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Research article

Statistical Gauge Convergence and Its Induced Topology in Metric Spaces

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Abstract: This paper introduces statistical gauge convergence as a refinement of statistical convergence in metric spaces, where deviations from the limit are controlled by positive continuous functions rather than fixed constants. We provide equivalent density-based characterizations and examine their relationship with both classical and statistical convergence, showing that the corresponding implications are strict in general. Furthermore, we investigate the topology generated by this convergence and prove that it is typically finer than the underlying metric topology. Several examples are included to clarify the hierarchical structure among the considered notions of convergence.

Mathematics Subject Classification: 54A20, 40A35, 54E35
Keywords: statistical convergence; Statistical gauge convergence; Sequential topology

1 Introduction and Preliminaries

The study of convergence constitutes one of the central themes of analysis and general topology. In metric and topological spaces, convergence of sequences serves as a fundamental tool for understanding structural properties such as compactness, completeness, and continuity.

Beyond classical convergence, various refined notions have been introduced to capture more flexible limiting behaviour. One of the most influential among these is *statistical convergence*, introduced independently by Fast [6] and Steinhaus [17], with conceptual roots in Zygmund's work on asymptotic density [19]. Instead of requiring convergence for all sufficiently large indices, statistical convergence demands that the exceptional set of indices has natural density zero. Since its introduction, the concept has been extensively developed in summability theory, approximation theory, and functional analysis [4,7]. It was further systematically studied by Šalát [16], who contributed to the early development and deeper understanding of statistically convergent sequences.

Several generalisations soon followed. Kostyrko, Šalát, and Wilczynski introduced I-convergence [9,10], providing a unifying framework based on ideals of $\mathbb{N}$, while Lahiri and Das [11,12] extended these ideas to more general topological settings. Systematic topological treatments of ideal-based convergence were presented in [8,18]. A significant milestone was achieved by Di Maio and Kočinac [5], who formulated statistical convergence within general topological spaces, thereby clarifying its interaction with sequential structures and induced topologies. Statistical convergence in uniform spaces and related continuity concepts were further investigated by Bilalov and Nazarova [2].

Parallel to these developments, statistical convergence has also been studied for sequences of functions and in function spaces [1,3], revealing that the choice of topology on the range space significantly influences the nature of convergence. Since classical continuity can be characterised via the preservation of convergent sequences, extending convergence notions naturally leads to corresponding generalisations of continuity [13–15]. These studies demonstrate that refined convergence concepts often induce nontrivial and sometimes strictly finer topologies.

Motivated by this interplay between convergence and topology, we introduce in this paper a refinement of statistical convergence in metric spaces, which we call *statistical gauge convergence*. A key feature of this notion is that the control of deviations from the limit is no longer governed by a fixed positive constant, but by positive continuous functions evaluated at the sequence terms.

This approach provides a locally adaptive tolerance mechanism. In classical statistical convergence, the deviation from the limit is measured against a constant threshold, which imposes a uniform global restriction. In contrast, by evaluating the control function at x_n, statistical gauge convergence allows the admissible deviation to vary depending on the position of the sequence in the space. This makes the notion sensitive to the local structure of the space and enables the detection of finer asymptotic behaviours that cannot be captured by classical statistical convergence.

Accordingly, for a sequence (x_n) in a metric space (X, d) and $x \in X$, statistical gauge convergence requires that for every $\varepsilon \in C^+(X)$, the set

$$n \in \mathbb{N} : d(x_n, x) \geq \varepsilon(x_n)$$

has natural density zero.

This definition preserves the asymptotic character of statistical convergence while strengthening it through a variable control mechanism. In particular, classical convergence implies statistical gauge convergence, whereas the converse fails in general. Moreover, statistical gauge convergence is strictly stronger than classical statistical convergence and gives rise to a naturally induced sequential topology.

The main objective of this paper is to develop the theory of statistical gauge convergence in metric spaces by establishing its fundamental properties and equivalent characterisations, comparing it with classical and statistical convergence through illustrative examples and counterexamples, and constructing the topology induced by statistical gauge sequential closure. In particular, we show that this topology is, in general, finer than the underlying metric topology.

If $A \subseteq \mathbb{N}$, then $A(n)$ denotes the set

$$A(n) = \{k \in A : k \leq n\}.$$

The natural (or asymptotic) density of A is given by

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n},$$

if it exists. The density takes values in the interval $[0,1]$. Clearly, finite subsets of $\mathbb{N}$ have natural density zero.

A subset $A \subseteq \mathbb{N}$ is called statistically dense when its natural density satisfies $\delta(A) = 1$. Also, for any $A \subseteq \mathbb{N}$, we have

$$\delta(\mathbb{N} \setminus A) = 1 - \delta(A).$$

Statistical convergence in metric spaces is formally defined as follows [2]:

Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be statistically convergent to a point $x \in X$ if for every $\varepsilon > 0$, the set

$$A = \{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon\}$$

has natural density zero, i.e.,

$$\delta(A) = 0.$$

Equivalently, there exists a statistically dense subset $A \subseteq \mathbb{N}$ such that the corresponding subsequence $(x_n)_{n\in A}$ converges to x [2]. We denote it by

$$x_n \xrightarrow{st} x \text{ or } st - \lim_{n\to\infty} x_n = x.$$

Throughout the paper, (X, d) denotes a metric space and $C^+(X)$ denotes the family of all positive continuous functions on X.

2 Statistical Gauge Convergence

Definition 1: *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The sequence (x_n) is said to be* statistically gauge convergent *to $x \in X$ if for every $\varepsilon \in C^+(X)$,*

$$\delta\big(\{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon(x_n)\}\big) = 0.$$

In this case, we write $x_n \xrightarrow{sg} x$.

Remark 1: *In statistical gauge convergence, the control function is evaluated at the sequence point x_n rather than at the limit point x. This feature makes the notion essentially non-uniform and, in general, stronger than classical statistical convergence.*

Statistical gauge convergence admits an equivalent characterization in terms of convergence along subsets of natural density one.

Proposition 1: *Let (x_n) be a sequence in X. Then $x_n \xrightarrow{sg} x$ if and only if for every $\varepsilon \in C^+(X)$ there exists a set $A_\varepsilon \subseteq \mathbb{N}$ with $\delta(A_\varepsilon) = 1$ such that*

$$d(x_n, x) < \varepsilon(x_n) \quad \textit{for all } n \in A_\varepsilon.$$

Proof: Assume $x_n \xrightarrow{sg} x$ and fix $\varepsilon \in C^+(X)$. Set

$$B_\varepsilon = \{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon(x_n)\}.$$

By definition, $\delta(B_\varepsilon) = 0$. Let $A_\varepsilon = \mathbb{N} \setminus B_\varepsilon$. Then $\delta(A_\varepsilon) = 1$ and the desired inequality holds.

Conversely, suppose that such a set A_ε exists for every $\varepsilon \in C^+(X)$. Then

$$\{n : d(x_n, x) \geq \varepsilon(x_n)\} \subset \mathbb{N} \setminus A_\varepsilon,$$

which has density zero. □

Theorem 1: *Let (X, d) be a metric space and (x_n) a sequence in X. If (x_n) converges to $x \in X$, then (x_n) is statistically gauge convergent to x.*

Proof: Suppose that (x_n) converges to x in the usual metric sense and let $\varepsilon \in C^+(X)$ be arbitrary.

Since ε is continuous at x and $\varepsilon(x) > 0$, there exists $\delta > 0$ such that

$$d(y, x) < \delta \implies |\varepsilon(y) - \varepsilon(x)| < \tfrac{1}{2}\varepsilon(x).$$

In particular,

$$d(y, x) < \delta \implies \varepsilon(y) > \tfrac{1}{2}\varepsilon(x).$$

Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, x) < \delta.$$

For such n we have

$$d(x_n, x) < \delta \quad \text{and} \quad \varepsilon(x_n) > \tfrac{1}{2}\varepsilon(x).$$

Choosing δ smaller if necessary so that $\delta < \frac{1}{2}\varepsilon(x)$, we obtain

$$d(x_n, x) < \varepsilon(x_n) \quad \text{for all } n \geq N.$$

Define

$A_\varepsilon = \{n \in \mathbb{N} : n \geq N\}.$

Then $\delta(A_\varepsilon) = 1$, and by Proposition 1,

$x_n \xrightarrow{sg} x.$

Hence, every sequence that converges in the usual metric sense is also statistically gauge convergent. □

The converse of Theorem 1 does not hold in general.

Example 1: *Let $X = \mathbb{R}$ and define the sequence (x_n) by*

$$x_n = \begin{cases} 0, & \text{if } n \text{ is not a perfect square,} \\ n, & \text{if } n \text{ is a perfect square.} \end{cases}$$

Observe that the set

$A = \{n \in \mathbb{N} : n \text{ is not a perfect square}\}$

has natural density $\delta(A) = 1$, because the perfect squares are sparse.

For any $\varepsilon \in C^+(\mathbb{R})$, all $n \in A$ satisfy

$d(x_n, 0) = |x_n - 0| = 0 < \varepsilon(x_n),$

so by Proposition 1, we have

$x_n \xrightarrow{sg} 0.$

However, (x_n) does not *converge to* 0 *in the usual metric sense, because the subsequence $x_{n^2} = n \to \infty$. Thus, this sequence is statistically gauge convergent but not classically convergent.*

We now compare statistical gauge convergence with classical statistical convergence.

Theorem 2: *Let (x_n) be a sequence in X. If a sequence (x_n) is statistically gauge convergent to x, then it is classically statistically convergent to x.*

Proof: Let $\eta > 0$ and define $\varepsilon(t) = \eta$ for all $t \in X$. Since constant positive functions belong to $C^+(X)$, statistical gauge convergence yields

$\delta\big(\{n : d(x_n, x) \geq \eta\}\big) = 0,$

which is exactly classical statistical convergence. □

The converse of Theorem 2 does not hold in general.

Example 2: *Let $X = \mathbb{R}$ with the usual metric and consider the function $\varepsilon(x) = |x|$. Although this function is not strictly positive at $x = 0$, the argument below remains valid if one replaces it with $\varepsilon(x) = |x| + \delta$ for any $\delta > 0$.*

Consider the sequence $x_n = \frac{1}{n}$. It is classically statistically convergent to 0. However,

$$d(x_n, 0) = \varepsilon(x_n) = \frac{1}{n} \quad \textit{for all } n,$$

and hence

$$\delta\big(n : d(x_n, 0) \geq \varepsilon(x_n)\big) = 1.$$

Thus (x_n) is not statistically gauge convergent to 0.

Theorem 3: *Let (x_n) be a sequence in X. If (x_n) is statistically gauge convergent to both x and y, then $x = y$.*

Proof: Assume $x \neq y$ and let $\eta = \frac{1}{3}d(x, y)$. Choose the constant gauge $\varepsilon(t) = \eta$. By statistical gauge convergence,

$$\delta(\{n : d(x_n, x) \geq \eta\}) = \delta(\{n : d(x_n, y) \geq \eta\}) = 0.$$

Hence the set

$$A = \{n : d(x_n, x) < \eta \text{ and } d(x_n, y) < \eta\}$$

has density one. For $n \in A$, the triangle inequality yields

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\eta = \tfrac{2}{3}d(x, y),$$

a contradiction. □

3 Topology Induced by Statistical Gauge Convergence

Definition 2: *Let $A \subset X$. The* statistical gauge-sequential closure *of A is*

$$\overline{A}^{sg} = \{x \in X : \exists (x_n) \subset A \textit{ such that } x_n \xrightarrow{sg} x\}.$$

Proposition 2: *For all $A, B \subset X$, the operator $A \mapsto \overline{A}^{sg}$ satisfies:*

(i) $A \subset \overline{A}^{sg}$

(ii) *If $A \subset B$, then $\overline{A}^{sg} \subset \overline{B}^{sg}$*

(iii) $\overline{\overline{A}^{sg}}^{sg} = \overline{A}^{sg}$
(iv) $\overline{A \cup B}^{sg} = \overline{A}^{sg} \cup \overline{B}^{sg}$

Proof: (i) Let $a \in A$. The constant sequence $x_n = a$ satisfies $d(x_n, a) = 0 < \varepsilon(x_n)$ for every $\varepsilon \in C^+(X)$. Hence $a \in \overline{A}^{sg}$.

(ii) If $A \subset B$ and $x \in \overline{A}^{sg}$, there exists $(x_n) \subset A$ with $x_n \xrightarrow{sg} x$. Since $(x_n) \subset B$, we obtain $x \in \overline{B}^{sg}$.

(iii) The inclusion $\overline{A}^{sg} \subset \overline{\overline{A}^{sg}}^{sg}$ follows from (i).

Conversely, let $x \in \overline{\overline{A}^{sg}}^{sg}$. Then there exists $(x_n) \subset \overline{A}^{sg}$ such that $x_n \xrightarrow{sg} x$. For each n, choose $(x_{n,k}) \subset A$ with $x_{n,k} \xrightarrow{sg} x_n$.

Let $\varepsilon \in C^+(X)$ be arbitrary. Since $x_n \xrightarrow{sg} x$, the set

$$D_\varepsilon = \{n \in \mathbb{N} \,:\, d(x_n, x) < \tfrac{1}{2}\varepsilon(x_n)\}$$

has natural density 1.

For each n, because $x_{n,k} \xrightarrow{sg} x_n$, the set

$$E_{n,\varepsilon} = \{k \in \mathbb{N} \,:\, d(x_{n,k}, x_n) < \tfrac{1}{2}\varepsilon(x_{n,k})\}$$

has density 1.

For $n \in D_\varepsilon$, choose $k(n) \in E_{n,\varepsilon}$ and define $y_n = x_{n,k(n)} \in A$.

Then for $n \in D_\varepsilon$,

$$d(y_n, x) \leq d(y_n, x_n) + d(x_n, x) < \tfrac{1}{2}\varepsilon(y_n) + \tfrac{1}{2}\varepsilon(y_n) = \varepsilon(y_n).$$

Since D_ε has density 1, it follows that $y_n \xrightarrow{sg} x$. Thus $x \in \overline{A}^{sg}$.

(iv) From (ii),

$$\overline{A}^{sg} \cup \overline{B}^{sg} \subset \overline{A \cup B}^{sg}.$$

Conversely, let $x \in \overline{A \cup B}^{sg}$. Then $(x_n) \subset A \cup B$ with $x_n \xrightarrow{sg} x$. Define

$$I_A = \{n \,:\, x_n \in A\}, \qquad I_B = \{n \,:\, x_n \in B\}.$$

Since $I_A \cup I_B = \mathbb{N}$, at least one of them has density 1. Assume $\delta(I_A) = 1$. Restricting (x_n) to I_A yields a sequence in A that still statistically gauge-converges to x. Hence $x \in \overline{A}^{sg}$. □

Theorem 4: *The family*

$$\tau_{sg} = \{U \subset X \,:\, X \setminus U \text{ is statistically gauge-closed}\}$$

defines a topology on X.

Proof: By Proposition 2, $\overline{(\cdot)}^{sg}$ satisfies the Kuratowski closure axioms. Hence the complements of closed sets form a topology. □

We now compare the topology τ_{sg} induced by statistical gauge convergence with the underlying metric topology τ on X.

Theorem 5: *Let (X, d) be a metric space. Then the topology τ_{sg} generated by statistical gauge convergence is in general finer than the metric topology τ:*

$$\tau \subseteq \tau_{sg}.$$

In other words, every τ-open set is also τ_{sg}-open.

Proof: Let $U \subset X$ be open in the metric topology, and let $x \in U$. Then there exists $\eta > 0$ such that the metric ball $B_\eta(x) \subset U$.

Let $(x_n) \subset X$ be any sequence with $x_n \xrightarrow{sg} x$. By definition, for every $\varepsilon \in C^+(X)$,

$$\delta\big(\{n : d(x_n, x) \geq \varepsilon(x_n)\}\big) = 0.$$

Choose the constant function $\varepsilon(y) = \eta$. Then

$$\delta\big(\{n : d(x_n, x) \geq \eta\}\big) = 0.$$

Equivalently,

$$\delta\big(\{n : d(x_n, x) < \eta\}\big) = 1.$$

Hence $x_n \in B_\eta(x) \subset U$ for a set of indices of natural density 1.

Therefore no statistically gauge-convergent sequence from $X \setminus U$ can converge to x, which shows that U is τ_{sg}-open. Consequently, $\tau \subseteq \tau_{sg}$. □

4 Conclusion

In this paper, we introduced statistical gauge convergence as a refinement of classical statistical convergence in metric spaces, where deviations from the limit are controlled by positive continuous functions rather than fixed constants. This variable control mechanism provides a locally adaptive framework, allowing the admissible deviation to depend on the position of the sequence, and thereby capturing finer asymptotic behaviour.

We established the fundamental properties of statistical gauge convergence, including its density-based characterization, uniqueness of limits, and its relationship with classical convergence and statistical convergence. In particular, we proved that classical convergence implies statistical gauge convergence, and that statistical gauge convergence implies statistical convergence, while the converses fail in general. These results position the new notion strictly between classical convergence and statistical convergence, while demonstrating that it is genuinely stronger than the latter.

Moreover, we constructed the topology τ_{sg} induced by statistical gauge sequential closure and showed that it defines a topology finer than the underlying metric topology. This reveals that statistical gauge convergence generates a strictly stronger sequential structure and leads to a nontrivial refinement of the ambient topology.

The concept introduced here suggests several directions for further investigation. In particular, it would be natural to study statistical gauge convergence in more general settings such as uniform spaces and function spaces, to examine completeness and compactness-type properties associated with τ_{sg}, and to explore its connections with ideal- and filter-based convergence structures. Such extensions may contribute to a deeper understanding of adaptive and locally controlled convergence methods in analysis and topology.

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