

Research article

Geometric characterizations of trans-Sasakian 3-manifolds admitting \ast -conformal η -Ricci solitons

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Abstract: In this paper, we study trans-Sasakian 3-manifolds admitting \ast -conformal η -Ricci solitons satisfying certain curvature conditions and obtain several interesting results.

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1 Introduction

In the Gray-Hervella classification of almost Hermitian manifolds [2], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [5] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4 . The class $C_6 \oplus C_5$ [4] coincides with the class of trans-Sasakian structures of type (α, β) . We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [1], β -Kenmotsu [3] and α -Sasakian [3], respectively. Therefore, trans-Sasakian manifolds generalize a large class of almost contact manifolds.

A solution to the Ricci flow [20]: $\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}$ is called the Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$\omega_V g + 2S + 2\lambda g = 0, \quad (1.1)$$

where S is the Ricci tensor, ω_V is the Lie derivative operator along the vector field V on M and $\lambda \in \mathbb{R}$ (the set of real numbers). The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.



As a generalization of Ricci soliton, the notion of η -Ricci soliton was introduced by Cho and Kimura [11] and is given by the equation

$$\omega_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \quad (1.2)$$

where λ and μ are real numbers.

In 2004, the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation was introduced by Fischer [12]. In classical Ricci flow equation the unit volume constraint plays an important role but in conformal Ricci flow equation the scalar curvature r is considered as constraint. The conformal Ricci flow on M is defined by the equation

$$\frac{\partial g}{\partial t} + 2(S + \frac{r}{n})g = -pg \quad (1.3)$$

and $r = -1$, where p is a scalar non-dynamical field (time dependent scalar field), r is the scalar curvature of the manifold and n is the dimension of manifold.

The conformal Ricci soliton equation is given by [9]:

$$\omega_V g + 2S + (2\lambda - (p + \frac{2}{n}))g = 0, \quad (1.4)$$

where λ is constant.

The notion of $*$ -Ricci tensor on almost Hermitian manifolds was introduced by Tachibana [24]. Later, T. Hamada [19] studied $*$ -Ricci flat real hypersurfaces of complex space forms and D. E. Blair [10] defined $*$ -Ricci tensor in contact metric manifolds given by

$$S^*(X, Y) = g(Q^*X, Y) = \text{Trace}\{\phi \circ R(X, \phi Y)\}, \quad (1.5)$$

where Q^* is the $*$ -Ricci operator and S^* is the $*$ -Ricci tensor of type $(0, 2)$.

Recently, the $*$ -Ricci solitons on almost contact metric manifolds have been studied by various authors such as [13,14,21–23] and many others.

The notion of $*$ -conformal η -Ricci soliton is defined as follows [26]:

$$\omega_V g + 2S^* + (2\lambda - (p + \frac{2}{n}))g + 2\mu\eta \otimes \eta = 0. \quad (1.6)$$

Recently, the author with his collaborators studied $*$ -conformal η -Ricci soliton in several ways to a different extent in [15–17].

The present paper is organized as follows: After a brief introduction, some useful results on trans-Sasakian 3-manifolds and basic definitions are given in Section 2. In Section 3, we study $*$ -conformal η -Ricci solitons in trans-Sasakian 3-manifolds. In Section 4, we confer $*$ -conformal η -Ricci solitons in trans-Sasakian 3-manifolds having cyclic η -recurrent Ricci tensor. Section 5 deals with the study of concircularly flat trans-Sasakian 3-manifolds admitting $*$ -conformal η -Ricci solitons. Sections 6 and 7 are dedicated to

conferring $*$ -conformal η -Ricci solitons in trans-Sasakian 3-manifolds confessing the constraint of curvature conditions $C(\xi, X) \cdot S = 0$ and $Q \cdot C = 0$, respectively. In Section 8, we study a φ -concurcularly flat trans-Sasakian 3-manifold admitting $*$ -conformal η -Ricci solitons.

2 Preliminaries

Let M be an almost contact metric manifold [1] with an almost contact metric structure (ϕ, ξ, η, g) , that is, ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \quad (2.3)$$

for all vector fields $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on M . The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \quad (2.4)$$

for any $X, Y \in \chi(M)$.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [5], if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [2], where J is the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$ for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \quad (2.5)$$

for some smooth functions α and β on M , and we say that the trans-Sasakian structure is of type (α, β) . From (2.5) it follows that

$$\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi), \quad (2.6)$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y), \quad (2.7)$$

where ∇ is the Levi Civita connection of g . In a trans-Sasakian 3-manifold, we have [6,8]

$$\begin{aligned} R(X, Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &\quad + 2\alpha\beta((\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad + (Y\alpha)\phi X - (X\alpha)\phi Y \\ &\quad + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \quad (2.8)$$

$$\begin{aligned} R(\xi, X)Y &= (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X) \\ &\quad + 2\alpha\beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \\ &\quad + (Y\alpha)\phi X + g(\phi Y, X)(grad\alpha) \\ &\quad + (Y\beta)(X - \eta(X)\xi) - g(\phi X, \phi Y)(grad\beta), \end{aligned} \quad (2.9)$$

$$2\alpha\beta + \xi\beta = 0, \quad (2.10)$$

$$S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \quad (2.11)$$

where R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature of the manifold M . Further, in a trans-Sasakian manifold of type (α, β) , we have [6]

$$\phi(grad\alpha) = (n - 1)grad\beta. \quad (2.12)$$

Using (2.10) and (2.12), for constants α and β , we have

$$R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X), \quad (2.13)$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X), \quad (2.14)$$

$$R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y), \quad (2.15)$$

$$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)), \quad (2.16)$$

$$S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X), \quad (2.17)$$

$$Q\xi = 2(\alpha^2 - \beta^2)\xi, \quad (2.18)$$

for all $X, Y, Z \in \chi(M)$. Throughout in the paper, we are using the fact that $\alpha = \beta = \text{constant}$. We define endomorphisms $R(X, Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.19)$$

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad (2.20)$$

respectively, where A is the symmetric $(0, 2)$ -tensor.

Definition 1: The concircular curvature tensor C in a trans-Sasakian n -manifold M is defined by [25]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (2.21)$$

where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

Definition 2: A trans-Sasakian 3-manifold is said to be an η -Einstein manifold if the Ricci tensor S is of the form [1]

$$S(X, Y) = pg(X, Y) + q\eta(X)\eta(Y),$$

where p and q are smooth functions on the manifold. If $q = 0$, then the manifold is said to be an Einstein manifold.

Lemma 1: In a trans-Sasakian 3-manifold the $*$ -Ricci tensor is given by [7]

$$S^*(Y, Z) = S(Y, Z) - (\alpha^2 - \beta^2)g(Y, Z) - (\alpha^2 - \beta^2)\eta(Y)\eta(Z) \quad (2.22)$$

for any $Y, Z \in \chi(M)$.

If a trans-Sasakian 3-manifold is $*$ -Ricci flat, then from (2.22), it follows that

$$S(Y, Z) = (\alpha^2 - \beta^2)g(Y, Z) + (\alpha^2 - \beta^2)\eta(Y)\eta(Z). \quad (2.23)$$

Thus, we state the following Lemma:

Lemma 2: A $*$ -Ricci flat trans-Sasakian 3-manifold is an Einstein manifold of the form (2.23).

Lemma 3: In a trans-Sasakian 3-manifold, the eigenvalue of the $*$ -Ricci operator Q^* corresponding to the eigenvector ξ is zero, i.e., $Q^*\xi = 0$.

Proof: From (2.22), we have

$$Q^*Y = QY - (\alpha^2 - \beta^2)Y - (\alpha^2 - \beta^2)\eta(Y)\xi, \quad (2.24)$$

which by putting $Y = \xi$, using (2.1) and (2.18) gives $Q^*\xi = 0$. \square

3 *-Conformal η -Ricci solitons on trans-Sasakian 3-manifolds

Let M be a trans-Sasakian 3-manifold admitting *-conformal η -Ricci solitons. Then (1.6) holds, and thus at $V = \xi$, we have

$$(\omega_\xi g)(X, Y) + 2S^*(X, Y) + (2\lambda - (p + \frac{2}{3}))g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \quad (3.1)$$

In a trans-Sasakian 3-manifold, we have

$$(\omega_\xi g)(X, Y) = 2\beta(g(X, Y) - \eta(X)\eta(Y)). \quad (3.2)$$

By using (3.2) in (3.1), we obtain

$$S^*(Y, Z) = -[\lambda + \beta - \frac{1}{2}(p + \frac{2}{3})]g(Y, Z) + (\beta - \mu)\eta(Y)\eta(Z). \quad (3.3)$$

In view of (2.22), (3.3) takes the form

$$S(Y, Z) = Ag(Y, Z) + B\eta(Y)\eta(Z), \quad (3.4)$$

where $A = -[\lambda + \beta - (\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3})]$ and $B = (\beta + \alpha^2 - \beta^2 - \mu)$.

Putting $Z = \xi$ in (3.4) and using (2.1) and (2.3), we have

$$S(Y, \xi) = C\eta(Y). \quad (3.5)$$

where $C = -[\lambda + \mu - 2(\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3})]$.

By virtue of (2.17), (3.5) gives

$$\lambda + \mu = \frac{1}{2}(p + \frac{2}{3}). \quad (3.6)$$

Thus, we have the following theorem:

Theorem 1: *Let M be a trans-Sasakian 3-manifold. If the manifold admits *-conformal η -Ricci solitons, then M is an η -Einstein manifold of the form (3.4). Moreover, λ and μ are related by $\lambda + \mu = \frac{1}{2}(p + \frac{2}{3})$.*

If we assume $\mu = 0$ in (3.6), then we have $\lambda = \frac{1}{2}(p + \frac{2}{3})$. Now, we state the following corollary:

Corollary 1: *Let M be a trans-Sasakian 3-manifold admitting $*$ -conformal η -Ricci solitons, then the soliton is expanding, steady or shrinking according to $p > -\frac{2}{3}$, $p = -\frac{2}{3}$ or $p < -\frac{2}{3}$, respectively.*

Now, let a trans-Sasakian 3-manifold admit a $*$ -conformal η -Ricci solitons, where V is pointwise collinear with ξ , i.e., $V = h\xi$ (h is a function). Then (1.6) holds, and thus we have

$$(\omega_{h\xi}g)(X, Y) + 2S^*(X, Y) + [2\lambda - (p + \frac{2}{3})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0. \quad (3.7)$$

Applying the property of the Lie derivative and the Levi-Civita connection in (3.7), we have

$$\begin{aligned} hg(\nabla_X\xi, Y) + (Xh)\eta(Y) + hg(X, \nabla_Y\xi) + (Yh)\eta(X) + 2S^*(X, Y) \\ + [2\lambda - (p + \frac{2}{3})]g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \end{aligned}$$

which by using (2.3) and (2.6) takes the form

$$\begin{aligned} 2S^*(X, Y) + (Xh)\eta(Y) + (Yh)\eta(X) + [2\lambda - (p + \frac{2}{3}) + 2\beta h]g(X, Y) \\ + 2(\mu - \beta h)\eta(X)\eta(Y) = 0. \end{aligned} \quad (3.8)$$

Taking $Y = \xi$ in (3.8), then using (2.1), (2.3), (3.6) and $S^*(X, \xi) = 0$, we lead to

$$(Xh) + (\xi h)\eta(X) = 0. \quad (3.9)$$

Now, putting $X = \xi$ in (3.9) and using (2.1), we get

$$(\xi h) = 0. \quad (3.10)$$

From (3.9) and (3.10), we obtain $X(h) = 0$, that is, h is constant. Thus, (3.8) takes the form

$$S^*(X, Y) = -[\lambda + \beta h - \frac{1}{2}(p + \frac{2}{3})]g(X, Y) + (\beta h - \mu)\eta(X)\eta(Y). \quad (3.11)$$

In view of (2.22), (3.11) takes the form

$$\begin{aligned} S(X, Y) = & -[\lambda + \beta h - (\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3})]g(X, Y) \\ & + (\beta h + \alpha^2 - \beta^2 - \mu)\eta(X)\eta(Y). \end{aligned} \quad (3.12)$$

Now putting $Y = \xi$ in (3.12) and using (2.1), (2.3) and (2.17), we lead to

$$\lambda + \mu = \frac{1}{2}(p + \frac{2}{3}).$$

Thus, we have the following theorem:

Theorem 2: *Let M be a trans-Sasakian 3-manifold. If the manifold admits $*$ -conformal η -Ricci solitons such that V is pointwise collinear with ξ , then M is an η -Einstein manifold of the form (3.12). Moreover, λ and μ are related by $\lambda + \mu = \frac{1}{2}(p + \frac{2}{3})$.*

4 $*$ -Conformal η -Ricci solitons on trans-Sasakian 3-manifolds admitting cyclic η -recurrent Ricci tensor

Definition 3: *A trans-Sasakian 3-manifold is said to have a cyclic η -recurrent Ricci tensor, if its Ricci tensor $S(\neq 0)$ of type $(0, 2)$ satisfies the following condition*

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \quad (4.1)$$

$$= \eta(X)S(Y, Z) + \eta(Y)S(Z, X) + \eta(Z)S(X, Y)$$

for all X, Y, Z on M .

Let the manifold admit $*$ -conformal η -Ricci solitons and has cyclic η -recurrent Ricci tensor, then (4.1) holds. Taking the covariant derivative of (3.4) and making use of (2.7), we find

$$\begin{aligned} (\nabla_X S)(Y, Z) &= B[-\alpha g(\phi X, Y)\eta(Z) - \alpha g(\phi X, Z)\eta(Y) \\ &\quad + \beta g(\phi X, \phi Y)\eta(Z) + \beta g(\phi X, \phi Z)\eta(Y)]. \end{aligned} \quad (4.2)$$

In view of (3.4) and (4.2), (4.1) takes the form

$$\begin{aligned} &2\beta B[g(\phi X, \phi Y)\eta(Z) + g(\phi Y, \phi Z)\eta(X) + g(\phi X, \phi Z)\eta(Y)] \\ &= A[g(Y, Z)\eta(X) + g(X, Z)\eta(Y) + g(X, Y)\eta(Z)] + 3B\eta(X)\eta(Y)\eta(Z), \end{aligned}$$

which can be written as

$$\begin{aligned} (2\beta B - A)[g(X, Y)\eta(Z) + g(Y, Z)\eta(X) + g(X, Z)\eta(Y)] \\ - 3B(2\beta + 1)\eta(X)\eta(Y)\eta(Z) = 0. \end{aligned} \quad (4.3)$$

Putting $Y = Z = \xi$ in (4.3), then using (2.1) and (2.3), we get $A + B = 0 \implies \lambda + \mu = 2(\alpha^2 - \beta^2) + \frac{1}{2}(p + \frac{2}{3})$, where $\eta(X) \neq 0$. Thus, we have the following theorem:

Theorem 3: *Let a trans-Sasakian 3-manifold admit $*$ -conformal η -Ricci solitons and has cyclic η -recurrent Ricci tensor, then λ and μ are related by $\lambda + \mu = 2(\alpha^2 - \beta^2) + \frac{1}{2}(p + \frac{2}{3})$.*

5 Conircularly flat trans-Sasakian 3-manifolds admitting *-conformal η -Ricci solitons

Let a trans-Sasakian 3-manifold admitting *-conformal η -Ricci solitons be conircularly flat, i.e. $C(X, Y)Z = 0$. Thus, from (2.21), it follows that

$$R(X, Y)Z = \frac{r}{6}[g(Y, Z)X - g(X, Z)Y]. \quad (5.1)$$

Taking the inner product of (5.1) with ξ , we have

$$\eta(R(X, Y)Z) = \frac{r}{6}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

which by using (2.16) takes the form

$$(\alpha^2 - \beta^2 - \frac{r}{6})[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] = 0. \quad (5.2)$$

Putting $Y = \xi$ in (5.2) and using (2.1) and (2.3), we have

$$(\alpha^2 - \beta^2 - \frac{r}{6})g(\phi X, \phi Z) = 0.$$

This gives $r = 6(\alpha^2 - \beta^2)$, where $g(\phi X, \phi Z) \neq 0$. Thus, we have the following theorem:

Theorem 4: *The scalar curvature of a conircularly flat trans-Sasakian 3-manifold admitting *-conformal η -Ricci solitons is $6(\alpha^2 - \beta^2)$.*

6 *-Conformal η -Ricci solitons on trans-Sasakian 3-manifolds satisfying $C(\xi, X) \cdot S = 0$

Let a trans-Sasakian 3-manifold admitting *-conformal η -Ricci solitons satisfies $C(\xi, X) \cdot S = 0$. Then, we have

$$S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0. \quad (6.1)$$

From (2.21), we find

$$C(\xi, X)Y = (\alpha^2 - \beta^2 - \frac{r}{6})(g(X, Y)\xi - \eta(Y)X). \quad (6.2)$$

By virtue of (6.2), (6.1) takes the form

$$(\alpha^2 - \beta^2 - \frac{r}{6})[S(X, Z)\eta(Y) - g(X, Y)S(\xi, Z) + S(X, Y)\eta(Z) - g(X, Z)S(Y, \xi)] = 0,$$

which by using (3.5) becomes

$$\begin{aligned} (\alpha^2 - \beta^2 - \frac{r}{6})[(\lambda + \mu - 2(\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3}))(g(X, Y)\eta(Z) \\ + g(X, Z)\eta(Y)) + S(X, Z)\eta(Y) + S(X, Y)\eta(Z)] = 0. \end{aligned} \quad (6.3)$$

Taking $Z = \xi$ in (6.3) and using (2.1), (2.3) and (3.5), we have

$$(\alpha^2 - \beta^2 - \frac{r}{6})[S(X, Y) + (\lambda + \mu - 2(\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3}))g(X, Y)] = 0.$$

Thus, we have either $r = 6(\alpha^2 - \beta^2)$, or

$$S(X, Y) = -(\lambda + \mu - 2(\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3}))g(X, Y). \quad (6.4)$$

Using (3.6) in (6.4), we lead to

$$S(X, Y) = 2(\alpha^2 - \beta^2)g(X, Y). \quad (6.5)$$

Thus, we state the following theorem:

Theorem 5: *If a trans-Sasakian 3-manifold admitting *-conformal η -Ricci solitons satisfies $C(\xi, X) \cdot S = 0$, then either the scalar curvature is constant, or the manifold is an Einstein manifold of the form (6.5).*

7 *-Conformal η -Ricci solitons on trans-Sasakian 3-manifolds satisfying $Q \cdot C = 0$

Let a trans-Sasakian 3-manifold admitting *-conformal η -Ricci solitons satisfies $Q \cdot C = 0$. Then, we have

$$Q(C(X, Y)Z) - C(QX, Y)Z - C(X, QY)Z - C(X, Y)QZ = 0 \quad (7.1)$$

for all X, Y, Z on M .

By making use of (2.21) in (7.1), we have

$$\begin{aligned} & Q(R(X, Y)Z) - R(QX, Y)Z - R(X, QY)Z - R(X, Y)QZ \\ & + \frac{2r}{6}(S(Y, Z)X - S(X, Z)Y) = 0, \end{aligned}$$

which by taking the inner product with ξ takes the form

$$\begin{aligned} & \eta(Q(R(X, Y)Z)) - \eta(R(QX, Y)Z) - \eta(R(X, QY)Z) - \eta(R(X, Y)QZ) \\ & + \frac{2r}{6}(S(Y, Z)\eta(X) - S(X, Z)\eta(Y)) = 0. \end{aligned} \quad (7.2)$$

Putting $Y = \xi$ in (7.2), we have

$$\begin{aligned} & \eta(Q(R(X, \xi)Z)) - \eta(R(QX, \xi)Z) - \eta(R(X, Q\xi)Z) - \eta(R(X, \xi)QZ) \\ & + \frac{2r}{6}(S(\xi, Z)\eta(X) - S(X, Z)) = 0. \end{aligned} \quad (7.3)$$

From (2.13) and (3.5), we find

$$\begin{cases} \eta(Q(R(X, \xi)Z)) = \eta(R(X, Q\xi)Z) = C(\alpha^2 - \beta^2)(\eta(X)\eta(Z) - g(X, Z)), \\ \eta(R(QX, \xi)Z) = \eta(R(X, \xi)QZ) = (\alpha^2 - \beta^2)(C\eta(X)\eta(Z) - S(X, Z)). \end{cases} \quad (7.4)$$

By using (3.5) and (7.4) in (7.3), we obtain

$$[(\alpha^2 - \beta^2) - \frac{r}{6}][S(X, Z) - C\eta(X)\eta(Z)] = 0.$$

Thus, we have either $r = 6(\alpha^2 - \beta^2)$, or

$$S(X, Z) = C\eta(X)\eta(Z), \quad (7.5)$$

where $C = -[\lambda + \mu - 2(\alpha^2 - \beta^2) - \frac{1}{2}(p + \frac{2}{3})]$.

Thus, we state the following theorem:

Theorem 6: *Let a trans-Sasakian 3-manifold admit $*$ -conformal η -Ricci solitons and satisfies $Q \cdot C = 0$. Then, either the scalar curvature is constant, or the manifold is a special type of η -Einstein manifold of the form (7.5).*

8 ϕ -concircularly semisymmetric trans-Sasakian 3-manifolds admitting $*$ -conformal η -Ricci solitons

Definition 4: *A Riemannian manifold is said to be ϕ -concircularly semisymmetric if [18]*

$$C(X, Y) \cdot \phi = 0 \quad (8.1)$$

for all X, Y on M .

Let a ϕ -concircularly semisymmetric trans-Sasakian 3-manifold admit $*$ -conformal η -Ricci solitons, then (8.1) holds. Thus, we have

$$(C(X, Y) \cdot \phi)Z = C(X, Y)\phi Z - \phi C(X, Y)Z = 0. \quad (8.2)$$

From (2.21), we find

$$C(X, Y)\phi Z = R(X, Y)\phi Z - \frac{r}{6}[g(Y, \phi Z)X - g(X, \phi Z)Y], \quad (8.3)$$

and

$$\phi C(X, Y)Z = \phi R(X, Y)Z - \frac{r}{6}[g(Y, Z)\phi X - g(X, Z)\phi Y]. \quad (8.4)$$

Using (8.3) and (8.4) in (8.2), we have

$$R(X, Y)\phi Z - \phi R(X, Y)Z + \frac{r}{6}[g(Y, Z)\phi X - g(X, Z)\phi Y - g(Y, \phi Z)X + g(X, \phi Z)Y] = 0. \quad (8.5)$$

Taking $Y = \xi$ in (8.5) and using (2.1) and (2.13), we find

$$\left[\frac{r}{6} - (\alpha^2 - \beta^2)\right](g(X, \phi Z)\xi - \eta(Z)\phi X) = 0. \quad (8.6)$$

Taking the inner product of (8.6) with ξ and using (2.1), it follows that

$$r = 6(\alpha^2 - \beta^2), \quad (8.7)$$

where $g(X, \phi Z) \neq 0$. Thus, we state the following theorem:

Theorem 7: *The scalar curvature of a ϕ -conircularly semisymmetric trans-Sasakian 3-manifold admitting $*$ -conformal η -Ricci solitons is $6(\alpha^2 - \beta^2)$.*

9 Conclusion

In this work, we have investigated trans-Sasakian 3-manifolds admitting $*$ -conformal η -Ricci solitons under various curvature conditions. By analyzing the interplay between the geometric structure and the soliton equation, several characterizations and constraints have been obtained. The results reveal that the existence of such solitons imposes significant restrictions on the underlying manifold, leading to notable geometric consequences. These findings contribute to a deeper understanding of the structure of trans-Sasakian manifolds and open avenues for further research in related geometric flows and contact metric geometry.

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