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Research article

On Interval-Valued Λ-Sets and λ-Closed Sets via Kernel Operators

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Abstract: In this study, we investigate specific kernel structures within interval-valued topological spaces. We introduce the notions of interval-valued Λ-sets and interval-valued λ-closed sets and discuss their essential properties. The connections between these concepts and existing notions in interval-valued topology are examined in detail. Various characterizations and foundational results are presented to clarify their structural behavior. This work aims to enhance and extend the theoretical framework of interval-valued topology.

Mathematics Subject Classification: 54A05; 54A40; 54A99
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1 Introduction

The theory of interval-valued sets, introduced by Yao [4], has emerged as an effective tool for modeling uncertainty and approximating vague or imprecise information. This concept was further developed by Kim et al. [3], who formalized interval-valued sets and investigated their topological properties, including neighborhood systems, closure, and interior operators. These studies established interval-valued topology as an important extension of classical topology with applications in areas such as fuzzy systems and information analysis. Later, Cheong et al. [1] expanded the framework by introducing interval-valued relations from a categorical perspective, thereby enhancing the structural and theoretical depth of the subject.

Although significant progress has been made in the study of interval-valued topological spaces, most existing works focus primarily on closure-based generalizations of sets. In contrast, kernel-based approaches, which provide a dual and often more structural viewpoint, have not been adequately explored in this setting. The investigation of kernel operators is important because they offer new ways to characterize sets and reveal finer relationships between different generalized notions.

Motivated by this gap, the present study introduces new classes of sets in interval-valued topological spaces based on kernel structures, namely interval-valued Λ-sets and interval-valued λ-closed sets. We analyze their fundamental properties and establish several characterization results that connect these new classes with existing generalized closed sets. The significance of this work lies in providing a unified framework that links kernel and closure operators, thereby enriching the theory of interval-valued topology and contributing to its further development.

2 Preliminaries

Definition 2.1: *[4] Let X be a non-empty set. Then, the form $[\mathcal{M}^-, \mathcal{M}^+] = \{[\mathcal{S} \subset X : \mathcal{M}^- \subset \mathcal{S} \subset \mathcal{M}^+\}$ is called an interval-valued set (briefly, $\mathcal{IVS}$) in X, if $\mathcal{M}^-, \mathcal{M}^+ \subset X$ and $\mathcal{M}^- \subset \mathcal{M}^+$.*

In this case, $\mathcal{M}^-$ [resp. $\mathcal{M}^+$] represent the set of minimum [resp. maximum] memberships of elements of X to $[\mathcal{M}^-, \mathcal{M}^+]$.

Here, $[\emptyset, \emptyset]$ $[resp.[X, X]]$ is called the interval-valued empty [resp. whole] set in X and denoted by $\tilde{\emptyset}$ [resp. $\tilde{X}$] The set of all $\mathcal{IVS}s$ in X is denoted by $\mathcal{IVS}(X)$.

Interval valued topological space by is denoted by $\mathcal{IVTS}$ or $\mathcal{T}^X$.

Definition 2.2: *[3] A non-empty set X with $[\mathcal{M}^-, \mathcal{M}^+]$ and $[\mathcal{S}^-, \mathcal{S}^+]$ are $\mathcal{IVS}$ respectively. Then*

1. $[\mathcal{M}^-, \mathcal{M}^+] \subset [\mathcal{S}^-, \mathcal{S}^+] \iff \mathcal{M}^- \subset \mathcal{S}^-$ *and* $\mathcal{M}^+ \subset \mathcal{S}^+$
2. $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \iff [\mathcal{M}^-, \mathcal{M}^+] \subset [\mathcal{S}^-, \mathcal{S}^+]$ *and* $[\mathcal{S}^-, \mathcal{S}^+] \subset [\mathcal{M}^-, \mathcal{M}^+]$
3. $([\mathcal{M}^-, \mathcal{M}^+])^c = [(\mathcal{M}^+)^c, (\mathcal{M}^-)^c]$
4. $[\mathcal{M}^-, \mathcal{M}^+] \cup [\mathcal{S}^-, \mathcal{S}^+] = [\mathcal{M}^- \cup \mathcal{S}^-, \mathcal{M}^+ \cup \mathcal{S}^+]$
5. $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{S}^-, \mathcal{S}^+] = [\mathcal{M}^- \cap \mathcal{S}^-, \mathcal{M}^+ \cap \mathcal{S}^+]$

Definition 2.3: *[3] Let X be a non-empty set and let τ be a non-empty family of $\mathcal{IVS}s$ on X. Then τ is called an interval-valued topology (briefly, $\mathcal{IVT}$) on X if it satisfies the following axioms:*

1. $\tilde{\emptyset}, \tilde{X} \in \tau$,
2. $[\mathcal{M}^-, \mathcal{M}^+] \cap [\mathcal{S}^-, \mathcal{S}^+] \in \tau$ *for any* $[\mathcal{M}^-, \mathcal{M}^+], [\mathcal{S}^-, \mathcal{S}^+] \in \tau$.
3. $\bigcup_{j \in J} \mathcal{S}_j \in \tau$ *for any family* $(\mathcal{S}_j)_{j \in J}$ *of members of* τ.

In this case, the pair $\mathcal{T}^X$ is called an interval-valued topological space (briefly, $\mathcal{IVTS}$) and each member of τ is called an interval-valued open set (briefly, $\mathcal{IVO}$s) in X. A $\mathcal{IVS}$ of $[\mathcal{M}^-, \mathcal{M}^+]$ is called an interval-valued closed set (briefly, $\mathcal{IVC}s$) in X.

Definition 2.4: *[3] If a $\mathcal{T}^X$ is space with respect to X and if $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \tilde{X}$, then*

1. *the an interval-valued interior of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as the union of all an interval-valued open subsets of $[\mathcal{M}^-, \mathcal{M}^+]$ and it is denoted by $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$.*
   That is, $\mathcal{IVI}([\mathcal{M}^-, \mathcal{M}^+])$ is the largest an interval-valued open subset of $[\mathcal{M}^-, \mathcal{M}^+]$.
2. *the an interval-valued closure of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as the intersection of all an interval-valued closed sets containing $[\mathcal{M}^-, \mathcal{M}^+]$ and it is denoted by $\mathcal{IVC}([\mathcal{M}^-, \mathcal{M}^+])$.*

That is, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$ is the smallest an interval-valued closed set containing $[\mathcal{M}^-, \mathcal{M}^+]$.

Definition 2.5: *[2]*
A subset $[\mathcal{M}^-, \mathcal{M}^+]$ from $\mathcal{IVS}(X)$ is an interval-valued generalized closed (briefly, $\mathcal{IV}g$-closed) if $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, whenever $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$ and $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}$-open.

3 New classes of sets and open sets in interval-valued spaces

Definition 3.1: *A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$ is called a*

1. *interval-valued-ker$([\mathcal{M}^-, \mathcal{M}^+]) = \bigcap\{[\mathcal{S}^-, \mathcal{S}^+] : [\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+], [\mathcal{S}^-, \mathcal{S}^+] \in \tau\}$ is called the interval-valued kernel of $[\mathcal{M}^-, \mathcal{M}^+]$ and is denoted by $\mathcal{IV}$-ker$([\mathcal{M}^-, \mathcal{M}^+])$.*
2. *interval-valued Λ-set (briefly, $\mathcal{IV}\Lambda$-set) if $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}$-ker$([\mathcal{M}^-, \mathcal{M}^+])$.*

Example 3.2: *Let $X = \{a, b, c\}$ with a space $\mathcal{T}^X$, where $\tau = \{\tilde{\emptyset}, \tilde{X}, [\{a\}, \{a, b\}]\}$ and $\tau^c = \{\tilde{X}, \tilde{\emptyset}, [\{c\}, \{b, c\}]\}$.*

1. *$[\mathcal{M}^-, \mathcal{M}^+] = [\{a\}, \{a\}]$ is $\mathcal{IV}$-ker$([\mathcal{M}^-, \mathcal{M}^+])$.*
2. *$[\mathcal{M}^-, \mathcal{M}^+] = [\{a\}, \{a, b\}]$ is $\mathcal{IV}\Lambda$-set.*

Definition 3.3: *A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$ is called an interval-valued λ-closed (briefly, $\mathcal{IV}\lambda$-closed) if $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap [\mathcal{R}^-, \mathcal{R}^+]$ where $[\mathcal{S}^-, \mathcal{S}^+]$ is an $\mathcal{IV}\Lambda$-set and $[\mathcal{R}^-, \mathcal{R}^+]$ is an $\mathcal{IV}$-closed.*

Example 3.4: *Let $X = \{a, b, c, d\}$ with a space $\mathcal{T}^X$, where*

$\tau = \{\tilde{\emptyset}, [\emptyset, \{a\}], [\{b\}, \{b, d\}], [\{b\}, \{a, b, d\}], \tilde{X}\}$

Hence $[\mathcal{S}^-, \mathcal{S}^+] = [\{b\}, \{b, c, d\}]$ is an $\mathcal{IV}\lambda$-closed set.

Remark 3.5: *For a subset of a space $\mathcal{T}^X$, we have the following implications:*

$$
\begin{array}{ccccc}
 & & \mathcal{IV}\textbf{-open} & & \\
 & & \downarrow & & \\
\mathcal{IV}\Lambda\textbf{-set} & \longrightarrow & \mathcal{IV}\lambda\textbf{-closed} & \longrightarrow & \mathcal{IV}\textbf{-closed}
\end{array}
$$

The converses of statements in Remark 3.5 are not necessarily true as seen from the following Examples.

Example 3.6: *In Example 3.4,*

1. *$[\emptyset, \{c\}]$ is an $\mathcal{IV}\lambda$-closed but not an $\mathcal{IV}\Lambda$-set.*
2. *$[\{a\}, \{a, c\}]$ is an $\mathcal{IV}\lambda$-closed but not an $\mathcal{IV}$-open.*
3. *$[\{b\}, \{b, d\}]$ is an $\mathcal{IV}\lambda$-closed but not an $\mathcal{IV}$-closed.*

Lemma 3.7: *For a subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$, the following conditions are equivalent:*

1. *$[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}\lambda$-closed set.*

2. $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$, *where* $[\mathcal{S}^-, \mathcal{S}^+]$ *is an* $\mathcal{IV}\Lambda$*-set.*
3. $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.

Proof:

(1) $\Longrightarrow$ (2) : Suppose $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}\lambda$-closed set. Then, by Definition, there exist an $\mathcal{IV}\Lambda$-set $[\mathcal{S}^-, \mathcal{S}^+]$ and an $\mathcal{IV}$-closed set $[\mathcal{R}^-, \mathcal{R}^+]$ such that $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap [\mathcal{R}^-, \mathcal{R}^+]$. Since $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{R}^-, \mathcal{R}^+]$ and $[\mathcal{R}^-, \mathcal{R}^+]$ is $\mathcal{IV}$-closed, we have $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{R}^-, \mathcal{R}^+]$. Thus, $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.

(2) $\Longrightarrow$ (3) : Since $[\mathcal{S}^-, \mathcal{S}^+]$ is an $\mathcal{IV}\Lambda$-set, we have $[\mathcal{S}^-, \mathcal{S}^+] = \mathcal{IV}\text{-}ker([\mathcal{S}^-, \mathcal{S}^+])$. Also, since $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$, it follows that $\mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$.

Hence, $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.

(3) $\Longrightarrow$ (1): Assume $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$. Since $\mathcal{IV}$-ker$([\mathcal{M}^-, \mathcal{M}^+])$ is an $\mathcal{IV}\Lambda$-set and $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$ is an $\mathcal{IV}$-closed, it follows that $[\mathcal{M}^-, \mathcal{M}^+]$ is the intersection of an $\mathcal{IV}\Lambda$-set and an $\mathcal{IV}$-closed set.

Therefore, $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}\lambda$-closed set.

□

Lemma 3.8: *A subset* $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{T}^X$ *is an* $\mathcal{IV}g$*-closed* $\iff$ $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.

Proof:

($\Longrightarrow$) Suppose $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}g$-closed. Let $[\mathcal{S}^-, \mathcal{S}^+]$ be any $\mathcal{IV}$-open set such that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. Since $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}g$-closed, we have $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. This holds for every an $\mathcal{IV}$-open superset of $[\mathcal{M}^-, \mathcal{M}^+]$. Hence, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$ is contained in the intersection of all such sets.

Therefore, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.

($\Longleftarrow$) Suppose $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$. Let $[\mathcal{S}^-, \mathcal{S}^+]$ be any $\mathcal{IV}$-open set such that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. By Definition of $\mathcal{IV}$-kernel, $\mathcal{IV}\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{U}^-, \mathcal{U}^+]$.

Thus, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{U}^-, \mathcal{U}^+]$.

Hence, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}g$-closed.

□

Definition 3.9: *A subset* $[\mathcal{M}^-, \mathcal{M}^+]$ *of a space* $\mathcal{T}^X$ *is called an*

1. *interval-valued regular-open (briefly,* $\mathcal{IV}r$*-open) if* $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}I(\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]))$.
2. *interval-valued* π*-open (briefly,* $\mathcal{IV}\pi$*-open) if the finite union of interval-valued regular-open sets.*

Definition 3.10: 1. *The union of all* $\mathcal{IV}rOs$ *(resp.* $\mathcal{IV}\pi Os$*) contained in* $[\mathcal{M}^-, \mathcal{M}^+]$ *is called an interval-valued regular-interior (resp.* $\mathcal{IV}\pi$*-interior) of* $[\mathcal{M}^-, \mathcal{M}^+]$*, which is denoted by* $\mathcal{IV}rI([\mathcal{M}^-, \mathcal{M}^+])$ *(resp.* $\mathcal{IV}\pi I([\mathcal{M}^-, \mathcal{M}^+])$*).*

2. *The intersection of all $\mathcal{IV}rCs$ (resp. $\mathcal{IV}\pi Cs$) containing $[\mathcal{M}^-, \mathcal{M}^+]$ is called an interval-valued regular-closure (resp. $\mathcal{IV}\pi$-closure) of $[\mathcal{M}^-, \mathcal{M}^+]$, which is denoted by $\mathcal{IV}rC([\mathcal{M}^-, \mathcal{M}^+])$ (resp. $\mathcal{IV}\pi C([\mathcal{M}^-, \mathcal{M}^+])$).*

Example 3.11: *Let $X = \{a, b, c, d\}$ with a space $\mathcal{T}^X$, where*

$\tau = \{\tilde{\emptyset}, [\emptyset, \{d\}], [\{b\}, \{a, b\}], [\{b\}, \{a, b, d\}], \tilde{X}\}$.

the set $[\{b\}, \{b, c, d\}]$ is an $\mathcal{IV}\pi$-open.

Theorem 3.12: *In a space $\mathcal{T}^X$, every $\mathcal{IV}r$-closed set is $\mathcal{IV}$-closed.*

Proof:
If $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}r$-closed, $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}C(\mathcal{IV}I([\mathcal{M}^-, \mathcal{M}^+])$.
Then $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) = \mathcal{IV}C(\mathcal{IV}C(\mathcal{IV}I([\mathcal{M}^-, \mathcal{M}^+]))) = \mathcal{IV}C(\mathcal{IV}I([\mathcal{M}^-, \mathcal{M}^+]) = [\mathcal{M}^-, \mathcal{M}^+]$. That is $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}$-closed □

Definition 3.13: *An $\mathcal{IVS}$ of $[\mathcal{M}^-, \mathcal{M}^+]$ is defined as an*

1. *interval-valued rg-closed (briefly, $\mathcal{IV}rg$-closed) if $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$ whenever $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$ and $[\mathcal{S}^-, \mathcal{S}^+]$ is $\mathcal{IV}r$-open.*
2. *interval-valued πg-closed (briefly, $\mathcal{IV}\pi g$-open) if $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$, whenever $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$ and $[\mathcal{S}^-, \mathcal{S}^+]$ is $\mathcal{IV}\pi$-open.*

Definition 3.14:

1. *The union of all $\mathcal{IV}rgOs$(resp. $\mathcal{IV}\pi gOs$) contained in $[\mathcal{M}^-, \mathcal{M}^+]$ is called an interval-valued rg-interior (resp. $\mathcal{IV}\pi g$-interior) of $[\mathcal{M}^-, \mathcal{M}^+]$, which is denoted by $\mathcal{IV}rgI([\mathcal{M}^-, \mathcal{M}^+])$ (resp. $\mathcal{IV}\pi gI([\mathcal{M}^-, \mathcal{M}^+])$.*
2. *The intersection of all $\mathcal{IV}rgCs$ (resp. $\mathcal{IV}\pi gCs$) containing $[\mathcal{M}^-, \mathcal{M}^+]$ is called an interval-valued rg-closure (resp. $\mathcal{IV}\pi g$-closure) of $[\mathcal{M}^-, \mathcal{M}^+]$, which is denoted by $\mathcal{IV}rgC([\mathcal{M}^-, \mathcal{M}^+])(resp.\mathcal{IV}\pi gC([\mathcal{M}^-, \mathcal{M}^+]))$.*

Example 3.15: *1. Let $X = \{a, b, c, d\}$ with a space $\mathcal{T}^X$, where $\tau = \{\tilde{\emptyset}, [\{a\}, \{a, b\}], \tilde{X}\}$*
Let $[\mathcal{M}^-, \mathcal{M}^+] = [\emptyset, \{a\}]$.
For every $\mathcal{IV}r$-open set $[\mathcal{S}^-, \mathcal{S}^+]$ such that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$, we have $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. Hence, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}rg$-closed.

2. *Let $X = \{a, b, c, d, e\}$ with a space $\mathcal{T}^X$, where $\tau = \{\tilde{\emptyset}, [\{b\}, \{a, b\}], [\emptyset, \{c, d\}], [\{b\}, \{a, b, c, d\}]\tilde{X}\}$*
Let $[\mathcal{M}^-, \mathcal{M}^+] = [\emptyset, \{a\}]$.
For every $\mathcal{IV}\pi$-open set $[\mathcal{S}^-, \mathcal{S}^+]$ such that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$, we have $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. Hence, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\pi g$-closed.

Remark 3.16: *For a subset of a space $\mathcal{T}^X$, we have the following implications:*

$$\begin{array}{ccccc} \mathcal{IV}\textbf{-closed} & \rightarrow & \mathcal{IV}g\textbf{-closed} & & \\ & & \downarrow & & \\ & & \mathcal{IV}\pi g\textbf{-closed} & \rightarrow & \mathcal{IV}rg\textbf{-closed} \end{array}$$

None of the above implications is reversible.

Example 3.17: 1. *In Example 3.4, the set $[\{b\},\{b,c\}]$ is an $\mathcal{IV}g$-closed but not an $\mathcal{IV}$-closed.*
2. *In Example 3.4, the set $[\emptyset,\{a\}]$ is an $\mathcal{IV}\pi g$-closed but not an $\mathcal{IV}g$-closed.*
3. *Let $X=\{a,b,c\}$ with a space $\mathcal{T}^X$, where $\tau=\{\tilde{\emptyset},[\emptyset,\{a\}],\tilde{X}\}$. The set $[\emptyset,\{a\}]$ is an $\mathcal{IV}rg$-closed but not an $\mathcal{IV}\pi g$-closed.*

Theorem 3.18: *For a subset $[\mathcal{M}^-,\mathcal{M}^+]$ of a space $\mathcal{T}^X$, the following conditions are equivalent.*
1. *$[\mathcal{M}^-,\mathcal{M}^+]$ is an $\mathcal{IV}$-closed.*
2. *$[\mathcal{M}^-,\mathcal{M}^+]$ is an $\mathcal{IV}g$-closed and an $\mathcal{IV}\lambda$-closed.*

Proof:

$(1)\Longrightarrow(2)$: Obvious by Remark 3.16 and 3.5.

$(2)\Longrightarrow(1)$: Since $[\mathcal{M}^-,\mathcal{M}^+]$ is an $\mathcal{IV}g$-closed, by Lemma 3.8, $\mathcal{IV}C([\mathcal{M}^-,\mathcal{M}^+])\subseteq\mathcal{IV}\text{-}ker([\mathcal{M}^-,\mathcal{M}^+])$. Since $[\mathcal{M}^-,\mathcal{M}^+]$ is an $\mathcal{IV}\lambda$-closed, by Lemma 3.7, $[\mathcal{M}^-,\mathcal{M}^+]=\mathcal{IV}\text{-}ker([\mathcal{M}^-,\mathcal{M}^+])\cap\mathcal{IV}C([\mathcal{M}^-,\mathcal{M}^+])=\mathcal{IV}C([\mathcal{M}^-,\mathcal{M}^+])$. Hence $[\mathcal{M}^-,\mathcal{M}^+]$ is an $\mathcal{IV}$-closed. □

Remark 3.19: *In an interval-valued topological space, the concepts of $\mathcal{IV}g$-closed sets and $\mathcal{IV}\lambda$-closed sets are independent as seen from the following Examples.*

Example 3.20: *In Example 3.4,*
1. *$[\emptyset,\{b,c\}]$ is an $\mathcal{IV}g$-closed but not an $\mathcal{IV}\lambda$-closed.*
2. *$[\emptyset,\{a\}]$ is an $\mathcal{IV}\lambda$-closed but not an $\mathcal{IV}g$-closed.*

Remark 3.21: *Theorem 3.18 together with Remark 3.19 and Example 3.20 gives a decomposition of $\mathcal{IV}$-closed set into a $\mathcal{IV}g$-closed set and a $\mathcal{IV}\lambda$-closed set.*

Definition 3.22: *Let $[\mathcal{M}^-,\mathcal{M}^+]$ be a subset of a space $\mathcal{T}^X$. Then*
1. *The interval-valued regular-kernel of the set $[\mathcal{M}^-,\mathcal{M}^+]$, denoted by $\mathcal{IV}r\text{-}ker([\mathcal{M}^-,\mathcal{M}^+])$, is the intersection of all $\mathcal{IV}r$-open supersets of $[\mathcal{M}^-,\mathcal{M}^+]$.*
2. *The interval-valued π-kernel of the set $[\mathcal{M}^-,\mathcal{M}^+]$, denoted by $\mathcal{IV}\pi\text{-}ker([\mathcal{M}^-,\mathcal{M}^+])$, is the intersection of all $\mathcal{IV}\pi$-open supersets of $[\mathcal{M}^-,\mathcal{M}^+]$.*

Example 3.23: *In Example 3.4,*
1. *the set $[\emptyset,\{a\}]$ is an $\mathcal{IV}r$-kernel*
2. *the set $[\{b\},\{b,d\}]$ is an $\mathcal{IV}\pi$-kernel*

Definition 3.24: *A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$ is called*

1. *interval-valued Λ_r-set (resp. $\mathcal{IV}\Lambda_r$-set) if $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.*
2. *interval-valued Λ_π-set (resp. $\mathcal{IV}\Lambda_\pi$-set) if $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}\pi\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.*

Example 3.25: *In Example 3.4,*

1. *$[\{b\}, \{b, d\}]$ is an $\mathcal{IV}\Lambda_r$-set.*
2. *$[\emptyset, \{a\}]$ is an $\mathcal{IV}\Lambda_\pi$-set.*

Definition 3.26: *A subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$ is called an*

1. *interval-valued λ_r-closed (briefly, $\mathcal{IV}\lambda_r$-closed) if $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap [\mathcal{K}^-, \mathcal{K}^+]$ where $[\mathcal{S}^-, \mathcal{S}^+]$ is a $\mathcal{IV}\Lambda_r$-set and $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}$-closed.*
2. *interval-valued λ_π-closed (briefly, $\mathcal{IV}\lambda_\pi$-closed) if $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap [\mathcal{K}^-, \mathcal{K}^+]$ where $[\mathcal{S}^-, \mathcal{S}^+]$ is a $\mathcal{IV}\Lambda_\pi$-set and $[\mathcal{K}^-, \mathcal{K}^+]$ is $\mathcal{IV}$-closed.*

Example 3.27: *In Example 3.4,*

1. *the set $[\{b\}, \{b, c, d\}]$ is an $\mathcal{IV}\lambda_r$-closed set.*
2. *the set $[\{a\}, \{a, c\}]$ is an $\mathcal{IV}\lambda_\pi$-closed set.*

Remark 3.28: *For a subset of a space $\mathcal{T}^X$, we have the following implications:*

$$\begin{array}{ccccc} \mathcal{IV}\lambda_\pi\textbf{-closed} & \longleftarrow & \mathcal{IV}\textbf{-closed} & \longrightarrow & \mathcal{IV}\lambda_r\textbf{-closed} \\ \uparrow & & & & \uparrow \\ \mathcal{IV}\Lambda_\pi\textbf{-set} & & & & \mathcal{IV}\Lambda_r\textbf{-set} \end{array}$$

The converses of the statements in Remark 3.28 are not necessarily true as seen from the following Examples.

Lemma 3.29: *For a subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$, the following are equivalent.*

1. *The following are equivalent:*
   - *(a)* $[\mathcal{M}^-, \mathcal{M}^+]$ *is an $\mathcal{IV}\lambda_r$-closed set.*
   - *(b)* $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$, *where $[\mathcal{S}^-, \mathcal{S}^+]$ is an $\mathcal{IV}\Lambda_r$-set.*
   - *(c)* $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.
2. *The following are equivalent:*
   - *(a)* $[\mathcal{M}^-, \mathcal{M}^+]$ *is an $\mathcal{IV}\lambda_\pi$-closed set.*
   - *(b)* $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$, *where $[\mathcal{S}^-, \mathcal{S}^+]$ is an $\mathcal{IV}\Lambda_\pi$-set.*
   - *(c)* $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}\pi\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.

Proof:

1. Proof for an $\mathcal{IV}\lambda_r$-closed:

$(a) \Longrightarrow (b)$: Suppose $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}\lambda_r$-closed. Then there exist an $\mathcal{IV}\Lambda_r$-set $[\mathcal{S}^-, \mathcal{S}^+]$ and an $\mathcal{IV}$-closed set $[\mathcal{K}^-, \mathcal{K}^+]$ such that $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap [\mathcal{K}^-, \mathcal{K}^+]$. Since $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{K}^-, \mathcal{K}^+]$, we have $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{K}^-, \mathcal{K}^+]$. Thus, $[\mathcal{M}^-, \mathcal{M}^+] = [\mathcal{S}^-, \mathcal{S}^+] \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.
$(2) \Longrightarrow (3)$: Since $[\mathcal{S}^-, \mathcal{S}^+]$ is an $\mathcal{IV}\Lambda_r$-set, $[\mathcal{S}^-, \mathcal{S}^+] = \mathcal{IV}r\text{-}ker([\mathcal{S}^-, \mathcal{S}^+])$. Also, $\mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. Hence, $[\mathcal{M}^-, \mathcal{M}^+] = \mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \cap \mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$.
$(c) \Rightarrow (a)$: Since $\mathcal{IV}r$-ker$([\mathcal{M}^-, \mathcal{M}^+])$ is an $\mathcal{IV}\Lambda_r$-set and $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$ is an $\mathcal{IV}$-closed, their intersection is an $\mathcal{IV}\lambda_r$-closed.

2. Proof for an $\mathcal{IV}\lambda_\pi$-closed:
The proof is similar.
$(a) \Longrightarrow (b)$: By Definition of an $\mathcal{IV}\lambda_\pi$-closed.
$(b) \Longrightarrow (c)$: Using the fact that an $\mathcal{IV}\Lambda_\pi$-set equals its an $\mathcal{IV}\pi$-kernel.
$(c) \Rightarrow (a)$: Since $\mathcal{IV}\pi$-ker$([\mathcal{M}^-, \mathcal{M}^+])$ is an $\mathcal{IV}\Lambda_\pi$-set and $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+])$ is $\mathcal{IV}$-closed, their intersection is an $\mathcal{IV}\lambda_\pi$-closed.

□

Lemma 3.30: *1. A subset $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{T}^X$ is an $\mathcal{IV}\pi g$-closed $\iff$ $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}\pi\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.*
2. A subset $[\mathcal{M}^-, \mathcal{M}^+] \subseteq \mathcal{T}^X$ is an $\mathcal{IV}rg$-closed $\iff$ $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.

Proof:

1. Proof for an $\mathcal{IV}\pi g$-closed :
$(\Longrightarrow)$: Assume $[\mathcal{M}^-, \mathcal{M}^+]$ is an $\mathcal{IV}\pi g$-closed. Let $[\mathcal{S}^-, \mathcal{S}^+]$ be any $\mathcal{IV}\pi$-open set such that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$.
Then, by Definition, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$.
This holds for all such $\mathcal{IV}\pi$-open supersets.
Hence, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \bigcap\{[\mathcal{S}^-, \mathcal{S}^+] : [\mathcal{S}^-, \mathcal{S}^+]$ is $\mathcal{IV}\pi$-open and $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]\}$.
Therefore, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}\pi\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.
$(\Longleftarrow)$: Assume $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}\pi\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$. Let $[\mathcal{S}^-, \mathcal{S}^+]$ be any $\mathcal{IV}\pi$-open set such that $[\mathcal{M}^-, \mathcal{M}^+] \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. By Definition of $\mathcal{IV}\pi$-kernel, $\mathcal{IV}\pi\text{-}ker([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. Thus, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq [\mathcal{S}^-, \mathcal{S}^+]$. Hence, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}\pi g$-closed.
2. Proof for $\mathcal{IV}rg$-closed :
The proof is similar.
$(\Longrightarrow)$: Assume $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}rg$-closed. Then for every $\mathcal{IV}r$-open set containing it, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq$ that set.
Hence, $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$.
$(\Longleftarrow)$: Assume $\mathcal{IV}C([\mathcal{M}^-, \mathcal{M}^+]) \subseteq \mathcal{IV}r\text{-}ker([\mathcal{M}^-, \mathcal{M}^+])$. Then for every $\mathcal{IV}r$-open superset, closure is contained in it. Hence, $[\mathcal{M}^-, \mathcal{M}^+]$ is $\mathcal{IV}rg$-closed.

□

Theorem 3.31: *For a subset $[\mathcal{M}^-, \mathcal{M}^+]$ of a space $\mathcal{T}^X$, the following are equivalent.*

1. $[\mathcal{M}^-, \mathcal{M}^+]$ *is* $\mathcal{IV}$*-closed.*
2. $[\mathcal{M}^-, \mathcal{M}^+]$ *is* $\mathcal{IV}\pi g$*-closed and* $\mathcal{IV}\lambda_\pi$*-closed.*

Proof:

(1)⇒(2) Proof follows by Remark 3.16 and 3.28.

(2)⇒(1) By Lemma 3.30 and Lemma 3.29(2), proof follows similar to the proof of Theorem 3.18. □

Theorem 3.32: *For a subset* $[\mathcal{M}^-, \mathcal{M}^+]$ *of a space* $\mathcal{T}^X$*, the following are equivalent.*

1. $[\mathcal{M}^-, \mathcal{M}^+]$ *is an* $\mathcal{IV}$*-closed.*
2. $[\mathcal{M}^-, \mathcal{M}^+]$ *is an* $\mathcal{IV}rg$*-closed and an* $\mathcal{IV}\lambda_r$*-closed.*

Proof:

(1) ⇒ (2) Proof follows by Remark 3.16 and 3.28.

(2) ⇒ (1) By Lemma 3.30 and Lemma 3.29(1), proof follows similar to the proof of Theorem 3.18. □

4 Conclusions

In this study, we introduced interval-valued Λ-sets and λ-closed sets using kernel operators in interval-valued topological spaces. We established their fundamental properties and examined their relationships with existing generalized closed sets. Several characterization and equivalence results were obtained, demonstrating the importance of kernel-based approaches as a dual to closure operators. These results enrich the framework of interval-valued topology and open directions for further research.

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References

1. M. Cheong, S. H. Han, J. I. Baek and K. Hur, Interval-valued relations and their application to category theory, *Ann. Fuzzy Math. Inform. 24(3)* **(2022)**, 281–299. https://doi.org/10.30948/afmi.2022.24.3.281
2. S. Dijitha and M. Navaneethakrishnan, A study on g-closed and g-open sets in interval-valued topology, *Proceedings of National Conference on INSTA'24* **(2024)**, 33–39.
3. J. Kim, Y. B. Jun, J. G. Lee and K. Hur, Topological structures based on interval-valued sets, *Ann. Fuzzy Math. Inform. 20(3)* **(2020)**, 273–295. https://doi.org/10.30948/afmi.2020.20.3.273
4. Y. Yao, Interval sets and interval-set algebras, *Proceedings of the* 8^{th} *IEEE International Conference on Cognitive Informatics* **(2009)**, 307–314.