graphs called the interior hop Roman domination and present some combinatorial results.

Mathematics Subject Classification: 05C69
Keywords: Interior domination, Roman domination, Hop Roman domination, Interior hop Roman domination.

1 Introduction

About 4^{th} century A.D., the Roman Emperor Constantine developed a defensive strategy after the Roman Empire was under attack by its enemies. The strategy is to place the army units (legions) so that every region is either secured by its own army (1 or 2 legions) or is securable by a neighbor with two legions, which one legion can be sent to the undefended region if an attack will happens. In that case, the Roman dominating function in graphs was initiated in the year 2004 by Cockayne et al. [9], which was modeled after the Roman Emperor Constantine's defensive strategy. A Roman dominating function is a strategy of labeling in graphs that assigns numbers (0, 1, or 2) to the vertices (locations) to ensure that every unsecured vertex (0) is adjacent to a vertex labeled with 2. At present, there are several published papers that involve various restricted parameter of Roman domination in graphs, which can be browsed in the following references: [1,5–7,12,14]. During the year 2017, a variant of Roman dominating function called the hop Roman domination has emerged as a motivating domination parameter which captures the interest of several discrete mathematicians [14]. Meanwhile, Casinillo [6] introduced a new parameter of Roman domination called the interior Roman dominating function, which is based on the papers of Cockayne et al. [9] and Kinsley and Selvaraj [13]. Given the two fascinating variants, namely hop Roman and interior Roman dominating functions, the author was inspired to fuse the two ideas and introduced a new study of domination parameter called the interior hop Roman dominating function in graphs. For some terminologies and definitions used in this paper that are not presented here, the readers are advised to browse the following: [7,10,11].

Let $G = (V(G), E(G))$ be any graph where $V(G)$ and $E(G)$ are the vertex and edge sets of G, respectively. Let $a, b \in V(G)$. The *distance* between a and b is the length of the shortest walk between the two vertices a and b in G, which is denoted by $d_G(a, b)$. If there is no such walk between a and b, then define the distance as $d_G(a, b) = \infty$. Let $c \in V$. If $d_G(a, b) = d_G(a, c) + d_G(c, b)$, then c is said to lie between a and b, and c is called *interior vertex* of G. In other words, $c \in V(G)$ is an interior vertex on G if there exists $a, b \in V(G)$ such that $d_G(a, b) = d_G(a, c) + d_G(c, b)$. Let $I \subseteq V(G)$. If for all $x \in I$ is an interior vertex on G, then I is called an interior vertex set on G. Let $z \in V(G)$. Then the set $N^2_G(z) = \{w \in V(G) : deg_G(w, z) = 2\}$ is so-called the *open hop-neighborhood* of z and for every $v \in N^2_G(z)$ is so-called hop-neighbor of a vertex z on G. In that case, the *closed hop-neighborhood* of $z \in V(G)$ is defined by $N^2_G[z] = N^2_G(z) \cup \{z\}$. A subset D of $V(G)$ is a *dominating set* of G provided that each vertex in $V(G) \setminus D$ is adjacent to some vertex in D [11]. The minimum cardinality of a dominating set D on G is so-called the *domination number*, which is denoted by $\gamma(G)$. Suppose that $|D| = \gamma(G)$. Then set D is so-called a γ-set on G. For some interesting papers involving domination in graphs, readers are advised to read the following references: [2–4,8]. A subset H_d of $V(G)$ is a *hop dominating set* of a graph G provided that for each $a \in V \setminus H_d$, there exists $b \in H_d$ such that $d_G(a, b) = 2$. The minimum cardinality of set H_d on G is called the *hop domination number* of G and is denoted by $\gamma_h(G)$. If $|H_d| = \gamma_h(G)$, then H_d is a γ_h-set on G.

Let $\xi : V(G) \to \{0, 1, 2\}$ be a function on G. Then consider the following sets below:

$$\begin{aligned} V_0 &= \{x \in V(G) : \xi(x) = 0\}; \\ V_1 &= \{x \in V(G) : \xi(x) = 1\}; \, and \\ V_2 &= \{x \in V(G) : \xi(x) = 2\}. \end{aligned}$$

Then a function ξ can be represented by $\xi = (V_0, V_1, V_2)$. A function $\xi = (V_0, V_1, V_2)$ is called a *hop Roman dominating function* (HRDF) on G provided that for each $a \in V_0$, there exists $b \in V_2$ such that $a \in N^2_G(b)$ [14]. The *weight* of function ξ is denoted by $\omega^{hR}_G(\xi)$ and is defined as $\omega^{hR}_G(\xi) = \sum_{x \in V(G)} \xi(x) = |V_1| + 2|V_2|$. The *hop Roman domination number* of G, denoted by $\gamma_{hR}(G)$, is defined as the minimum weight of an HRDF ϕ on G. Thus, we have $\gamma_{hR}(G) = min\{\omega^{hR}_G(\xi) : \xi \; is \; an$ HRDF $on \; G\}$. Every HRDF ξ on any graph G with $\omega^{hR}_G(\xi) = \gamma_{hR}(G)$ is called a γ_{hR}-function on G. A function $\xi = (V_0, V_1, V_2)$ is an *interior hop Roman dominating function* (InHRDF) on G provided that for each $v \in V_0$ there exists $u \in V_2$ such that $v \in N^2_G(u)$, and either $V_1 = V(G)$ or for every $w \in V_2$, w is an interior vertex on G. The *weight* of InHRDF ξ is denoted by $\omega^{InhR}_G(\xi)$ and is defined to be $\omega^{InhR}_G(\xi) = \sum_{x \in V(G)} \xi(x) = |V_1| + 2|V_2|$. The *interior hop Roman domination number* of G is denoted by $\gamma_{InhR}(G)$, and is defined to be the minimum weight of an InHRDF ξ

on G, that is, $\gamma_{InhR}(G) = min\{\omega_G^{InhR}(\xi) : \xi \text{ is an InHRDF on } G\}$. Any InHRDF ξ on G with the restriction $\omega_G^{InhR}(\xi) = \gamma_{InhR}(G)$ is called a γ_{InhR}-function on G. This paper aims to introduce a newly restricted parameter of a hop Roman domination in graphs called the interior hop Roman dominating function. In addition, some graph-theoretic and combinatorial properties of the interior hop Roman dominating function in some graph classes were investigated.

2 Results

In this section, we present some combinatorial properties of the interior hop Roman dominating function in graphs. We start with an important proposition.

Proposition 1: *Let $G = K_n$ where $n \in \mathbb{N}$ and $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Then $V_2 = \emptyset$ and $\gamma_{InhR}(G) = n$.*

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on $G = K_n$. Then for each $v \in V(G)$, $d_G(v, u) = 1$ for all $u \in V(G) \setminus \{v\}$. It follows that there does not exist $x, y \in V(G) \setminus \{v\}$ such that $d_G(x, y) = d_G(x, v) + d_G(v, y)$. Thus, v is not an interior vertex. Since v is arbitrary, it follows that $V_2 = \emptyset$. Since ξ is an γ_{InhR}-function on G, it implies that $V_1 = V(G)$. Therefore, we conclude that $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. This completes the proof. □

As a consequence of Proposition 1, we exclude complete graphs in our investigation. Next is an important remark as a direct consequence of the definition of the interior hop Roman dominating function.

Remark 1: *Let G be a non-complete graph of order $n \in \mathbb{N}$ and let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. If $\gamma_{InhR}(G) < n$, then $|V_0| \neq 0$ and $|V_2| \neq 0$, and for every $u \in V_2$, u is an interior vertex in $V(G)$.*

In Remark above, it is worth noting that if $\gamma_{InhR}(G) < n$, V_1 is not necessarily empty. The next results are important graph-theoretical properties of the interior hop Roman dominating function.

Theorem 1: *Let G be a non-complete graph and let $\xi = (V_0, V_1, V_2)$ is a γ_{InhR}-function on G. If $V_1 = \emptyset$, then $V_2 \neq \emptyset$ is an interior hop dominating set on G.*

Proof: Let $\xi = (V_0, V_1, V_2)$ is a γ_{InhR}-function on a non-complete graph G. Then ξ is an InHRDF on G. By definition of InHRDF, it implies that for every $v \in V_0$, there exists $u \in V_2$ such that $d_G(u, v) = 2$, and either $V_1 = V(G)$ or for every $x \in V_2$, x is an interior vertex in G. Assume that $V_1 = \emptyset$. Then $V_2 \neq \emptyset$ is a hop dominating set on G, that is, $V(G) = N_G^2[V_2]$. Therefore, we concluded that V_2 is an interior hop dominating set on G. This proves the assertion. □

Theorem 2: *Let G be a non-complete graph of order n and let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Then*

(i) $V_0 = \emptyset$ *if and only if* $V_2 = \emptyset$, *hence* $\gamma_{InhR}(G) = n$;
(ii) $|V_0| = |V_2|$ *if and only if* $\gamma_{InhR}(G) = n$; *and*
(iii) $0 < |V_2| < |V_0|$ *if and only if* $\gamma_{InhR}(G) < n$.

Proof: Let $\xi = (V_0, V_1, V_2)$ is a γ_{InhR}-function on G with $|V(G)| = n \in \mathbb{N}$. Assume that $V_0 = \emptyset$. Seeking a contradiction. Suppose for a moment that $V_2 \neq \emptyset$. Let $u \in V_2$ and let $M_0 = V_0$, $M_1 = V_1 \cup \{u\}$, and $M_2 = V_2 \setminus \{u\}$. Then it is easy to see that $\xi' = (M_0, M_1, M_2)$ is an InHRDF on G. Hence, we have $\omega_G^{InhR}(\xi') = |M_1| + 2|M_2| = (|V_1|+1)+2(|V_2|-1) = |V_1|+2|V_2|-1 < \omega_G^{InhR}(\xi) = \gamma_{InhR}(G)$. A contradiction to the fact that ξ is a γ_{InhR}-function on G. Therefore, we get $V_2 = \emptyset$. Conversely, assume that $V_2 = \emptyset$. Seeking a contradiction. Suppose for a moment that $V_0 \neq \emptyset$. Let $v \in V_0$. Since $V_2 = \emptyset$, it follows that v is an undefended vertex in G. A contradiction. Therefore, $V_0 = \emptyset$. Moreover, we have $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1|+2|V_2| = |V_1| = |V(G)| = n$ and thus, (i) is satisfied. Next, assume that $|V_0| = |V_2|$. If $|V_0| = |V_2| = 0$, then by (i), it directly follows that $\gamma_{InhR}(G) = n$. Consider that $|V_0| = |V_2| \geq 1$. Then we obtain $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. Conversely, assume that $\gamma_{InhR}(G) = n$. Seeking a contradiction. Suppose for a moment that $|V_0| \neq |V_2|$. Then either $|V_0| > |V_2|$ or $|V_0| < |V_2|$. Let $|V_0| > |V_2|$. Then we obtain $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$. A contradiction to the assumption. Let $|V_0| < |V_2|$. Then we get $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$. Again, a contradiction. Accordingly, $|V_0| = |V_2|$. Therefore, (ii) is satisfied. Lastly, the proof of (iii) directly follows from (ii). This completes the proof. □

Corollary 1: *Let G be a non-complete graph of order n. If $\gamma_{InhR}(G) < n$ and $|V_2| = 1$, then $V_1 \neq \emptyset$.*

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Assume that $\gamma_{InhR}(G) < n$ and $|V_2| = 1$. By Theorem 2(iii), we have $|V_2| < |V_0|$. Then let $V_2 = \{x\}$ and $y \in V_0$. In that case, $d_G(x, y) = 2$. Seeking a contradiction. Suppose for a moment that $V_1 = \emptyset$. Then there exists $w \in N_G(x) \cap N_G(y)$ such that $w \in V_0$. However, there does not exist a vertex in V_2 such that it hop dominates w. Thus, it follows that $V(G) \neq N_G^2[V_2]$. A contradiction. Consequently, $w \in V_1$. Therefore, we concluded that $V_1 \neq \emptyset$. This proves the assertion. □

Theorem 3: *Let G be a non-complete graph and let $\xi = (V_0, V_1 = \emptyset, V_2)$ be an InHRDF on G. Then $\gamma_{Inh}(G) = |V_2|$ if and only if ξ is a γ_{InhR}-function on G.*

Proof: Let $\xi = (V_0, V_1 = \emptyset, V_2)$ be an InHRDF on a non-complete graph G. Assume that $\gamma_{Inh}(G) = |V_2|$. Then it means that V_2 is a γ_{Inh}-set on G. Now, since $V_1 = \emptyset$, it implies that $V_0 = V(G) \setminus V_2$. In that case, for each $a \in V_0$, there exists $b \in V_2$ such that $d_G(a, b) = 2$ and for every $c \in V_2$, c is an interior vertex in G. This follows that ξ is an InHRDF on G. Suppose for a moment that ξ is not a γ_{InhR}-function on G. Then it follows that there exists an InHRDF $\xi' = (X_0, X_1, X_2)$ such that ξ' is a γ_{InhR}-function on G and $X_1 = \emptyset$. Hence, we get $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi') = |X_1| + 2|X_2| = 2|X_2| < 2|V_2| = |V_1| + 2|V_2| = \omega_G^{InhR}(\xi)$. Accordingly, we obtain $|U_2| < |V_2| = \gamma_{Inh}(G)$. This is a contradiction. Therefore, we concluded that ξ is a γ_{InhR}-function on G. On the other hand, assume that ξ is a γ_{InhR}-function on G. By Theorem 1, V_2 is an interior hop dominating set on G. Suppose for a moment that V_2 is not a γ_{Inh}-set on G. Then it means that there exists an interior hop dominating set U_2 such that U_2 is a γ_{Inh}-set on G. Hence, we get $\gamma_{Inh}(G) = |U_2| < |V_2|$. Let $\xi'' = (Y_0, Y_1, Y_2)$ be an HRDF on G where $Y_0 = V(G) \setminus U_2$, $|Y_1| = 0$ and $Y_2 = U_2$. Since U_2 is a γ_{Inh}-set on G, it implies that for every $v \in Y_0$, there exists $u \in Y_2$ such that $d_G(u, v) = 2$ and for every $c \in Y_2$, c is an interior vertex in G. Thus, ξ'' is an RHRDF on G. In that case, we obtain

$\omega_G^{InhR}(\xi'') = |Y_1| + 2|Y_2| = 2|Y_2| = 2|U_2| < 2|V_2| = \omega_G^{InhR}(\xi) = \gamma_{InhR}(G)$. A contradiction since ξ is a γ_{InhR}-function on G. Therefore, it suffices to conclude that V_2 is a γ_{Inh}-set on G and so, $\gamma_{Inh}(G) = |V_2|$. This proves the assertion. □

The following results are a consequence of Theorem 3.

Corollary 2: *Let G be a non-complete graph and let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. If $V_1 = \emptyset$, then $\gamma_{InhR}(G) = 2\gamma_{Inh}(G)$.*

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Assume that $V_1 = \emptyset$. Then by Theorem 3, it implies that V_2 is a γ_{Inh}-set on G. Thus, we have $\gamma_{Inh}(G) = |V_2|$. Therefore, we conclude that $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = 2|V_2| = 2\gamma_{Inh}(G)$. This proves the assertion. □

Remark 2: *Let G be a non-complete graph and let $\xi = (V_0, V_1, V_2)$ be a InHRDF on G. If $V_1 = \emptyset$, then for all $x \in V(G)$ with $deg_G(x) = 1$, $x \in V_0$.*

The Theorem characterizes the InHRDF on a graph G that contains several components.

Theorem 4: *Let $G = G_1 \cup G_2 \cup \ldots \cup G_k$ be a graph. Then $\xi = (V_0, V_1, V_2)$ is an InHRDF on G if and only if $\xi|_{G_i} = (V_0^i, V_1^i, V_2^i)$ (restriction function with respect to each component) is an InHRDF on G_i for all $i \in \{1, 2, \ldots k\}$.*

Proof: Assume that $\xi = (V_0, V_1, V_2)$ is an InHRDF on $G = G_1 \cup G_2 \cup \ldots \cup G_k$. For each $i \in \{1, 2, \ldots, k\}$, let $V_0^i = V_0 \cap V(G_i)$, $V_1^i = V_1 \cap V(G_i)$ and $V_2^i = V_2 \cap V(G_i)$. Then the restriction function for each component is given by $\xi|_{G_i} = (V_0^i, V_1^i, V_2^i)$ for all $i \in \{1, 2, \ldots, k\}$. Let $v \in V_0^s$ where $s \in \{1, 2, \ldots, k\}$. Since $V_0^i \subseteq V_0$ for all $i \in \{1, 2, \ldots, k\}$, it implies that $v \in V_0$. Since ξ is an interior hop Roman dominating function on G, it follows that there exists $u \in V_2$ such that $d_G(u, v) = 2$ and either $V_1 = V(G)$ or for every $w \in V_2$, w is an interior vertex in G. Note that $d_G(x, y) = \infty$ for any $x \in G_i$ and for any $y \in G_l$ where $i \neq l$ and for all $i, l \in \{1, 2, \ldots, k\}$. This follows that $u \in V_2^s$ and $d_{G_s}(u, v) = 2$, and either $V_1^s = V(G_s)$ or for every $w \in V_2^s$, w is an interior vertex in G_s where $s \in \{1, 2, \ldots, k\}$. Accordingly, it suffices to say that $\xi|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is an InHRDF on G_i for all $i \in \{1, 2, \ldots, k\}$. On the other hand, assume that $\xi|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is an InHRDF on G_i for all $i \in \{1, 2, \ldots k\}$. Then let $V_0 = \bigcup_{i=1}^k V_0^i$, $V_1 = \bigcup_{i=1}^k V_1^i$ and $V_2 = \bigcup_{i=1}^k V_2^i$. Hence, it is easy to see that $\xi = (V_0, V_1, V_2)$ is a function on $G = G_1 \cup G_2 \cup \ldots \cup G_k$. Let $x \in V_0$. Then it implies that $x \in V_0^i$ for some $i \in \{1, 2, \ldots k\}$. Since $\xi|_{G_i}$ is an InHRDF on G_i for all $i \in \{1, 2, \ldots k\}$, it means that there exists $y \in V_2^i$ such that $d_{G_i}(x, y) = 2$ and either $V_1^i = V(G_i)$ or for all $z \in V_2^i$, z is an interior vertex in G_i. Now, since $V_2 = \bigcup_{i=1}^k V_2^i$, it implies that $y \in V_2$ and $d_G(x, y) = 2$ and either $V_1 = V(G)$ or for all $z \in V_2$, z is an interior vertex in G. Accordingly, we concluded that ξ is an InHRDF on G. This completes the proof. □

The next result is a consequence of Theorem 4 above.

Corollary 3: *Let $G = G_1 \cup G_2 \cup \ldots \cup G_k$ be a graph. Then, $\gamma_{InhR}(G) = \sum_{i=1}^{k} \gamma_{InhR}(G_i)$.*

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on $G = G_1 \cup G_2 \cup \ldots \cup G_k$. Then for each $i \in \{1, 2, ..., k\}$, let $V_0^i = V_0 \cap V(G_i)$, $V_1^i = V_1 \cap V(G_i)$ and $V_2^i = V_2 \cap V(G_i)$. Thus, $\xi|_{G_i} = (V_0^i, V_1^i, V_2^i)$ is a function on G_i for all $i \in \{1, 2, ..., k\}$. Invoking Theorem 4, it implies that $\phi|_{G_i}$ is an InHRDF on G_i for all $i \in \{1, 2, ..., k\}$. Hence, we have

$$\begin{aligned}
\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| \\
&= \sum_{i=1}^{k} |V_1^i| + 2\sum_{i=1}^{k} |V_2^i| \\
&= \sum_{i=1}^{k} \left(|V_1^i| + 2|V_2^i|\right) \\
&\geq \sum_{i=1}^{k} \gamma_{InhR}(G_i).
\end{aligned}$$

Now, let $\xi|_{G_i} = (X_0^i, X_1^i, X_2^i)$ be an InHRDF on G_i for all $i \in \{1, 2, ...k\}$. Then let $X_0 = \bigcup_{i=1}^k X_0^i$, $X_1 = \bigcup_{i=1}^k X_1^i$ and $X_2 = \bigcup_{i=1}^k X_2^i$. In that case, it is easy to see that $\xi = (X_0, X_1, X_2)$ is a function on G. In view of Theorem 4, it follows that ξ is an InHRDF on G. Hence, we obtain

$$\begin{aligned}
\sum_{i=1}^{k} \gamma_{InhR}(G_i) = \sum_{i=1}^{k} \omega_G^{InhR}(\xi|_{G_i}) &= \sum_{i=1}^{k} \left(|X_1^i| + 2|X_2^i|\right) \\
&= \sum_{i=1}^{k} |X_1^i| + 2\sum_{i=1}^{k} |X_2^i| \\
&= |X_1| + 2|X_2| \\
&\geq \gamma_{InhR}(G).
\end{aligned}$$

Therefore, we end up with $\gamma_{InhR}(G) = \sum_{i=1}^{k} \gamma_{InhR}(G_i)$. This completes the proof. □

Corollary 4: *Let G be a graph with several components and $|V(G)| = n$. If each of the components of G is a complete graph, then $\gamma_{InhR}(G) = n$. In particular, $\gamma_{InhR}(\overline{K}_n) = n$.*

Proof: The proof follows from the concepts of Proposition 1 and Corollary 3. □

Remark 3: *Let G be a non-complete graph and let $\xi = (V_0, V_1, V_2)$ be a InHRDF on G. If there exists $x \in V(G)$ such that $deg_G(x) = 0$, then $x \in V_1$.*

The next theorem presents lower and upper bounds of the interior hop Roman domination number for any non-complete graph.

Theorem 5: *Let G be a non-complete graph of order $n \geq 1$. Then the following is true:*

$$\gamma_R(G) \leq \gamma_{InhR}(G) \leq min\{2\gamma_{Inh}(G), n\}.$$

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Since every InHRDF is an HRDF on G, it follows that $\gamma_R(G) \leq \gamma_{InhR}(G)$. Hence, we obtain $max\{\gamma_{Inh}(G), \gamma_R(G)\} \leq \gamma_{InhR}(G)$. Let $V_0 = \emptyset$. Since ξ is a γ_{InhR}-function on G, by Theorem 2(i), we have $V_2 = \emptyset$. Hence, $\phi = (V_0 = \emptyset, V_1 = V(G), V_2 = \emptyset)$ is an InHRDF on G. Hence, we have $\gamma_{InhR}(G) \leq \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Suppose that ξ is a γ_{InhR}-function on G such that $V_1 = \emptyset$. This implies that $V_1 \cup V_2 = V_2$ is a interior hop dominating set on G by Theorem 1. By Theorem 3, we have $\gamma_{Inh}(G) = |V_2|$. This follows that $\gamma_{InhR}(G) \leq \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = 2|V_2| = 2\gamma_{Inh}(G)$. Accordingly, we get $\gamma_{InhR}(G) \leq min\{n, 2\gamma_{Inh}(G)\}$. This completes the proof. □

The following Corollary 5 is a direct consequence of Theorem 5.

Corollary 5: *Let G be a graph of order $n \geq 1$. If $\xi = (V_0, V_1, V_2)$ is a γ_{InhR}-function on G, then* $0 \leq |V_2| \leq |V_0|$.

Proof: Suppose that $\xi = (V_0, V_1, V_2)$ is a γ_{InhR}-function on G. Then, in view of Theorem 5, we get $\gamma_{InhR}(G) \leq n$. Consider that $\gamma_{InhR}(G) = n$. Then by Theorem 2(ii), it implies that $|V_2| = |V_0| \geq 0$. Now, if we consider that $\gamma_{InhR}(G) < n$, then invoking Theorem 2(iii), we have that $0 < |V_2| < |V_0|$. And the conclusion follows. □

The result depicts the lower bound of the interior hop Roman domination number for some graphs that contain an interior hop dominating set.

Theorem 6: *Let G be a non-complete graph that only contains an interior hop dominating set and $|V(G)| = n \geq 1$. Then the following holds:*

$$\gamma_{Inh}(G) \leq \gamma_{InhR}(G)$$

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G for which $V_1 = \emptyset$. Then by Theorem 1, V_2 is an interior hop dominating set on G. This implies that $\gamma_{Inh}(G) \leq |V_2| \leq 2|V_2| = |V_1| + 2|V_2| = \omega_G^{InhR}(\phi) = \gamma_{InhR}(G)$. This proves the assertion. □

However, the sharpness of the lower bound in Theorem 6 only holds for graphs without an interior vertex. For instance, let $G = K_n$ for all $n \geq 1$. Then we get that $\gamma_{Inh}(G) = \gamma_{InhR}(G) = n$ for all $n \geq 1$. The next Theorem is a characterization of the interior hop Roman domination number for small values.

Theorem 7: *Let G be a graph. Then*

(i) $\gamma_{InhR}(G) = 1$ *if and only if* $G = K_1$;

(ii) $\gamma_{InhR}(G) = 2$ *if and only if* $G \in \{K_2, \overline{K_2}\}$; *and*

(iii) $\gamma_{InhR}(G) = 3$ *if and only if* $G \in \{\overline{K_3}, K_1 \cup K_2, P_3, K_3\}$.

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Assume that $\gamma_{InhR}(G) = 1$. Then it implies that $|V_1| + 2|V_2| = 1$. In that case, we obtain $|V_2| = 0$ and $|V_1| = 1$. Invoking Theorem 2(i), it follows that $|V_0| = 0$. Hence, we get that $\gamma_{InhR}(G) = \omega_G^{InhR} = |V_1| + 2|V_2| = |V_1| = |V(G)| = 1$. Therefore, we have $G = K_1$. The converse follows directly from Proposition 1. Next, assume that $\gamma_{InhR}(G) = 2$. Then it implies that $|V_1|+2|V_2| = 2$. In that case, we have $|V_2| \leq 1$. Suppose that $|V_2| = 1$. Then it means that $|V_1| = 0$ and $|V_0| \geq 1$. Let $V_2 = \{x\}$ and let $y \in V_0$. Then we have that $d_G(x, y) = 2$. Now, let $z \in N_G(x) \cap N_G(y)$. Since $|V_1| = 0$, it follows that $z \in V_0$. But $N_G^2(z) \cap V_2 = \emptyset$, which leads to a contradiction. Thus, we have that $|V_2| = 0$. By Theorem 2(i), we obtain $|V_0| = 0$. Hence, we end up with $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + |V_2| = |V_1| = |V(G)| = 2$. Therefore, we concluded that $G \in \{K_2, \overline{K_2}\}$. By Proposition 1, the converse follows. Lastly, we assume that $\gamma_{InhR}(G) = 3$. Then we have that $|V_1| + 2|V_2| = 3$. Thus, we get $|V_2| \leq 1$. Suppose that $|V_2| = 0$. Hence, we have $|V_1| = 3$. In that case, we obtain $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 3$. This follows that $G \in \{\overline{K_3}, K_1 \cup K_2, P_3, K_3\}$. Suppose that $|V_2| = 1$. Then we get $|V_1| = 1$. Let $V_2 = \{u\}$ and $V_1 = \{w\}$. Since ξ is an InHRDF on G, it follows that u is an interior vertex on G and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Then $d_G(u, z) = 2$ and $d_G(w, z) = 1$ for every $z \in V_0$. Thus, it implies that $G = \langle V_1 \rangle + \langle V_0 \cup V_2 \rangle$. However, there does not exist $a, b \in V(G)$ such that $d_G(a, b) = d_G(a, u) + d_G(u, b)$. Hence, u cannot be an interior vertex in G. A contradiction. Thus, $|V_2| \neq 1$. As for the converse, we let $G \in \{\overline{K_3}, K_1 \cup K_2, P_3, K_3\}$, then it follows that $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = |V_1| = |V(G)| = 3$. This completes the proof. □

Theorem 8: *Let G be a graph and let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Then $|V_1| = 0$ and $\gamma_{InhR}(G) = 4$ if and only if there exists hop dominating set $\{x, y\}$ such that x and y are interior vertices on G.*

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Assume that $|V_1| = 0$ and $\gamma_{InhR}(G) = 4$. Then it follows that $\omega_G^{InhR}(\xi) = 2|V_2| = 4$. Thus, we have $|V_2| = 2$. Since $V_1 = \emptyset$, by Theorem 3, V_2 is a γ_{InhR}-set on G. Let $V_2 = \{x, y\}$ and let $V_0 = V(G) \setminus V_2 = \{x, y\}$. Then we have that $\{x, y\}$ is an interior hop dominating set on G. Hence, $\{x, y\}$ is a hop dominating set on G such that x and y are interior vertices on G. Conversely, assume that there exists a hop dominating set $\{x, y\}$ such that x and y are interior vertices on G. Then it follows that $\{x, y\}$ is an interior hop dominating set on G. Then we obtain $V_0 = V(G) \setminus \{x, y\}$ and $|V_1| = 0$. In view of Theorem 3, we have V_2 is a γ_{Inh}-set on G. Thus, we obtain $2 = |\{x, y\}| \geq |V_2|$. Suppose that $|V_2| = 0$. Then we obtain $\gamma_{InhR}(G) = 0$. This is a contradiction since $V(G) \neq \emptyset$. Suppose that $|V_2| = 1$. Let $V_2 = \{u\}$. Since $d_G(u, x) = 2$ for all $x \in V_0$, it follows that there exists $y \in V(G)$ such that $d_G(u, y) = 1$. Hence, $V_1 \neq \emptyset$. This is a contradiction since $|V_1| = 0$. Consequently, we get $|V_2| \geq 2$. Hence, we have that $|V_2| = 2$. Therefore, $\gamma_{InhR}(G) = \omega_G^{InhR}(\xi) = |V_1| + 2|V_2| = 2|V_2| = 4$. This completes the proof. □

Proposition 2: *If $G = K_{m,n}$ where $m, n \geq 2$, then $\gamma_{InhR}(G) = 4$.*

Proof: Let $G = K_{m,n} = (U(G), V(G), E(G))$ be a bipartite graph where $U(G)$ and $V(G)$ denotes the partition sets, that is, $|U(G)| = m$ and $|V(G)| = n$, and $E(G)$ is the edge set of G. Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Let $U(G) = \{u_1, u_2, ..., u_m\}$ and $V(G) = \{v_1, v_2, ..., v_k\}$. Then if we pick $u_j \in V_2$ for some $j \in \{1, 2, 3, ..., m\}$, then $d_G(u_j, u_i) = 2$ for all $i \neq j \in \{1, 2, 3, ..., m\}$. This follows that $U(G) \subseteq N_G^2(u_j)$ for all $j \in \{1, 2, 3, ..., m\}$. Similarly, we can say that $V(G) \subseteq N_G^2(v_k)$ for each $k \in \{1, 2, 3, ..., n\}$. In that case, we let

$V_2 = \{u_j, v_k\}$ for some $j \in \{1,2,3,...,m\}$ and $k \in \{1,2,3,...,n\}$. Thus, it implies that $V(G) \subseteq N_G^2(u_j, v_k)$. Since $m, n \geq 2$, it follows that $V(G)$ is a vertex set on G. Hence, we have that $V_0 = V(G) \setminus V_2$ and $V_1 = \emptyset$. Hence, V_2 is a hop dominating set on G. Since V_2 is a interior vertex set on G, by Theorem 8, we conclude that $\gamma_{InhR}(G) = 4$. This proves the assertion. □

Theorem 9: *Let G be a graph of order $n \geq 5$ for which $\gamma_{Inh}(G) < \gamma_{InhR}(G)$. If there exists a non-empty set $I \subseteq V(G)$ and a vertex $u \in V(G)$ such that $I \cup \{u\}$ is a γ_{Inh}-set on G and $V(G) \setminus (I \cup \{u\}) \subseteq N_G^2(u)$, then $\gamma_{InhR}(G) = \gamma_{Inh}(G) + 1$.*

Proof: Let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G and let $\gamma_{Inh}(G) < \gamma_{InhR}(G)$. Suppose that there exist a non-empty set $I \subseteq V(G)$ and $u \in V(G)$ such that $R \cup \{u\}$ is a γ_{Inh}-set on G and $V(G) \setminus (I \cup \{u\}) \subseteq N_G^2(u)$. Let $V_0 = V(G) \setminus (I \cup \{u\})$, $V_1 = I$, and $V_2 = \{u\}$. Then we set $\xi' = (V_0, V_1, V_2)$. Since $I \cup \{u\}$ is a γ_{Inh}-set on G, it implies that ξ' is an InHRDF on G. Hence, we obtain $\gamma_{InhR}(G) \leq \omega_G^{InhR}(\xi') = |V_1| + 2|V_2| = |I| + 2(1) = (\gamma_{Inh}(G) - 1) + 2 = \gamma_{Inh}(G) + 1$. Now, since $\gamma_{Inh}(G) < \gamma_{InhR}(G)$, we have that $\gamma_{Inh}(G) = \gamma_{InhR}(G) + 1$. This completes the proof. □

The converse of Theorem 9 is not always true since by the definition of InHRDF, there is a possibility that some elements of V_1 are not interior vertices on G.

Theorem 10: *Let $G \in \{P_{n\geq 1}, C_{n\geq 3}\}$. Then*

$$\gamma_{InhR}(G) = \begin{cases} n, & \textit{if } n \leq 4, \\ n - k, & \textit{if } n \geq 5, \end{cases}$$

where $k = \lfloor \frac{n}{5} \rfloor$.

Proof: First, we let $G = P_n$ with $n \geq 1$ and let $\xi = (V_0, V_1, V_2)$ be a γ_{InhR}-function on G. Suppose that $n \leq 4$. If $V_1 = V(G)$, then we are done. Assume for a contrary that $\gamma_{InhR}(G) < n$. Then it follows that $V_1 \neq V(G)$. By Theorem 2(*iii*), it implies that $0 < |V_2| < |V_0|$. Hence, there exists $x \in V_2$ such that $|N_G^2(x) \cap V_0| = 2$. Since G is a path graph and x is an interior hop dominating vertex, then it follows that there exists $a, b \in V_1$ such that $a, b \in N_G(x)$. In that case, $n \geq 5$, a contradiction. Hence, it suffices to say that $\gamma_{InhR}(G) = n$ whenever $n \leq 4$. Now, suppose that $n \geq 5$. Let $G = P_n = [x_1, x_2, ..., x_n]$ and consider the following cases:
Case 1: $n \equiv 0 \pmod 5$

Let $n = 5k$ where $k \in \mathbb{N}$. Then $k = \lfloor \frac{n}{5} \rfloor = \frac{n}{5}$. For each $i \in \{1,2,...,k\}$, we can set $A_i = \{x_{2+5(i-1)}, x_{4+5(i-1)}\}$ and $B_i = \{x_{3+5(i-1)}\}$. Then we let $U_1 = \bigcup_{i=1}^{k} A_i$, $U_2 = \bigcup_{i=1}^{k} B_i$ and hence, we have $U_0 = V(G) \setminus (U_1 \cup U_2)$. This follows that $\xi' = (U_0, U_1, U_2)$ is an HRDF on G. Since for every $v \in V_2$, there exist $x, y \in V(G)$ such that $d_G(x,y) = d_G(x,v) + d_G(v,y)$, it indicates that for all $v \in V_2$, v is an interior vertex on G. Hence, ξ' is an InHRDF on G. By construction, ξ' is a γ_{InhR}-function on G. Hence, we get

$$\begin{aligned}\gamma_{InhR}(G) = \omega_G^{InhR}(\xi') &= |U_1| + 2|U_2| \\ &= \sum_{i=1}^{k} |A_i| + 2\sum_{i=1}^{k} |B_i| \\ &= \sum_{i=1}^{k} |\{x_{2+5(i-1)}, x_{4+5(i-1)}\}| + 2\sum_{i=1}^{k} |\{x_{3+5(i-1)}\}| \\ &= 4k \\ &= 5k - k \\ &= n - k\end{aligned}$$

where $k = \frac{n}{5}$.
Case 2: $n \equiv r \pmod 5$ where $1 \le r \le 4$

Let $n = 5k + r$ where $k \in \mathbb{N}$ and $1 \le r \le 4$. Then $k = \lfloor \frac{n}{5} \rfloor = \frac{n-r}{5}$ where $r \in \{1,2,3,4\}$. By *Case 1*, we can set $A_i' = \{x_{2+5(i-1)}, x_{4+5(i-1)}\}$ and $B_i' = \{x_{3+5(i-1)}\}$, for each $i \in \{1,2,...,k\}$. Then we let $U_1' = \left(\bigcup_{i=1}^{k} A_i'\right) \cup \{x_{n-r+1},...,x_n\}$ where $1 \le r \le 4$, $U_2' = \bigcup_{i=1}^{k} B_i'$ and $U_0' = V(G) \setminus (U_1' \cup U_2')$. Let $\xi'' = (U_0', U_1', U_2')$. Then ξ'' is an HRDF on G. Since for all $v' \in U_2'$, there exist $x', y' \in V(G)$ such that $d_G(x',y') = d_G(x',v') + d_G(v',y')$, it follows that for all $v' \in U_2'$, v' is an interior vertex on G. Thus, we have that ξ'' is an InHRDF on G. By construction, it means that ξ'' is a γ_{InhR}-function on G. So, we have

$$\begin{aligned}\gamma_{InhR}(G) = \omega_{InhR}(\xi'') &= |U_1'| + 2|U_2'| \\ &= \Big(\sum_{i=1}^{k} |A_i'| + r\Big) + 2\sum_{i=1}^{k} |B_i'| \\ &= \Big(\sum_{i=1}^{k} |\{x_{2+5(i-1)}, x_{4+5(i-1)}\}| + r\Big) + 2\sum_{i=1}^{k} |\{x_{3+5(i-1)}\}| \\ &= 2k + r + 2k \\ &= (5k + r) - k \\ &= n - k\end{aligned}$$

where $k = \frac{n-r}{5}$ for all $r \in \{1,2,3,4\}$.
Secondly, we let $G = C_n$ with $n \ge 3$. Then applying the similar argument, we arrived with the same conclusion. This completes the proof. □

Theorem 11: *Let G and H be non-complete graphs. Then, $\xi = (V_0, V_1, V_2)$ is an InHRDF on $G + H$ if and only if $\xi|_G = (V_0^G, V_1^G, V_2^G)$ and $\xi|_H = (V_0^H, V_1^H, V_2^H)$ are InHRDF on G and H, respectively, where $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for each $i \in \{0,1,2\}$.*

Proof: Assume that $f = (V_0, V_1, V_2)$ is an InHRDF on $G + H$. For each $i \in \{0, 1, 2\}$, we let $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$. Then it implies that $\xi|_G = (V_0^G, V_1^G, V_2^G)$ and $\xi|_H = (V_0^H, V_1^H, V_2^H)$. Let $v \in V_0^G$. Then $v \in V_0$. Since ξ is an InHRDF on $G + H$, it means that there exists $u \in V_2$ such that $v \in N^2_{G+H}(u)$ and $V_1 = V(G + H)$ or for every $z \in V_2$, z is an interior vertex on $G + H$. However, since $d_G(v, x) = 1$ for all $x \in V(H)$, it means that $u \in V_2^G$ and $V_1 = V(G)$ or for every $z \in V_2^G$, z is an interior vertex on G. Hence, $\xi|_G$ is an InHRDF on G. Now, let $v \in V_0^H$. Then, using a similar argument, it suffices to conclude that $\xi|_H$ is an InHRDF on H. On the other hand, we asumme that $\xi|_G = (V_0^G, V_1^G, V_2^G)$ and $\xi|_H = (V_0^H, V_1^H, V_2^H)$ are InHRDF on G and H, respectively. Let $V_i = V_i^G \cup V_i^H$ for all $i \in \{0, 1, 2\}$. Then $\xi = (V_0, V_1, V_2)$ is a function on $G + H$. Let $a \in V_0$. Then either $a \in V_0^G$ or $a \in V_0^H$. Without loss of generality, we consider $a \in V_0^G$. Since $\xi|_G$ is an InHRDF on G, it means there exists $b \in V_2^G$ such that $a \in N^2_G(b)$ and $V_1 = V(G)$ or for every $z \in V_2^G$, z is an interior vertex on G. Since $V_i^G \subseteq V_i$ for each $i \in \{0, 1, 2\}$, it follows that $b \in V_2$ for which $a \in N^2_{G+H}(b)$ and $V_1 = V(G + H)$ or for every $z \in V_2$, z is an interior vertex on $G + H$. Therefore, $\xi = (V_0, V_1, V_2)$ is an InHRDF on $G + H$. This completes the proof. □

The corollaries below are an immediate consequence of Theorem 11.

Corollary 6: *Let G and H be non-complete graphs. Then, $\gamma_{InhR}(G + H) = \gamma_{InhR}(G) + \gamma_{InhR}(H)$.*

Corollary 7: *Let G and H be complete graphs of order n and m, respectively. Then $\gamma_{InhR}(G + H) = n + m$.*

Theorem 12: *Let G be a graph and let $\xi = (V_0, V_1 = \emptyset, V_2)$ be an InHRDF on G. Then, V_2 is a minimal interior hop dominating set on G if and only if for every $u \in V_2$, there exists $v \in V_0$ such that $N^2_G(v) \cap V_2 = \{u\}$ or $d_G(u, z) \neq 2$ for any $z \in V_2 \setminus \{u\}$.*

Proof: Let $f = (V_0, V_1 = \emptyset, V_2)$ be an InHRDF on G. By Theorem 1, V_2 is an interior hop dominating set on G. Assume that V_2 is a minimal interior hop dominating set on G. This follows that for any $u \in V_2$, $V_2 \setminus \{u\}$ is not an interior hop dominating set on G. This implies that there exists $v \in V(G) \setminus (V_2 \setminus \{u\})$ such that $d_G(v, z) \neq 2$ for all $z \in V_2 \setminus \{u\}$. Suppose that $v \neq u$. Since V_2 is an interior hop dominating set, v is hop dominated by V_2. So, $d_G(u, v) = 2$, which indicates that $N^2_G(v) \cap V_2 = \{u\}$. Suppose that $v = u$. Then it follows that $d_G(u, z) \neq 2$ for any $z \in V_2 \setminus \{u\}$. As for the converse, assume that for every $u \in V_2$, there exists $v \in V_0$ such that $N^2_G(v) \cap V_2 = \{u\}$. Then $v \in V_0$ is not hop dominated by the set $V_2 \setminus \{u\}$. This means that V_2 is a minimal interior hop dominating set on G. Next, we assume that for every $u \in V_2$, $d_G(u, z) \neq 2$ for any $z \in V_2 \setminus \{u\}$. This implies that u is not hop dominated for any vertex $z \in V_2 \setminus \{u\}$. Hence, $V_2 \setminus \{u\}$ is not an interior hop dominating set on G. Therefore, V_2 is a minimal interior hop dominating set on G. This completes the proof. □

3 Conclusions

This study has introduced a new restricted parameter of a hop Roman domination called the interior hop Roman dominating function in graphs. This study has investigated some graph-theoretic and combinatorial properties of the interior hop dominating function in any graph. Exact values of the interior hop Roman domination number of some graph classes were also given. Moreover, characterizations were given for small values of the interior hop Roman domination number. Furthermore, characterization of the interior hop Roman dominating function on a graph with several components and the join of two graphs was also presented. For future studies, the newly defined variation of hop Roman domination in graphs can be investigated for some binary operations, which include corona, Cartesian, and lexicographic products.

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