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Research article

Correction of a Proof of an Inequality involving the Polygamma Function

Kwara Nantomah

Department of Mathematics, School of Mathematical Sciences, C. K. Tedam University of Technology and Applied Sciences, P. O. Box 24, Navrongo, Upper-East Region, Ghana.

*Corresponding Author: Kwara Nantomah. Email: knantomah@cktutas.edu.gh

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Abstract: The proof of Lemma 5 in the paper "Corrigendum to A Harmonic Mean Inequality for the Polygamma Function" [Math. Inequal. Appl., 27(2)(2024), 273-274] contains an error. The purpose of this paper is to correct the error.

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1 Introduction

The digamma (or psi) function is defined for $x > 0$ as

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x)$$

where $\Gamma(x)$ is the Euler's gamma function. The polygamma function is defined for $x > 0$ and $m \in \mathbb{N}$ as

$$\begin{aligned} \psi^{(m)}(x) = \frac{d^m}{dx^m} \psi(x) &= (-1)^{m+1} \int_0^\infty \frac{t^m e^{-xt}}{1-e^{-t}} dt, \\ &= (-1)^{m+1} \sum_{k=0}^{\infty} \frac{m!}{(k+x)^{m+1}}. \end{aligned}$$

In 2020, Das and Swaminathan [2] provided a proof of Lemma 1.

Lemma 1: *For $x > 0$, $x \neq 1$ and $m \in \mathbb{N}$, we have*

$$\psi^{(m)}(x)\psi^{(m)}(1/x) > \left(\psi^{(m)}(1)\right)^2.$$

In 2024, Zhang and Yin [5] identified an error in the proof by Das and Swaminathan and subsequently provided a new proof of Lemma 1 which is labeled as Lemma 5 in their work.

We have also identified an error in the proof by Zhang and Yin [5] and the objective of this paper is to correct the error. The error involves omission of some terms which consequentially invalidates their proof. We shall first explain the error and then provide a correction to it.

Lemma 2 ([1])**:** *For positive real numbers x, and $m \geq 2$, $m \in \mathbb{N}$, the following holds*

$$\frac{m-1}{m} < \frac{\left(\psi^{(m)}(x)\right)^2}{\psi^{(m+1)}(x)\psi^{(m-1)}(x)} < \frac{m}{m+1}. \tag{1.1}$$

The lower and upper bounds are best possible.

Based on Lemma 2, the proof by Zhang and Yin [5] is given as follows.

Proof by Zhang and Yin : Let

$$N(x) = \psi^{(m)}(x)\psi^{(m)}(1/x),$$

then

$$N'(x) = \psi^{(m+1)}(x)\psi^{(m)}(1/x) - \frac{1}{x^2}\psi^{(m)}(x)\psi^{(m+1)}(1/x), \tag{1.2}$$

$$\begin{aligned} N''(x) = {} & \psi^{(m+2)}(x)\psi^{(m)}(1/x) - \frac{2}{x^2}\psi^{(m+1)}(x)\psi^{(m+1)}(1/x) \\ & + \frac{1}{x^4}\psi^{(m)}(x)\psi^{(m+2)}(1/x). \end{aligned} \tag{1.3}$$

So, $N'(1) = 0$ and

$$N''(1) = 2\left[\psi^{(m+2)}(1)\psi^{(m)}(1) - \left(\psi^{(m+1)}(1)\right)^2\right]. \tag{1.4}$$

Then by Lemma 2, they concluded that $N''(1) > 0$ and therefore $N(1)$ is the minimum of $N(x)$ concluding the proof of Lemma 1. □

The errors have to do with (1.3) and (1.4) as some terms were omitted. Our checks reveal that (1.3) and (1.4) should be

$$\begin{aligned} N''(x) = {} & \psi^{(m+2)}(x)\psi^{(m)}(1/x) - \frac{2}{x^2}\psi^{(m+1)}(x)\psi^{(m+1)}(1/x) \\ & + \frac{2}{x^3}\psi^{(m)}(x)\psi^{(m+1)}(1/x) + \frac{1}{x^4}\psi^{(m)}(x)\psi^{(m+2)}(1/x) \end{aligned} \tag{1.5}$$

and

$$N''(1) = 2\Big[\psi^{(m+2)}(1)\psi^{(m)}(1) - \big(\psi^{(m+1)}(1)\big)^2 + \psi^{(m)}(1)\psi^{(m+1)}(1)\Big]. \tag{1.6}$$

Indeed, the right-hand side inequality of Lemma 2 implies that

$$\psi^{(m+2)}(1)\psi^{(m)}(1) - \big(\psi^{(m+1)}(1)\big)^2 > 0. \tag{1.7}$$

However, considering (1.6), the inequality (1.7) is not enough to conclude that $N''(1) > 0$ since the term $\psi^{(m)}(1)\psi^{(m+1)}(1)$ is negative. Therefore, the evidence available in [5] does not support the conclusion that $N''(1) > 0$. To resolve this, we are only required to prove that

$$\psi^{(m+2)}(1)\psi^{(m)}(1) - \big(\psi^{(m+1)}(1)\big)^2 + \psi^{(m)}(1)\psi^{(m+1)}(1) > 0. \tag{1.8}$$

We shall now proceed to do that. In [3], it has been shown that the function

$$\mathcal{K}(z) = z + \frac{\psi^{(m)}(z)\psi^{(m+1)}(z)}{\psi^{(m)}(z)\psi^{(m+2)}(z) - \big(\psi^{(m+1)}(z)\big)^2} \tag{1.9}$$

is increasing from $(0,\infty)$ onto $(0,\frac{1}{2})$. This means that $\mathcal{K}(z)$ is positive on $(0,\infty)$. In particular, $\mathcal{K}(1)$ implies that

$$1 + \frac{\psi^{(m)}(1)\psi^{(m+1)}(1)}{\psi^{(m)}(1)\psi^{(m+2)}(1) - \big(\psi^{(m+1)}(1)\big)^2} > 0$$

which simplifies to (1.8). Hence, $N''(1) > 0$ and therefore $N(1)$ is the minimum of $N(x)$. By this correction, the proof is now complete.

Remark: *A different proof of Lemma 1 is given in Theorem 3.2 of [4]. This reaffirms the validity of Lemma 1.*

2 Conclusion

In this study, the author has identified and corrected an error in an earlier proof of an inequality involving the polygamma function. This correction reaffirms the validity of the inequality in question.

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