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Research article

Outer-convex hop Roman dominating function in graphs

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Abstract: Let $G = (V(G), E(G))$ be a connected graph and let $f : V(G) \to \{0, 1, 2\}$ be a hop Roman dominating function (HRDF) on G. If for each $k \in \{0, 1, 2\}$, $V_k = \{x \in V(G) : f(x) = k\}$, then $f = (V_0, V_1, V_2)$. A function f is an outer-convex hop Roman dominating function (OConHRDF) on G provided that for every $v \in V_0$, there exists $u \in V_2$ such that $v \in N_G^2(u)$ and V_0 is a convex set. The *weight* of OConHRDF f on G is denoted by $\widetilde{\omega}_G^{conhR}(f)$ and is defined as $\widetilde{\omega}_G^{conhR}(f) = \sum_{v \in V(G)} f(v)$. The smallest weight of an OConHRDF f on G, denoted by $\widetilde{\gamma}_{conhR}(G)$ is called the *outer-convex hop Roman domination number*, which can be written as $\widetilde{\gamma}_{conhR}(G) = min\{\widetilde{\omega}_G^{conhR}(f) : f\ is\ an$ OConHRDF $on\ G\}$. Every OConHRDF f on G satisfying the condition $\widetilde{\omega}_G^{conhR}(f) = \widetilde{\gamma}_{conhR}(G)$ is so-called a $\widetilde{\gamma}_{conhR}$-function on G. This paper introduces a new parameter of a hop Roman dominating function in graphs, called outer-convex hop Roman dominating function and presents initial investigation.

Mathematics Subject Classification: 05C69
Keywords: Outer-convex domination; Hop domination; Roman dominating function; Outer-convex hop Roman dominating function.

1 Introduction

Dominating sets have become interesting and growing topics in graph theory. Currently, the Roman dominating function is one of the intriguing parameters of dominating sets in graphs, which was pioneered by Cockayne et al. [9] in the year 2004. Some studies that dealt with Roman dominating function can be found in [1,2,7,10,13]. Later, the hop Roman domination in graphs was initiated in the year 2017 by Shabani [14], and this topic has become an interest of many graph theorists. Another interesting topic that captures many mathematicians is the outer-convex domination in graphs, which was introduced in the year 2020 by Dayap and Enriquez [11]. In that case, the authors are inspired to fuse the two concepts, such as hop Roman and outer-convex dominations in graphs to form a stronger parameter of the Roman domination scheme and establish a new interesting network security. In fact, the new parameter may be useful in designing a network security system with a stronger connection of digital infrastructures as an application in computer science. For the definitions involved in this study, the readers may refer to [4,5,12].

Let $G = (V(G), E(G))$ be a connected graph of order $n \in \mathbb{N}$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. Let $x, y \in V(G)$. The distance between x and y, denoted by $d_G(x, y)$, is the length of the shortest path between x and y in G. The set $N_G^2(x) = \{z \in V(G) : deg_G(x, z) = 2\}$ is called the *open hop-neighborhood* and every element of $N_G^2(x)$ is called hop-neighbor of x. For $A \subseteq V(G)$, $N_G^2(A) = \bigcup_{x \in H} N_G^2(x)$ and $N_G^2[A] = N_G^2(A) \cup A$. Let $x, y \in G$. An x-y path with distance $d_G(x, y)$ is called x-y geodesic. A closed interval denoted by $I_G[x, y]$ is a set that consists of all vertices that lie on an x-y geodesic on G. Let $U \subset V(G)$ where $x, y \in U$. The union of all sets $I_G[x, y]$ is written as $\cup_{x,y \in U} I_G[x, y] = I_G[U]$. In that case, $v \in I_G[U]$ if and only if v is in some x-y geodesic for any $x, y \in U$. A set U is a convex set in G if and only if $I_G[U] = U$. In other words, a convex set in a connected graph G is a set $U \subseteq V(G)$ in which for any $x, y \in U$ every x-y geodesic is entirely contained within the set U. Accordingly, $V(G)$ is a convex set if G is connected. The largest cardinality of a set U is called the convex number of G, denoted by $con(G)$.

A subset D of $V(G)$ is called a *dominating set* of G provided that for each $x \in V(G) \setminus D$, there exists $y \in D$ such that $d_G(x, y) = 1$ [12]. The smallest cardinality of a dominating set D on G is called the *domination number* on G and is denoted by $\gamma(G)$. The dominating set D that satisfies $|D| = \gamma(G)$ is called γ-set on G. For some studies on dominating sets, we refer to [3–6,8]. A subset C of $V(G)$ is called an *outer-convex dominating set* on G provided that for every $v \in V(G) \setminus C$, there exists $u \in C$ such that $d_G(u, v) = 1$ and $V(G) \setminus C$ is a convex set on G [11]. The minimum cardinality of C is called *outer-convex domination number* and is denoted by $\widetilde{\gamma}_{con}(G)$. An outer-convex dominating set C with $|C| = \widetilde{\gamma}_{con}(G)$ is called $\widetilde{\gamma}_{con}$-set on G. A set $H \subseteq V(G)$ is called a *hop dominating set* of G provided that for each vertex in $x \in V(G) \setminus H$, there exists $y \in H$ such that $x \in N_G^2(y)$. The minimum cardinality of a hop dominating set on G is called the *hop domination number* of G and is denoted $\gamma_h(G)$. A hop dominating set that satisfies $|H| = \gamma_h(G)$ is called a γ_h-set on G.

Let $f : V(G) \to \{0, 1, 2\}$ be a function on G. for each $k \in \{0, 1, 2\}$. For each $k \in \{0, 1, 2\}$, we set $V_k = \{x \in V(G) : f(x) = k\}$. In that case, we can have $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a *Roman dominating function* (RDF) on G provided that for each vertex $x \in V_0$ there exists $y \in V_2$ such that $d_G(x, y) = 1$ [9]. The *weight* of f on G is denoted by $\omega_G^R(f)$ and is defined by $\omega_G^R(f) = \sum_{x \in V(G)} f(x) = |V_1| + 2|V_2|$. The *Roman domination number* of G is the minimum weight of f on G, that is, $\gamma_R(G) = min\{\omega_G^R(\lambda) : \lambda \textit{ is an } \text{RDF } on\ G\}$. Every RDF f on G satisfying $\omega_G^R(\lambda) = \gamma_R(G)$ is a γ_R-function on G. A function $f = (V_0, V_1, V_2)$ is a *hop Roman dominating function* (HRDF) on G provided that for each $x \in V_0$, there exists $y \in V_2$ such that $d_G(x, y) = 2$ [14]. The *weight* of f denoted by $\omega_G^{hR}(f)$ and is defined as $\omega_G^{hR}(f) = \sum_{z \in V(G)} f(z)$. The *hop Roman domination number* of G is defined as the smallest weight of an HRDF f on G denoted $\gamma_{hR}(G)$, that is, $\gamma_{hR}(G) = min\{\omega_G^{hR}(f) : f \textit{ is an } \text{HRDF } on\ G\}$. So, every HRDF f on G that satisfies $\omega_G^{hR}(f) = \gamma_{hR}(G)$ is called a γ_{hR}-function on G.

Let $G = (V(G), E(G))$ be a connected graph. A function $f = (V_0, V_1, V_2)$ is an outer-convex hop Roman dominating function (OConHRDF) on G provided that for every $v \in V_0$, there exists $u \in V_2$ such that $v \in N_G^2(u)$ and V_0 is a convex set. The *weight* of OConHRDF f on G is denoted by $\widetilde{\omega}_G^{conhR}(f)$ and is defined as $\widetilde{\omega}_G^{conhR}(f) = \sum_{v \in V(G)} f(v) = |V_1| + 2|V_2|$. The smallest weight of an OConHRDF f on G, denoted by $\widetilde{\gamma}_{conhR}(G)$ is called the *outer-convex hop Roman domination number* which can be written as

$\widetilde{\gamma}_{conhR}(G) = min\{\widetilde{\omega}_G^{conhR}(f) \,:\, f \; is \; an$ OConHRDF $on \; G\}$. Every OConHRDF f on G satisfying the condition $\widetilde{\omega}_G^{conhR}(f) = \widetilde{\gamma}_{conhR}(G)$ is so-called a $\widetilde{\gamma}_{conhR}$- function on G. This paper aims to introduce a new variant of a hop Roman domination in graphs called outer-convex hop Roman domination. Important theoretical properties of an outer-convex hop Roman dominating function in some classes of connected graphs, bounds of outer-convex hop Roman domination number, the realization problem, and characterization in a join of two graphs were investigated.

2 Results

This section presents some important results involving outer-convex hop Roman dominating function in connected graphs.

Lemma 1: *Let G be a connected graph of order $n \in \mathbb{N}$ and $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Then $V_1 \cup V_2$ is an outer-convex hop dominating set on G. Moreover, $V_0 = \emptyset$ if and only if $V_2 = \emptyset$. In that case, $\widetilde{\gamma}_{conhR}(G) = n$.*

Proof: Assume that $f = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conhR}$-function on G. This follows that f is an OConHRDF on G. Let $v \in V_0$. Then there exists $u \in V_2$ such that $v \in N_G^2(u)$ and V_0 is a convex set on G. Hence, $V_1 \cup V_2$ is an outer-convex hop dominating set on G. Suppose $V_0 = \emptyset$. Seeking a contradiction. Assume that $V_2 \neq \emptyset$. Let $u \in V_2$. Set $W_0 = V_0 = \emptyset$, $W_1 = V_1 \cup \{u\}$, and $W_2 = V_2 \setminus \{u\}$. This follows that $g = (W_0, W_1, W_2)$ is an OConHRDF on G. Thus, $\widetilde{\omega}_G^{conhR}(g) = |W_1| + 2|W_2| = (|V_1| + 1) + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 < \widetilde{\omega}_G^{conhR}(f) = \widetilde{\gamma}_{conhR}(G)$. A contradiction. Accordingly, $V_2 = \emptyset$. Conversely, suppose $V_2 = \emptyset$. Since f is a $\widetilde{\gamma}_{conhR}$-function on G, it is clear that $V_0 = \emptyset$. To this end, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. This completes the proof. □

Corollary 1: *Let $G \in \{P_n, K_n\}$ where $n \in \mathbb{N}$. Then $\widetilde{\gamma}_{conhR}(G) = n$.*

Proof: Let $f = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conhR}$-function on G. First, we assume that $G = P_n$ where $n \in \mathbb{N}$. Let $G = P_n = [x_1, x_2, ..., x_n]$. For $n = 1, 2, 3$, it is clear that $\widetilde{\gamma}_{conhR}(G) = 3$. Now, consider $n \geq 4$. Since $con(P_n) = 2$ for $n \geq 4$, we can let $V_0 = \{x_1, x_2\}$, $V_1 = \{x_5, x_6, ..., x_n\}$, and $V_2 = \{x_3, x_4\}$. In this case, we have $V_0 \subseteq N_G^2[V_2]$ and V_0 is a convex set on G. Hence, by construction, we obtain $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = |\{x_5, x_6, ..., x_n\}| + 2|\{x_3, x_4\}| = (n-4) + 2(2) = n$. Equivalently, we can set $|V_0| = |V_2| = 0$. By Lemma 1, it implies that $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| = n$. Secondly, assume that $G = K_n$ where $n \in \mathbb{N}$. By Lemma 1, $V_1 \cup V_2$ is an outer-convex hop dominating set on G. This implies that $V_1 \cup V_2 = V(G)$. Thus $V_0 = \emptyset$. Since f $\widetilde{\gamma}_{conhR}$-function on G, it follows that $V_2 = \emptyset$ by Lemma 1. Therefore, we conclude that $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| = n$. This completes the proof. □

Theorem 1: *Let G be a connected graph of order $n \in \mathbb{N}$. Then the following holds:*

$$max\{\gamma_R(G), \widetilde{\gamma}_{conh}(G)\} \leq \widetilde{\gamma}_{conhR}(G) \leq min\{2\widetilde{\gamma}_{conh}(G), n\}.$$

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on a connected graph G. By Lemma 1, it implies that $V_1 \cup V_2$ is an outer-convex hop dominating set on G. Thus, $\widetilde{\gamma}_{conh}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\omega}_G^{conhR}(f) = \widetilde{\gamma}_{conhR}(G)$. Moreover, since every outer-convex hop Roman dominating function is a Roman dominating function on G, we obtain $\gamma_R(G) \leq \widetilde{\gamma}_{conhR}(G)$. Hence, we have $max\{\gamma_R(G), \widetilde{\gamma}_{conh}(G)\} \leq \widetilde{\gamma}_{conhR}(G)$. Meanwhile, let $V_0 = \varnothing$. Since f is a $\widetilde{\gamma}_{conhR}$-function on G, by Lemma 1, it implies that $V_2 = \varnothing$. Consequently, $f = (V_0 = \varnothing, V_1 = V(G), V_2 = \varnothing)$ is an OConHRDF on G. Thus, $\widetilde{\gamma}_{conhR}(G) \leq \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = |V_1| = |V(G)| = n$. Suppose that f is a $\widetilde{\gamma}_{conhR}$-function on G with $V_1 = \varnothing$. Seeking a contradiction. Assume that V_2 is not a γ_{conh}-set on G. Let U_2 be a γ_{conh}-set on G. Then $V(G) \setminus U_2$ is a convex set on G and $|U_2| < |V_2|$. Set $W_0 = V(G) \setminus U_2$, $W_1 = \varnothing$ and $W_2 = U_2$. This implies that $g = (W_0, W_1, W_2)$ is an OConHRDF on G. So, we have $\widetilde{\omega}_G^{conhR}(g) = 2|W_2| < 2|V_2| = \widetilde{\omega}_G^{conhR}(f) = \widetilde{\gamma}_{conhR}(G)$. A contradiction to the assumption that f is a $\widetilde{\gamma}_{conhR}$-function on G. Hence, V_2 is not a γ_{conh}-set on G, that is, $\widetilde{\gamma}_{conh}(G) = |V_2|$. So, $\widetilde{\gamma}_{conhR}(G) \leq \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 2|V_2| = 2\widetilde{\gamma}_{conh}(G)$. Accordingly, it suffices to say that $\widetilde{\gamma}_{conhR}(G) \leq min\{n, 2\widetilde{\gamma}_{conh}(G)\}$. Therefore, $max\{\gamma_R(G), \widetilde{\gamma}_{conh}(G)\} \leq \widetilde{\gamma}_{conhR}(G) \leq min\{2\widetilde{\gamma}_{conh}(G), n\}$. This completes the proof. □

Theorem 2: *Let G be a connected graph of order $n \in \mathbb{N}$ and let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Then the following hold:*

(i) $|V_0| = |V_2| \geq 0$ *if and only if* $\widetilde{\gamma}_{conhR}(G) = n$; *and*

(ii) $1 \leq |V_2| < |V_0|$ *if and only if* $\widetilde{\gamma}_{conhR}(G) < n$.

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on a connected graph G. Assume that $|V_0| = |V_2| \geq 0$. Suppose $|V_0| = |V_2| = 0$. By Lemma 1, it follows that $\widetilde{\gamma}_{conhR}(G) = n$. Suppose $|V_0| = |V_2| \geq 1$. This implies that $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = |V_0| + |V_1| + |V_2| = |V(G)| = n$. Conversely, assume that $\widetilde{\gamma}_{conhR}(G) = n$. Seeking a contradiction. Suppose $|V_0| \neq |V_2|$. Then either $|V_0| > |V_2|$ or $|V_0| < |V_2|$. If $|V_0| > |V_2|$, then $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| < |V_0| + |V_1| + |V_2| = |V(G)| = n$, a contradiction to the assumption. If $|V_0| < |V_2|$, then $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| > |V_0| + |V_1| + |V_2| = |V(G)| = n$, a contradiction to the Theorem 1 that $\widetilde{\gamma}_{conhR}(G) \leq n$. Hence, we get $|V_0| = |V_2| \geq 0$. Thus, (i) holds.

Meanwhile, assume that $\widetilde{\gamma}_{conhR}(G) < n$. Then $\widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| < n$ and so, $V_0 \neq \varnothing$. By the definition of OConHRDF on G, it implies that $|V_2| \geq 1$. By (i), we have $|V_0| \neq |V_2|$. This follows that we have either $|V_0| > |V_2|$ or $|V_0| < |V_2|$. Suppose $|V_2| > |V_0|$. Then we have $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$. This is a contradiction to the Theorem 1 that $\widetilde{\gamma}_{conhR}(G) \leq n$. Consequently, $1 \leq |V_2| < |V_0|$. Conversely, assume that $1 \leq |V_2| < |V_0|$. This implies that $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. Therefore, ($ii$) holds. This completes the proof. □

Theorem 3: *Let G be a connected graph. Then the following hold:*

(i) $\widetilde{\gamma}_{conhR}(G) = 1$ *if and only if* $G = K_1$; *and*

(ii) $\widetilde{\gamma}_{conhR}(G) = 2$ *if and only if* $G = K_2$.

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Assume that $\widetilde{\gamma}_{conhR} = 1$. Then $|V_1| + 2|V_2| = 1$. This implies that $|V_2| = 0$. By Lemma 1, we get $|V_0| = 0$. Thus, we have $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| =$

1. Hence, $G = K_1$. Conversely, let $G = K_1$. By Corollary 1, it follows that $\widetilde{\gamma}_{conhR}(G) = 1$. Therefore, (i) holds. On the other hand, assume that $\widetilde{\gamma}_{conhR}(G) = 2$. Then $\widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 2$. Thus, $|V_2| \leq 1$. Suppose $|V_2| = 1$. Since $f = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conhR}$-function on G, it follows that $|V_1| = 0$ and $|V_0| \neq 0$. Let $V_2 = \{u\}$ and let $v \in V_0$. Then $v \in N_G^2(u)$. Now, let $x \in N_G(u) \cap N_G(v)$. Since $|V_1| = 0$, it implies that $x \in V_0$. This follows that $N_G^2(x) \cap V_2 = \emptyset$. So, V_2 is not a hop dominating set on G, a contradiction. Hence, it means that $|V_2| = 0$. By Lemma 1, $|V_0| = 0$. So, we get $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| = 2$. Since G is a connected graph, we have $G = K_2$. Conversely, let $G = K_2$. By Corollary 1, it implies that $\widetilde{\gamma}_{conhR}(G) = 2$. Therefore, (ii) holds. This completes the proof. □

Proposition 1: *Let G be a connected graph and let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. If $|V_2| = 1$, then $|V_1| \neq 0$ and $deg_G(u) \leq |V_1|$ where $u \in V_2$.*

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Suppose $|V_2| = 1$. Seeking a contradiction. Assume for a moment that $|V_1| = 0$. Then by Lemma 1, it implies that V_2 is an outer-convex hop dominating set on G. Hence, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 2(1) = 2$. By Theorem 3, we get $G = K_2$ and it means that for any $x, y \in V(G)$, $d_G(x, y) \leq 1$. This is a contradiction since V_2 is a hop dominating set on G. Hence, we conclude that $|V_1| \neq 0$. Now, since $|V_2| = 1$ and f is a $\widetilde{\gamma}_{conhR}$-function on G, it follows that $|V_0| \geq 1$. Let $V_2 = \{u\}$ and $v \in V_0$ be an arbitrary. Then $v \in N_G^2(u)$. This implies that there exists $x \in V(G)$ such that $x \in V_1$ and $x \in N_G(u)$. Therefore, it suffices to say that $deg_G(u) \leq |V_1|$. This completes the proof. □

Theorem 4: *Let G be a connected graph of order $n \in \mathbb{N} \setminus \{1, 2\}$ and $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. If $|V_2| = 1$ and $|V_1| = 1$, then for any $x_1, x_2 \in V_0$, $d_{\langle V_0 \rangle}(x_1, x_2) \leq 2$.*

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on a connected graph of order $n \in \mathbb{N} \setminus \{1, 2\}$. Assume that $|V_2| = 1$ and $|V_1| = 1$. Since $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G, it implies that $|V_0| \geq 1$. Suppose $|V_0| = 1$. Then V_0 is a trivial convex set on G and so, $d_{\langle V_0 \rangle}(z, z) = 0 = d_G(z, z)$ where $V_0 = \{z\}$. On the other hand, we suppose that $|V_0| \geq 2$. Let $x_1, x_2 \in V_0$ be arbitrary vertices. If $x_1 = x_2$, then $d_{\langle V_0 \rangle}(x_1, x_2) = 0 = d_G(x_1, x_2)$. Let $x_1 \neq x_2$. If $x_1x_2 \in E(G)$, then we have $d_{\langle V_0 \rangle}(x_1, x_2) = 1 = d_G(x_1, x_2)$. Let $x_1x_2 \notin E(G)$, let $u \in V_2$, and let $v \in V_1$. Then we have $v \in N_G(u)$, $\{x_1, x_2\} \subseteq N_G^2(u)$ and $\{x_1, x_2\} \subseteq N_G(v)$. This implies that $xv, vy \in E(G)$. Thus, we have $d_G(x_1, x_2) = 2$. Since f is $\widetilde{\gamma}_{conhR}$-function on G, it implies that V_0 is a convex set on G and we obtain $d_{\langle V_0 \rangle}(x_1, x_2) = 2 = d_G(x_1, x_2)$. Therefore, for any $x_1, x_2 \in V_0$, $d_{\langle V_0 \rangle}(x_1, x_2) \leq 2$ whenever $|V_2| = 1$ and $|V_1| = 1$. This completes the proof. □

Theorem 5: *Let G be a connected graph and $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Then $\gamma_{conhR}(G) = 3$ if and only if $G = K_1 + H$ where $H \in \{K_2, K_1 \cup K_{n\geq 1}, K_1 \cup C_{4\leq n\leq 5}, K_1 \cup P_3\}$.*

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on graph G. Suppose that $\gamma_{conhR}(G) = 3$. Then it follows that $\widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 3$. Thus, either $|V_1| = 3$ or $|V_2| = 1$ and $|V_1| = 1$. If $|V_1| = 3$, then we obtain $|V(G)| = 3$. Since G is connected, then $G = K_3$ or $G = P_3$. Hence, we have $G = K_1 + H$ where $H \in \{K_2, K_1 \cup K_1\}$. Now, consider that $|V_2| = 1$ and $|V_1| = 1$. By Theorem 4, for any $x_1, x_2 \in V_0$, $d_{\langle V_0 \rangle}(x_1, x_2) \leq 2$. Let $G = K_1 + H$

where H is any graph with isolated vertex $z \in V(H)$. Since $|V_2| = 1$ and $|V_1| = 1$, we let $V_2 = \{z\}$ and $V_1 = \{v\} = V(K_1)$. If $|V(H)| = 2$, then $d_{\langle V_0\rangle}(x, x) = 0$ where $x \in V(H) \setminus \{z\}$. This follows that $G = K_1 + H$ where $H = \overline{K_2} = K_1 \cup K_1$. Consider $|V(H)| \geq 3$ and for any $x_1, x_2 \in V(H) \setminus \{z\}$ for which $x_1 \neq x_2$. Suppose $d_{\langle V_0\rangle}(x_1, x_2) = 1$. Then we obtain $G = K_1 + H$ where $H = K_1 \cup K_{n\geq 1}$. Suppose $1 \leq d_{\langle V_0\rangle}(x, y) \leq 2$. Hence, this implies that $G = K_1 + H$ where $H \in \{K_1 \cup C_{4\leq n\leq 5}, K_1 \cup P_3\}$ since $I_G[V(C_{4\leq n\leq 5})] = V(C_{4\leq n\leq 5})$ and $I_G[V(P_3)] = V(P_3)$.

Conversely, suppose that $G = K_1 + H$ where $H \in \{K_2, K_1 \cup K_{n\geq 1}, K_1 \cup C_{4\leq n\leq 5}, K_1 \cup P_3\}$. Then consider the following cases:

Case 1. $H = K_2$

This implies that $G = K_1 + H = K_1 + K_2 = K_3$. In that case, by Corollary 1, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| = 3$.

Case 2. $H = K_1 \cup K_n$ for all $n \geq 1$

Let z_1 be the isolated vertex in H. In that case, $V(H)\setminus\{z_1\} \subseteq N_G^2(z_1)$. Let $V_2 = \{z_1\}$, and set $V_0 = V(H)\setminus\{z_1\}$. Since $G = K_1 + H$ and $V(K_1) = N_G(z_1)$, we can let $V_1 = V(K_1)$. This follows that $V_0 = V(G) \setminus (V_1 \cup V_2)$ and $f = (V_0, V_1 = V(K_1), V_2 = \{z_1\})$ is an HRDF on G. Since $\langle V_0\rangle = G - (V_1 \cup V_2) = K_n$ for all $n \geq 1$ is a complete graph, it means that V_0 is a convex set in G. Thus, f is an OConHRDF on G. Hence, we have $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 1 + 2(1) = 3$.

Case 3. $H = K_1 \cup C_n$ where $4 \leq n \leq 5$

We let z_2 be the isolated vertex in the graph H. By similar argument as in Case 2, set $V_2 = \{z_2\}$, $V_0 = V(H) \setminus \{z_2\}$. Since $G = K_1 + H$, it follows that $V_1 = V(K_1)$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. This implies that $f = (V_0, V_1 = V(K_1), V_2 = \{z_2\})$ is an HRDF on G. Observe that $\langle V_0\rangle = G - (V_1 \cup V_2) = C_{4\leq n\leq 5}$. Then let $x, y \in V(C_{4\leq n\leq 5})$ and let $u \in V_1$. Since $xu, uy \in E(G)$, it follows that $1 \leq d_{C_{4\leq n\leq 5}}(x, y) \leq 2$. This implies that $I_G[V(C_{4\leq n\leq 5})] = V(C_{4\leq n\leq 5})$. Thus, $V_0 = V(C_{4\leq n\leq 5})$ is a convex set on G. Hence, f is an OConHRDF on G. Accordingly, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 1 + 2(1) = 3$.

Case 4. $H = K_1 \cup P_3$

Again, we let z_3 be the isolated vertex in the graph H. Using a similar argument as in Case 2, we can set $V_2 = \{z_3\}$, $V_0 = V(H) \setminus \{z_3\}$. Since $G = K_1 + H$, we have $V_1 = V(K_1)$ and $V_0 = V(G) \setminus (V_1 \cup V_2)$. Thus, $f = (V_0, V_1 = V(K_1), V_2 = \{z_3\})$ is an HRDF on G. Note that $\langle V_0\rangle = G - (V_1 \cup V_2) = P_3$ and $I_G[V(P_3)] = V(P_3)$. Consequently, $V_0 = V(P_3)$ is a convex set on G. Hence, f is an OConHRDF on G. To this end, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = 1 + 2(1) = 3$.

This completes the proof. □

Theorem 6: *Let $G \neq K_1 + H$ where $H \in \{K_2, K_1 \cup K_{n\geq 1}, K_1 \cup C_{4\leq n\leq 5}, K_1 \cup P_3\}$ be a connected graph of order $n \in \mathbb{N}$ and let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Then $V_1 = \emptyset$ and $|V_2| = 2$ if and only if there exists a hop dominating set $\{u, v\}$ such that $V(G) \setminus \{u, v\}$ is a convex set on G.*

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on $G \neq K_1+H$ where $H \in \{K_2, K_1\cup K_{n\geq 1}, K_1\cup C_{4\leq n\leq 5}, K_1\cup P_3\}$. Assume that $|V_2| = 2$ and $V_1 = \emptyset$. As in the proof of Theorem 1, it implies that V_2 is a $\widetilde{\gamma}_{conh}$-set on G. Thus, V_2 is an outer-convex hop dominating set on G. Let $V_2 = \{u, v\}$. Then $\{u, v\}$ is a hop dominating set on G. Set $V_0 = V(G) \setminus V_2$. Hence, it follows that $V_0 = V(G) \setminus \{u, v\}$ is a convex set in G. Conversely, assume that there exists a hop dominating set $\{u,v\}$ such that $V(G)\setminus\{u,v\}$ is a convex set on G. This implies that $C=\{u,v\}$ is an outer-convex hop dominating set on G and so, $V_0=V(G)\setminus\{u,v\}$. Since $f=(V_0,V_1=\emptyset,V_2)$ is a $\widetilde{\gamma}_{conhR}$-function on G, it implies that V_2 is a $\widetilde{\gamma}_{conh}$-set on G as in the proof of Theorem 1. In that case, we have $2=|C|\geq|V_2|$. Suppose $|V_2|=1$. By Theorem 5, we get $|V_1|=1$ and $G=K_1+H$ where $H\in\{K_2,K_1\cup K_{n\geq1},K_1\cup C_{4\leq n\leq5},K_1\cup P_3\}$. This is a contradiction to our assumption. Hence, we obtain $|V_2|\geq 2$. Consequently, we therefore conclude that $V_1=\emptyset$ and $|V_2|=2$. This completes the proof. □

Corollaries 2 and 3 below are direct consequence of Theorem 6.

Corollary 2: *Let G be a connected graph of order $n\in\mathbb{N}\setminus\{1,2,3\}$ and let $f=(V_0,V_1,V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. If $|V_2|=2$ and $V_1=\emptyset$, then $V_0=V(G)\setminus\{x,y\}$ where $V_2=\{x,y\}$ and $d_G(x,y)=1$ or 3.*

Proof: Let $f=(V_0,V_1,V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Suppose that $|V_2|=2$ and $V_1=\emptyset$. Then by Theorem 6, there exists a hop dominating set $\{x,y\}$ such that $V(G)\setminus\{x,y\}$ is a convex set on G. Thus, it follows that $V_0=V(G)\setminus\{x,y\}$ where $V_2=\{x,y\}$. To prove that $d_G(x,y)=1$ or 3. We seek a contradiction. Assume for a moment that $d_G(x,y)\neq 1$ and $d_G(x,y)\neq 3$. Suppose that $d_G(x,y)=0$. Then it follows that $x=y$ and so, $|V_2|=1$. A contradiction. Suppose that $d_G(x,y)=2$. Then there exists $z_1\in V(G)$ such that $z_1\in N_G(x)\cap N_G(y)$. Since $V_1=\emptyset$, it implies that $z_1\in V_0$. Thus, $\{x,y\}$ is not a hop dominating set on G. A contradiction. Suppose that $d_G(x,y)\geq 4$. Then there exists $z_2\in V(G)$ such that $z_2\in N^2_G[\{x,y\}]$ which forces that $z_2\in V_1$. This follows that $V_1\neq\emptyset$ and $V_0\neq V(G)\setminus\{x,y\}$. This is again a contradiction. Therefore, it suffices to conclude that $d_G(x,y)=1$ or 3. This completes the proof. □

Corollary 3: *Let $G=K_{m,n}$ where $m,n\in\mathbb{N}\setminus\{1\}$. Then $\widetilde{\gamma}_{conhR}(G)=4$.*

Proof: Let $G=K_{m,n}=(U_1,U_2,E)$ where V_1 and V_2 denote the bipartite partition parts of vertex sets and E is the edge set of G. Let $f=(V_0,V_1,V_2)$ be a $\widetilde{\gamma}_{conh}$-function on G. Let $U_1=\{x_1,x_2,...,x_m\}$ and $U_2=\{y_1,y_2,...,y_n\}$. Let $x_i\in U_1$ and $y_j\in U_2$ where $i\in\{1,2,3,...,m\}$ and $j\in\{1,2,3,...,n\}$. Then it follows that $U_1\subseteq N^2_G[x_i]$ for each $i\in\{1,2,3,...,m\}$ and $U_2\subseteq N^2_G[y_j]$ for each $j\in\{1,2,3,...,n\}$. Hence, $\{x_i,y_j\}$ is hop dominating set on G for each $i\in\{1,2,3,...,m\}$ and $j\in\{1,2,3,...,n\}$, that is, $V(G)\subseteq N^2_G[x_i,y_j]$. Observe that $I_G[V(G)\setminus\{x_i,y_j\}]=V(G)\setminus\{x_i,y_j\}$ for each $i\in\{1,2,3,...,m\}$ and $j\in\{1,2,3,...,n\}$ implying that $V(G)\setminus\{x_i,y_j\}$ is a convex set on G. By Theorem 6, it implies that $|V_1|=0$ and $|V_2|=2$. Therefore, $\widetilde{\gamma}_{conhR}(G)=\widetilde{\omega}^{conhR}_G(f)=|V_1|+2|V_2|=2(2)=4$. This completes the proof. □

The next theorem is a realization problem for hop Roman and outer-convex hop Roman domination numbers.

Theorem 7: *Let p,q and n be positive integers for which $1\leq p\leq q\leq n$. Then there exists a connected graph G of order n such that $\gamma_{hR}(G)=p$ and $\widetilde{\gamma}_{conhR}(G)=q$.*

Proof: Let G be a graph of order $n \geq 1$. Let $f' = (U_0, U_1, U_2)$ be a γ_{hR}-function and $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G. Then consider the following three cases below:
Case 1. $1 \leq p \leq q = n$
Let $G = K_1$. Then by Theorem 3, we obtain $1 = \gamma_{hR}(G) = \widetilde{\gamma}_{conhR}(G) = |V(G)| = n$. Let $G = S_n$ where $n \in \mathbb{N} \setminus \{1, 2\}$. Note that $G = S_n = K_1 + \overline{K_n}$. Let $u \in V(\overline{K_n})$. Then it implies that $V(\overline{K_n}) \setminus \{u\} \subseteq N_G^2[u]$. So, we can set $U_0 = V(\overline{K_n}) \setminus \{u\}$, $U_1 = V(K_1)$, and $U_2 = \{u\}$. Hence, we get $\gamma_{hR}(G) = \omega_G^{hR}(f') = |U_1| + 2|U_2| = |V(K_1)| + 2|\{u\}| = 1 + 2(1) = 3$. Now, let S be a collection of subsets of $V(\overline{K_n})$ for which S does not contain an empty set or singleton set, that is, $S = \{A : A \subseteq V(\overline{K_n}), |A| \neq 0, 1\}$. Observe that $I_G[A] \neq A$ for all $A \in S$. This follows that S does not contain a convex set on G. Hence, we can set $V_0 = \emptyset$. By Lemma 1, we get $V_2 = \emptyset$. Consequently, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| = n$ and so, $1 < \gamma_{hR}(G) < \widetilde{\gamma}_{conhR}(G) = n$. Therefore, Case 1 holds.
Case 2. $1 < p < q < n$
Consider the graph $G = K_1 + H$ where $H = K_1^H \cup K_1^{H'} \cup K_n^H$ for $n \geq 2$. In this case, K_1^H, $K_1^{H'}$ and K_n^H are complete graphs that are components of H. Let $z_1 \in V(K_1^H)$ and $z_2 \in V(K_1^{H'})$. Then z_1 and z_2 are isolated vertices of H. Since $G = K_1 + H$, it follows that $V(H) \setminus \{z_1\} \subseteq N_G^2[z_1]$ and $V(H) \setminus \{z_2\} \subseteq N_G^2[z_2]$. Thus, in particular, we can set $V_0 = V(H) \setminus \{z_1\}$, $V_1 = V(K_1)$ and $V_2 = \{z_1\}$. In that case, we get $\gamma_{hR}(G) = \omega_G^{hR}(f') = |U_1| + 2|U_2| = |V(K_1)| + 2|\{z_1\}| = 1 + 2(1) = 3$. Observe that $I_G[V(G) \setminus (V(K_1) \cup \{z_1, z_2\})] = V(G) \setminus (V(K_1) \cup \{z_1, z_2\}) = V(H) \setminus \{z_1, z_2\}$. This implies that $V(H) \setminus \{z_1, z_2\}$ is a convex set on G. Hence, we can set $V_0 = V(H) \setminus \{z_1, z_2\}$, $V_1 = V(K_1) \cup \{z_2\}$, and $V_2 = \{z_1\}$. So, we have $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = |V(K_1) \cup \{z_2\}| + 2|\{z_1\}| = 2 + 2(1) = 4$, and thus, $1 < \gamma_{hR}(G) < \widetilde{\gamma}_{conhR}(G) < n$. Therefore, Case 2 holds.
Case 3. $1 < p = q < n$
Let $G = K_1 + H$ where $H = K_1^H \cup K_n^H$ for $n \geq 2$. Again, K_1^H and K_n^H are complete graphs that are two components of H. Let $z \in V(K_1^H)$. Observe that $V(H) \setminus \{z\} \subseteq N_G^2[z]$. This implies that we can set $U_0 = V(H) \setminus \{z\}$, $U_1 = V(K_1)$ and $U_2 = \{z\}$. Thus, we obtain $\gamma_{hR}(G) = \omega_G^{hR}(f') = |U_1| + 2|U_2| = |V(K_1)| + 2|\{z\}| = 1 + 2(1) = 3$. Note that $I_G[V(H) \setminus \{z\}] = V(H) \setminus \{z\} = K_n^H$, implying that K_n^H is a convex set on G. In view of Theorem 5, it follows that $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| + 2|V_2| = |V(K_1)| + 2|\{z\}| = 3$. Accordingly, we have $1 < \gamma_{hR}(G) = \widetilde{\gamma}_{conhR}(G) < n$. Therefore, Case 3 holds.
This completes the proof. □

The following corollaries are immediate from Theorem 7.

Corollary 4: *Let G be a connected graph of order $n \in \mathbb{N}$. Then the difference $\widetilde{\gamma}_{conhR}(G) - \gamma_{hR}(G)$ can be made arbitrarily large.*

Corollary 5: *Let G be a connected graph of order $n \in \mathbb{N}$. Then the difference $n - \widetilde{\gamma}_{conhR}(G)$ can be made arbitrarily large.*

Theorem 8: *Let G be a connected graph of order $n \in \mathbb{N} \setminus \{1, 2\}$ and $\widetilde{\gamma}_{conh}(G) < \widetilde{\gamma}_{conhR}(G)$. Then $\widetilde{\gamma}_{conhR}(G) = \widetilde{\gamma}_{conh}(G) + 1$ if and only if there exist a set $O \subseteq V(G)$ and vertex $u \in V(G)$ such that $O \cup \{u\}$ is a $\widetilde{\gamma}_{conh}(G)$-set on G and $V(G) \setminus (O \cup \{u\}) \subseteq N_G^2(u)$.*

Proof: Let $f = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conhR}$-function on G for which $\widetilde{\gamma}_{conh}(G) < \widetilde{\gamma}_{conhR}(G)$. Then f is an OConHRDF on G. This follows that $\widetilde{\gamma}_{conh}(G) \leq |V_1| + |V_2|$. Assume that $\widetilde{\gamma}_{conhR}(G) = \widetilde{\gamma}_{conh}(G) + 1$. If $\widetilde{\gamma}_{conh}(G) = |V_1| + |V_2|$, that is, $V_1 \cup V_2$ is a $\widetilde{\gamma}_{conh}$-set on G, then it implies that $|V_1| + |V_2| + 1 = |V_1| + 2|V_2|$. This follows that $|V_2| = 1$ and $|V_1| = \widetilde{\gamma}_{conh}(G) - 1$. Now, let $V_2 = \{u\}$ and $O = V_1$. Then it implies that $V_0 = V(G) \setminus (V_1 \cup V_2) = V(G) \setminus (O \cup \{u\})$. Let $v \in V_0$. Since f is an OConHRDF on G, it means that $d_G(u, v) = 2$ and V_0 is a convex set on G. Consequently, $V(G) \setminus (O \cup \{u\}) \subseteq N_G^2(u)$. On the other hand, if $\widetilde{\gamma}_{conh}(G) < |V_1| + |V_2|$, then it follows that $\widetilde{\gamma}_{conh}(G) + 1 \leq |V_1| + |V_2|$. Since $\widetilde{\gamma}_{conhR}(G) = \widetilde{\gamma}_{conh}(G) + 1$, it implies that $|V_1| + 2|V_2| \leq |V_1| + |V_2|$ and so, we get $|V_2| = 0$. By Lemma 1, we obtain $|V_0| = 0$. Hence, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\omega}_G^{conhR}(f) = |V_1| = |V(G)| = n$ and so, $\widetilde{\gamma}_{conh}(G) = n - 1$. Let $D = V(G) \setminus \{v\}$ be a $\widetilde{\gamma}_{conh}$-set on G. In this case, there exists $u \in D$ such that $d_G(u, v) = 2$. Let $O = D \setminus \{u\}$. Hence, $O \cup \{u\} = D$ is $\widetilde{\gamma}_{conh}$-set on G and $V(G) \setminus D = V(G) \setminus (O \cup \{u\}) = \{v\} \subseteq N_G^2(u)$. Conversely, assume that there exist a set $O \subseteq V(G)$ and vertex $u \in V(G)$ such that $O \cup \{u\}$ is a $\widetilde{\gamma}_{conh}$-set on G and $V(G) \setminus (O \cup \{u\}) \subseteq N_G^2(u)$. Then we set $V_0 = V(G) \setminus (O \cup \{u\})$, $V_1 = F$, and $V_2 = \{u\}$. This implies that $f = (V_0, V_1, V_2)$ is an OConHRDF on G. Hence, we have $\widetilde{\gamma}_{conhR}(G) \leq \omega_G^{conhR}(f) = |V_1| + 2|V_2| = |O| + 2(1) = (\widetilde{\gamma}_{conh}(G) - 1) + 2 = \widetilde{\gamma}_{conh}(G) + 1$. Therefore, $\widetilde{\gamma}_{conhR}(G) = \widetilde{\gamma}_{conh}(G) + 1$ since $\widetilde{\gamma}_{conh}(G) < \widetilde{\gamma}_{conhR}(G)$. This completes the proof. □

Theorem 9: *Let J be a non-complete connected graph. Then $f = (V_0, V_1, V_2)$ is an OConHRDF on $G = K_n + J$ if and only if the following conditions hold:*

(i) $V(K_n) \subseteq V_1 \cup V_2$; *and*

(ii) $f|_J$ *is an OConHRDF on* J.

Proof: Assume that $f = (V_0, V_1, V_2)$ is an OConHRDF on $G = K_n + J$ where J is a non-complete connected graph. Then f is an HRDF on G. Let $a \in V(K_n)$. Then $ab \in E(G)$ for all $b \in V(G) \setminus \{a\}$. Since f is an HRDF on G, we have that $a \notin V_0$. This implies that $a \in V_1 \cup V_2$. Hence, (i) holds. In addition, it follows that $V_0 \subseteq V(J)$. Set $V_0^J = V_0$, $V_1^J = V_1 \cap V(J)$, and $V_2^J = V_2 \cap V(J)$. Then we have $f|_J = (V_0^J, V_1^J, V_2^J)$. Let $x \in V_0^J$. Then there exists $y \in V_2 \cap N_G^2(x)$ and V_0 is a convex set on G since f is an OConHRDF on G. This means that $y \in V_2^J$ and V_0^J is a convex set on G. Thus, it suffices to say that $f|_J$ is an OConHRDF on J. Hence, (ii) holds. Conversely, assume that (i) and (ii) hold. Let $V_1^J = V_1 \cap V(J)$, $V_2^J = V_2 \cap V(J)$, $V_1^n = V_1 \cap V(K_n)$, and $V_2^n = V_2 \cap V(K_n)$. In view of (i), $V_0 = V_0^J \subseteq V(J)$. Suppose $V_0 = \emptyset$. Then it follows that $f = (V_0 = \emptyset, V_1^J \cup V_1^n, V_2^J \cup V_2^n)$ is an OConHRDF on G. On the other hand, suppose $V_0 \neq \emptyset$. Let $v \in V_0$. Since $f|_J = (V_0, V_1^J, V_2^J)$ is an OConHRDF on J in view of (ii), there exists $u \in V_2^H$ such that $d_G(u, v) = 2$ and V_0 is a convex set on G. Since $V_2^H \subseteq V_2$, it implies that $u \in V_2 \cap N_G^2(v)$ and V_0 is a convex set on G. Therefore, we conclude that $f = (V_0, V_1, V_2)$ is an OConHRDF on G. This completes the proof. □

The next result is a consequence of Theorem 9.

Corollary 6: *Let G be a non-complete connected graph. Then $\widetilde{\gamma}_{conhR}(K_n + G) = n + \widetilde{\gamma}_{conhR}(G)$.*

Proof: Let $f' = (V_0', V_1', V_2')$ be a $\widetilde{\gamma}_{conhR}$-function on a non-complete connected graph G and let $C = V(K_n)$. Set $V_0 = V_0'$, $V_1 = C \cup V_1'$, and $V_2 = V_2'$. In view of Theorem 9, $f = (V_0, V_1, V_2)$ is an OConHRDF on $K_n + G$. Thus, $\widetilde{\gamma}_{conhR}(K_n + G) \leq \widetilde{\omega}^{conhR}_{K_n+G}(f) = |V_1| + 2|V_2| = (|C \cup V_1'|) + 2|V_2'| = (|C| + |V_1'|) + 2|V_2'| = |C| + (|V_1'| + 2|V_2'|) = n + \widetilde{\gamma}_{conhR}(G)$. Now, let $g = (U_0, U_1, U_2)$ be a $\widetilde{\gamma}_{conhR}$-function on $K_n + G$. Invoking Theorem 9, we have that $V(K_n) \subseteq U_1 \cup U_2$ and $g|_G$ is an OConHRDF on G. Since g is a $\widetilde{\gamma}_{conhR}$-function, it is force that $V(K_n) \subseteq U_1$. This implies that $U_1 = V(K_n) \cup (U_1 \cap V(G))$ and $g|_G = (U_0, U_1 \cap V(G), U_2)$. Hence, we obtain $\widetilde{\gamma}_{conhR}(K_n + G) = \widetilde{\omega}^{conhR}_{K_n+G}(g) = |U_1| + 2|U_2| = (|V(K_n)| + |U_1 \cap V(G)|) + 2|U_2| = n + |U_1 \cap V(G)| + 2|U_2| = n + \widetilde{\omega}^{conhR}_{K_n+G}(g|_G) \geq n + \widetilde{\gamma}_{conhR}(G)$. Therefore, we conclude that $\widetilde{\gamma}_{conhR}(K_n + G) = n + \widetilde{\gamma}_{conhR}(G)$. This completes the proof. □

Theorem 10: *Let G and H be non-complete connected graphs. Then, $f = (V_0, V_1, V_2)$ is an OConHRDF on $G + H$ if and only if $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$ where $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$ are OConHRDF on G and H, respectively, for which $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for each $i \in \{0, 1, 2\}$.*

Proof: Assume that $f = (V_0, V_1, V_2)$ is an OConHRDF on $G + H$ where G and H are both non-complete connected graphs. For each $i \in \{0, 1, 2\}$, we set $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$. In that case, $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$. Let $x \in V_0^G$. Then $x \in V_0$. Since f is an OConHRDF on $G + H$, it follows that there exists $y \in V_2$ such that $d_{G+H}(x, y) = 2$ and V_0 is a convex set on G. Note that $d_G(x, v) = 1$ for all $v \in V(H)$. This implies that $y \in V_2^G$ and V_0^G is a convex set on G. Hence, we conclude that $f|_G$ is an OConHRDF on G. Using a similar argument, we also have $f|_H$ is an OConHRDF on H. Accordingly, it suffices to have $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$.

Conversely, suppose $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$ where $f|_G = (V_0^G, V_1^G, V_2^G)$ and $f|_H = (V_0^H, V_1^H, V_2^H)$ are OConHRDF on G and H, respectively, for which $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$ for each $i \in \{0, 1, 2\}$. Then it implies that $f = (V_0, V_1, V_2)$ is an HRDF on $G + H$. Let $x \in V_0$ be an arbitrary element. This follows that either $x \in V_0^G$ or $x \in V_0^H$. First, we suppose that $x \in V_0^G$. Since $f|_G$ is an OConHRDF on G, there exists $y \in V_2^G$ such that $d_G(x, y) = 2$ and V_0^G is a convex set on G. Now, since $V_i^G \subseteq V_i$ for all $i \in \{0, 1, 2\}$, it means that $y \in V_2$ for which $d_{G+H}(x, y) = 2$ and V_0 is a convex set on $G + H$, implying that $f = (V_0, V_1, V_2)$ is an OConHRDF on $G + H$. Suppose $x \in V_0^H$. Using similar argument, the same conclusion is arrived. This completes the proof. □

The next corollary is a consequence of Theorem 10.

Corollary 7: *Let G and H be non-complete connected graphs. Then $\widetilde{\gamma}_{conhR}(G + H) = \widetilde{\gamma}_{conhR}(G) + \widetilde{\gamma}_{conhR}(H)$.*

Proof: Let $\phi = (V_0^G, V_1^G, V_2^G)$ and $\lambda = (V_0^H, V_1^H, V_2^H)$ be $\widetilde{\gamma}_{conhR}$-functions on non-complete connected graphs G and H, respectively. Set $V_i = V_i^G \cup V_i^H$ for each $i \in \{0, 1, 2\}$. Then it follows that $\phi = f|_G$ and

$\lambda = f|_H$ where $f = (V_0, V_1, V_2)$ is an HRDF on $G + H$. Invoking Theorem 10, it implies that f is an OConHRDF on $G + H$. In that case, since $V_i = V_i^G \cup V_i^H$, we have

$$\begin{aligned}
\widetilde{\gamma}_{conhR}(G+H) \leq \widetilde{\omega}^{conhR}_{G+H}(f) &= |V_1| + 2|V_2| \\
&= |V_1^G \cup V_1^H| + 2|V_2^G \cup V_2^H| \\
&= (|V_1^G| + 2|V_2^G|) + (|V_1^H| + 2|V_2^H|) \\
&= \widetilde{\omega}^{conhR}_{G+H}(f)(\phi) + \widetilde{\omega}^{conhR}_{G+H}(f)(\phi'') \\
&= \widetilde{\gamma}_{conhR}(G) + \widetilde{\gamma}_{conhR}(H).
\end{aligned}$$

On the other hand, let $f = (U_0, U_1, U_2)$ be a $\widetilde{\gamma}_{conhR}$-function on $G + H$ where $U_i = \{z \in V(G) : f(z) = i\}$ for each $i \in \{0, 1, 2\}$. Now set $U_i^G = U_i \cap V(G)$ and $U_i^H = U_i \cap V(H)$ for each $i \in \{0, 1, 2\}$. Invoking Theorem 10, it means that $f|_G = (U_0^G, U_1^G, U_2^G)$ and $f|_H = (U_0^H, U_1^H, U_2^H)$ are OConHRDF on G and H, respectively. Hence, it follows that

$$\begin{aligned}
\widetilde{\gamma}_{conhR}(G+H) = \widetilde{\omega}^{conhR}_{G+H}(\lambda) &= |U_1| + 2|U_2| \\
&= |U_1^G \cup U_1^H| + 2|U_2^G \cup U_2^H| \\
&= (|U_1^G| + 2|U_2^G|) + (|U_1^H| + 2|U_2^H|) \\
&= \widetilde{\omega}^{conhR}_{G}(f|_G) + \widetilde{\omega}^{conhR}_{H}(f|_H) \\
&\geq \widetilde{\gamma}_{conhR}(G) + \widetilde{\gamma}_{conhR}(H).
\end{aligned}$$

Therefore, $\widetilde{\gamma}_{conhR}(G + H) = \widetilde{\gamma}_{conhR}(G) + \widetilde{\gamma}_{conhR}(H)$. This completes the proof. □

Remark 1: *Let G and H be complete graphs of order $n \in \mathbb{N}$ and $m \in \mathbb{N}$, respectively. Then $\widetilde{\gamma}_{conhR}(G + H) = n + m$.*

3 Conclusions

A new restricted parameter of hop Roman domination in graphs, called the outer-convex hop Roman dominating function, has been introduced. Conclusively, important properties of outer-convex hop Roman dominating function in some classes of connected graphs were obtained and discussed, which include lower and upper bounds, the realization problem, and characterizations. As for future research, one may consider characterizing the outer-convex hop Roman dominating function under product operations in graphs.

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