Annals of Communications in Mathematics
Volume 9, Number 1 (2026), 14
DOI: `10.62072/acm.2026.09014`
ISSN: 2582-0818
`https://www.technoskypub.com/journal/acm/`

Research article

Stability of Euler–Lagrange additive inequality in Banach spaces

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Received: 09 February 2026; Accepted: 12 March 2026; Published: 31 March 2026

Citation: M. H. Han, H. M. Kim, J. M. Rassias, E. Son, Stability of Euler–Lagrange additive inequality in Banach spaces. Ann. Commun. Math. 9 **(2026)**, 14. https://doi.org/10.62072/acm.2026.09014

Abstract: In the paper, we investigate the Hyers–Ulam stability theorem of an Euler–Lagrange additive functional inequality

$$\Big\|\sum_{j=1}^{n}\Big[\Big(\sum_{k=j}^{n}\lambda_k\Big)f(x_j-x_{j-1})\Big]\Big\| \le \Big\|f\Big(\sum_{j=1}^{n}\lambda_j x_j\Big)\Big\| + \varphi(x_1,\cdots,x_n),$$
$$\text{where}\quad x_0\equiv 0,\quad n\ge 3,$$

subject to control function φ in (non-Archimedean) Banach spaces.

Mathematics Subject Classification: 39B82, 16W25, 39B72
Keywords: Euler–Lagrange additive mappings; generalized Hyers–Ulam stability; contractively subadditive; non-Archimedean Banach spaces;

1 Introduction

In 1940, S.M. Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems, and it was first published in 1960 [1]. The stability problem of functional equations has been originated from the question of S.M. Ulam [1] concerning the stability of group homomorphisms. In 1941, D.H. Hyers [2] proved the following stability result for the case of approximate additive mappings between Banach spaces. The method which was provided by D.H. Hyers, and which produces the additive mapping, is called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' theorem was generalized by T. Aoki [3] and D.G. Bourgin [4] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [5] also provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded, and then Rassias' theorem was extended

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by P. Găvruta [6] for mappings with general controlled perturbing functions by considering unbounded Cauchy difference. During the last three decades, the stability problems of several functional equations have been intensively and extensively investigated by a number of authors and there are many interesting volumes containing these stability problems of several functional equations [7–13] and of additive functional inequalities [14, 15].

K. Hensel [16] has introduced special normed spaces which do not have the Archimedean property: for any real $x, y > 0$, there exists a natural number n such that $x < ny$. Thus, we recall from [17] that a non-Archimedean valuation in a field $\mathbb{K}$ is a function $|\cdot|_v : \mathbb{K} \to [0,\infty)$ equipped with

(i) $|r|_v = 0$ if and only if $r = 0$;

(ii) $|rs|_v = |r|_v\,|s|_v$ for all $r, s \in \mathbb{K}$;

(iii) $|r+s|_v \le \max\{|r|_v, |s|_v\}$ for all $r, s \in \mathbb{K}$.

Any field $\mathbb{K}$ endowed with a non-Archimedean valuation is said to be a non-Archimedean field. In any non-Archimedean field, we remark that $|1|_v = |-1|_v = 1$, $|n|_v \le 1$ for all nonzero integers $n \in \mathbb{Z}$, and $|na|_v \le |ka|_v \le |a|_v \le |\frac{a}{k}|_v \le |\frac{a}{n}|_v$ for any $a \in \mathbb{K}$ and $n, k \in \mathbb{Z}$ with $k|n$. On the other hand, let Y be a linear space over the non-Archimedean field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|_v$. A function $\|\cdot\|_v : Y \to [0,\infty)$ is said to be a non-Archimedean norm on Y if it satisfies the following conditions:

(i) $\|x\|_v = 0$ if and only if $x = 0$;

(ii) $\|rx\|_v = |r|_v \|x\|_v$ for all $r \in \mathbb{K}$ and $x \in Y$;

(iii) $\|x+y\|_v \le \max\{\|x\|_v, \|y\|_v\}$ for all $x, y \in Y$.

In this case, $(Y, \|\cdot\|_v)$ is called a non-Archimedean normed space. They say that a non-Archimedean normed space $(Y, \|\cdot\|_v)$ is complete, that is, non-Archimedean Banach space, if and only if every Cauchy sequence in Y is convergent in the space Y by the norm $\|\cdot\|_v$ [8, 10, 18]. It follows from the strong triangle inequality that

$$\|x_n - x_m\|_v \le \max\{\|x_{j+1} - x_j\|_v : m \le j \le n-1\}$$

for all $x_n, x_m \in Y$ and all $m, n \in \mathbb{N}$ with $n > m$. Therefore, a sequence $\{x_n\}$ is a Cauchy sequence in non-Archimedean normed space $(Y, \|\cdot\|_v)$ if and only if the sequence $\{x_{n+1} - x_n\}$ converges to zero in the space $(Y, \|\cdot\|_v)$.

Now let $n \ge 3$ be any fixed positive integer and $(\lambda_1, \cdots, \lambda_n)$ any n-tuple of fixed positive real numbers λ_j for all $1 \le j \le n$. Then, we now consider a modified and generalized Euler–Lagrange additive functional inequality

$$\Big\|\sum_{j=1}^{n}\Big[\Big(\sum_{k=j}^{n}\lambda_k\Big) f(x_j - x_{j-1})\Big]\Big\| \le \Big\|f\Big(\sum_{j=1}^{n}\lambda_j x_j\Big)\Big\|,\ x_0 := 0, \tag{1.1}$$

of which the general solution exactly contains Cauchy additive mappings in the sequel. In this article, we first investigate generalized Hyers–Ulam stability via direct method of the inequality (1.1) in Banach spaces in Section 2, and then we alternatively study generalized Hyers–Ulam stability in non-Archimedean Banach spaces in Section 3.

2 Stability of Ineq. (1.1) in Banach spaces.

First of all, we present the general solution of the functional inequality (1.1) between linear spaces.

Lemma 2.1. *Let both X and Y be linear spaces. A mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (1.1) if and only if f is additive.*

Proof: Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (1.1). Then, by letting $x_1 := y_1$, $x_j := \sum_{k=1}^{j} y_k$, $(j = 2, \cdots, n)$, and $\overline{\lambda_j} := \sum_{k=j}^{n} \lambda_k$ for all $j = 1, 2, \cdots, n$, one obtains that $x_j - x_{j-1} = y_j$ $(j = 1, 2, \cdots, n)$, where $x_0 \equiv 0$ in case $j = 1$, and thus

$$\begin{aligned} \Big\| \sum_{j=1}^{n} \Big[\Big(\sum_{k=j}^{n} \lambda_k \Big) f(x_j - x_{j-1}) \Big] \Big\| &\leq \Big\| f\Big(\sum_{j=1}^{n} \lambda_j x_j \Big) \Big\|, \\ \Leftrightarrow \Big\| \sum_{j=1}^{n} \overline{\lambda_j}\, f(y_j) \Big\| &\leq \Big\| f\Big(\sum_{j=1}^{n} \lambda_j \big(\sum_{k=1}^{j} y_k \big) \Big) \Big\|, \\ \Leftrightarrow \Big\| \sum_{j=1}^{n} \overline{\lambda_j}\, f(y_j) \Big\| &\leq \Big\| f\Big(\sum_{j=1}^{n} \overline{\lambda_j}\, y_j \Big) \Big\|, \end{aligned}$$

for all vectors $y_j \in X$ $(1 \leq j \leq n)$. Therefore, we figure out this inequality

$$\|\overline{\lambda_1} f(y_1) + \overline{\lambda_2} f(y_2) + \overline{\lambda_3} f(y_3)\| \leq \|f(\overline{\lambda_1} y_1 + \overline{\lambda_2} y_2 + \overline{\lambda_3} y_3)\|$$

for all $y_1, y_2, y_3 \in X$, from which we conclude that f is additive by Lemma 2.1 [19].

The proof of converse is trivial. □

Corollary 2.2. *Let both X and Y be linear spaces. A mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation*

$$\sum_{j=1}^{n} \Big[\Big(\sum_{k=j}^{n} \lambda_k \Big) f(x_j - x_{j-1}) \Big] = f\Big(\sum_{j=1}^{n} \lambda_j x_j \Big),$$

for all vectors $x_1, \cdots, x_n$ in X, where $x_0 \equiv 0$, if and only if f is additive.

From now on, we assume that X is a linear space and Y is a Banach space with norm $\|\cdot\|$ unless we give any specific reference. For notational convenience, given a mapping $f : X \to Y$ we consider the following Euler–Lagrange functional inequality with a perturbing term $\varphi : X^n \to [0, \infty)$ as follows:

$$\begin{aligned} \Big\| \sum_{j=1}^{n} \Big[\Big(\sum_{k=j}^{n} \lambda_k \Big) f(x_j - x_{j-1}) \Big] \Big\| \quad &\leq \quad \Big\| f\Big(\sum_{j=1}^{n} \lambda_j x_j \Big) \Big\| \\ &\quad + \varphi(x_1, \cdots, x_n) \end{aligned} \tag{2.1}$$

for all vectors $x_1, \cdots, x_n$ in a linear space X, where $x_0 \equiv 0$. In the following, we investigate a generalized Hyers–Ulam stability via direct method of the inequality (2.1) controlled by the perturbing term φ in the Banach space Y with norm $\|\cdot\|$.

Theorem 2.3. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) and the perturbing function $\varphi : X^n \to [0, \infty)$ satisfies*

$$\sum_{i=0}^{\infty} \frac{1}{2^i} \varphi(2^i x_1, \cdots, 2^i x_n) < \infty, \tag{2.2}$$

for all $x_1, \cdots, x_n \in X$. Then, there exists a unique Euler–Lagrange additive mapping $A_1 : X \to Y$ defined by $A_1(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}$ such that the mapping A_1 satisfies the approximation

$$\|f(x) - A_1(x)\| \le \frac{1}{2} \sum_{i=0}^{\infty} \frac{\overline{\varphi}(2^i x, 2^i x)}{2^i} \tag{2.3}$$

for all $x \in X$, where $\overline{\varphi}$ is defined as in (2.8), and $\overline{\lambda_j} := \sum_{k=j}^{n} \lambda_k$ for all $j = 1, 2, \cdots, n$.

Proof: Replacing $x_1 := y_1$, $x_j := \sum_{k=1}^{j} y_k$, $(j = 2, \cdots, n)$ in the functional inequality (2.1), one obtains $x_j - x_{j-1} = y_j$ $(j = 1, 2, \cdots, n)$, and thus

$$\begin{aligned} \Big\| \sum_{j=1}^{n} \overline{\lambda_j}\, f(y_j) \Big\| \quad &\le \quad \Big\| f\Big(\sum_{j=1}^{n} \overline{\lambda_j}\, y_j \Big) \Big\| \\ &+ \quad \varphi\Big(y_1, y_1 + y_2, \cdots, x_j := \sum_{k=1}^{j} y_k, \cdots, \sum_{k=1}^{n} y_k \Big), \end{aligned} \tag{2.4}$$

for all vectors $y_j \in X$. Letting $(y_1, \cdots, y_n) := (x, 0, -\overline{\lambda_1} x, 0, \cdots, 0)$ in (2.4), we have

$$\|\overline{\lambda_1} f(x) + \overline{\lambda_3} f(-\overline{\lambda_1} x)\| \le \varphi(x, x, (1 - \overline{\lambda_1})x, \cdots, (1 - \overline{\lambda_1})x) \tag{2.5}$$

for all $x \in X$. Letting $(y_1, \cdots, y_n) := (0, y, -\overline{\lambda_2} y, 0, \cdots, 0)$ in (2.4), one obtains

$$\|\overline{\lambda_2} f(y) + \overline{\lambda_3} f(-\overline{\lambda_2} y)\| \le \varphi(0, y, (1 - \overline{\lambda_2})y, \cdots, (1 - \overline{\lambda_2})y) \tag{2.6}$$

for all $y \in X$. Letting $(y_1, \cdots, y_n) := (x, y, -\overline{\lambda_1} x - \overline{\lambda_2} y, 0, \cdots, 0)$ in (2.4), we arrive at

$$\begin{aligned} &\|\overline{\lambda_1} f(x) + \overline{\lambda_2} f(y) + \overline{\lambda_3} f(-\overline{\lambda_1} x - \overline{\lambda_2} y)\| \\ &\quad \le \varphi(x, x + y, (1 - \overline{\lambda_1})x + (1 - \overline{\lambda_2})y, \cdots, (1 - \overline{\lambda_1})x + (1 - \overline{\lambda_2})y) \end{aligned} \tag{2.7}$$

for all $x, y \in X$. Thus, it follows from three inequalities above that

$$\begin{aligned} &\overline{\lambda_3} \|f(-\overline{\lambda_1} x) + f(-\overline{\lambda_2} y) - f(-\overline{\lambda_1} x - \overline{\lambda_2} y)\| \\ &\quad \le \varphi(x, x, (1 - \overline{\lambda_1})x, \cdots, (1 - \overline{\lambda_1})x) \\ &\qquad + \varphi(0, y, (1 - \overline{\lambda_2})y, \cdots, (1 - \overline{\lambda_2})y) \\ &\qquad + \varphi(x, x + y, (1 - \overline{\lambda_1})x + (1 - \overline{\lambda_2})y, \cdots, (1 - \overline{\lambda_1})x + (1 - \overline{\lambda_2})y), \end{aligned}$$

which yields the approximate Cauchy functional inequality

$$\|f(x)+f(y)-f(x+y)\| \le \overline{\varphi}(x,y) \tag{2.8}$$

for all $x, y \in X$, where

$$\begin{aligned}\overline{\varphi}(x,y) \quad := \quad & \frac{1}{\lambda_3}\Big\{\varphi\Big(-\frac{x}{\lambda_1}, -\frac{x}{\lambda_1}, (1-\frac{1}{\lambda_1})x, \cdots, (1-\frac{1}{\lambda_1})x\Big) \\ & +\varphi\Big(0, -\frac{y}{\lambda_2}, (1-\frac{1}{\lambda_2})y, \cdots, (1-\frac{1}{\lambda_2})y\Big) \\ & +\varphi\Big(-\frac{x}{\lambda_1}, -\frac{x}{\lambda_1}-\frac{y}{\lambda_2}, (1-\frac{1}{\lambda_1})x+(1-\frac{1}{\lambda_2})y, \\ & \qquad \cdots, (1-\frac{1}{\lambda_1})x+(1-\frac{1}{\lambda_2})y\Big)\Big\},\end{aligned}$$

for all $x, y \in X$. Now, applying the iterative process and direct method to the approximate Cauchy functional inequality (2.8) with controlled condition (2.2), we see from the reference [6] that there exists a unique mapping $A_1 : X \to Y$ defined by $A_1(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}$ such that the mapping A_1 satisfies the approximation (2.3).

In addition, it follows from (2.1), (2.2) and the definition of A_1 that

$$\begin{aligned}\Big\|\sum_{j=1}^{n}\Big[\Big(\sum_{k=j}^{n}\lambda_k\Big)\frac{f(2^m(x_j-x_{j-1}))}{2^m}\Big]\Big\| \quad \le \quad & \Big\|\frac{1}{2^m}f\Big(2^m\sum_{j=1}^{n}\lambda_j x_j\Big)\Big\| \\ + \quad & \frac{1}{2^m}\varphi(2^m x_1, \cdots, 2^m x_n)\end{aligned}$$

for all $x_1, \cdots, x_n \in X$, and all positive integers $m \in \mathbb{N}$. Letting $m \to \infty$,

$$\Big\|\sum_{j=1}^{n}\Big[\Big(\sum_{k=j}^{n}\lambda_k\Big)A_1(x_j-x_{j-1})\Big]\Big\| \le \Big\|A_1\Big(\sum_{j=1}^{n}\lambda_j x_j\Big)\Big\|$$

for all $x_1, \cdots, x_n \in X$. Therefore, the mapping A_1 satisfies the Euler–Lagrange functional inequality (1.1), and so it is additive by Lemma 2.1. This completes the proof. □

The following theorem is an alternative stability result of Theorem 2.3 concerning the stability problem of functional inequality (2.1).

Theorem 2.4. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) and the perturbing function $\varphi : X^n \to [0,\infty)$ satisfies*

$$\sum_{i=1}^{\infty} 2^i \varphi(\frac{x_1}{2^i}, \cdots, \frac{x_n}{2^i}) < \infty, \tag{2.9}$$

for all $x_1, \cdots, x_n \in X$. Then, there exists a unique Euler–Lagrange additive mapping $A_2 : X \to Y$ defined by

$A_2(x) = \lim_{m\to\infty} 2^m f(\frac{x}{2^m})$ such that the mapping A_2 satisfies the approximation

$$\|f(x) - A_2(x)\| \le \frac{1}{2}\sum_{i=1}^{\infty} 2^i \overline{\varphi}(\frac{x}{2^i}, \frac{x}{2^i}) \tag{2.10}$$

for all $x \in X$, where $\overline{\varphi}$ is defined as in (2.8).

Proof: The proof goes similarly through the corresponding part of Theorem 2.3, and thus we complete the proof. □

We recall from [19] that a subadditive function is a function $\phi : U \to V$, having a domain U and a codomain $(V, \leq)$ that are both closed under addition, with the following property:

$$\phi(x + y) \le \phi(x) + \phi(y),\ \forall x, y \in U.$$

Now, we say that a function $\phi : U \to V$ is contractively subadditive if there exists a constant L_1 with $0 < L_1 < 1$ such that

$$\phi(x + y) \le L_1[\phi(x) + \phi(y)],\ \forall x, y \in U.$$

Thus, ϕ satisfies these properties $\phi(2x) \le 2L_1\phi(x)$ and so $\phi(2^n x) \le (2L_1)^n\phi(x)$ for all $x \in U$. Similarly, we say that a function $\phi : U \to V$ is expansively superadditive if there exists a constant L_2 with $0 < L_2 < 1$ such that

$$\phi(x + y) \ge \frac{1}{L_2}[\phi(x) + \phi(y)],\ \forall x, y \in U.$$

Thus, ϕ satisfies the inequalities $\phi(\frac{x}{2}) \le \frac{L_2}{2}\phi(x)$ and so $\phi(\frac{x}{2^n}) \le (\frac{L_2}{2})^n\phi(x)$ for all $x \in U$. Now, we investigate the generalized Hyers–Ulam stability of the Euler–Lagrange functional inequality (2.1) with a perturbing term $\varphi(x_1, \cdots, x_n)$ subject to either contractively subadditive or expansively superadditive conditions, respectively.

Theorem 2.5. *Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.1) and the perturbing function $\varphi : X^n \to [0, \infty)$ satisfies contractively subadditive condition*

$$\varphi(2x_1, \cdots, 2x_n) \le 2L_1\varphi(x_1, \cdots, x_n), \tag{2.11}$$

for all $x_1, \cdots, x_n \in X$ and for some L_1 with $0 < L_1 < 1$. Then, there exists a unique Euler–Lagrange additive mapping $A_1 : X \to Y$ defined by $A_1(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}$ such that the mapping A_1 satisfies the approximation

$$\|f(x) - A_1(x)\| \le \frac{1}{2(1 - L_1)}\overline{\varphi}(x, x) \tag{2.12}$$

for all $x \in X$, where $\overline{\varphi}$ is defined as in (2.8).

Proof: We observe that

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{1}{2^i}\varphi(2^i x_1,\cdots,2^i x_n) &\leq \sum_{i=0}^{\infty}\frac{1}{2^i}(2L_1)^i\varphi(x_1,\cdots,x_n)\\ &= \frac{1}{1-L_1}\varphi(x_1,\cdots,x_n)<\infty,\end{aligned}$$

for all $x_1,\cdots,x_n \in X$, and thus one obtains the desired results. □

Theorem 2.6. *Assume that a mapping $f : X \to Y$ with $f(0)=0$ satisfies the functional inequality (2.1) and the perturbing function $\varphi : X^n \to [0,\infty)$ satisfies expansively superadditive condition*

$$\varphi(\frac{x_1}{2},\cdots,\frac{x_n}{2}) \leq \frac{L_2}{2}\varphi(x_1,\cdots,x_n), \tag{2.13}$$

for all $x_1,\cdots,x_n \in X$ and for some L_2 with $0<L_2<1$. Then, there exists a unique Euler–Lagrange additive mapping $A_2 : X \to Y$ defined by $A_2(x)=\lim_{m\to\infty}2^m f(2^{-m}x)$ such that the mapping A_2 satisfies the approximation

$$\|f(x)-A_2(x)\| \leq \frac{L_2}{2(1-L_2)}\overline{\varphi}(x,x) \tag{2.14}$$

for all $x \in X$, where $\overline{\varphi}$ is defined as in (2.8).

Proof: In fact, we figure out that

$$\begin{aligned}\sum_{i=1}^{\infty}2^i\varphi(2^{-i}x_1,\cdots,2^{-i}x_n) &\leq \sum_{i=1}^{\infty}2^i(\frac{L_2}{2})^i\varphi(x_1,\cdots,x_n)\\ &= \frac{L_2}{1-L_2}\varphi(x_1,\cdots,x_n)<\infty,\end{aligned}$$

for all $x_1,\cdots,x_n \in X$, and thus we arrive at the desired results. □

As a corollary of Theorem 2.3 and Theorem 2.5, we obtain the following stability result of the inequality (1.1), which generalizes stability result of equation in complete normed spaces.

Corollary 2.7. *Assume that a mapping $f : X \to Y$ with $f(0)=0$ satisfies the functional inequality*

$$\Big\|\sum_{j=1}^{n}\Big[\Big(\sum_{k=j}^{n}\lambda_k\Big)f(x_j-x_{j-1})\Big]\Big\| \leq \Big\|f\Big(\sum_{j=1}^{n}\lambda_j x_j\Big)\Big\|+\delta$$

for all $x_1,\cdots,x_n \in X$ and for some constant $\delta \geq 0$. Then, there exists a unique Euler–Lagrange additive mapping $A_1 : X \to Y$ defined by $A_1(x)=\lim_{m\to\infty}\frac{f(2^m x)}{2^m}$ such that the mapping A_1 satisfies the approximation

$$\|f(x)-A_1(x)\| \leq \frac{3\delta}{\overline{\lambda_3}}$$

for all $x \in X$.

3 Stability of Ineq. (1.1) in non-Archimedean Banach spaces.

In this section, let X be a linear space and Y a non-Archimedean complete normed space. Now, we will investigate the generalized the Hyers–Ulam stability problem for the functional inequality (1.1) in a non-Archimedean Banach space Y.

Theorem 3.1. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality*

$$\Big\| \sum_{j=1}^{n} \Big[\Big(\sum_{k=j}^{n} \lambda_k \Big) f(x_j - x_{j-1}) \Big] \Big\|_v \le \Big\| f\Big(\sum_{j=1}^{n} \lambda_j x_j \Big) \Big\|_v + \varphi(x_1, \cdots, x_n) \tag{3.1}$$

and that the perturbing function $\varphi : X^n \to [0, \infty)$ satisfies

$$\lim_{m\to\infty} \frac{1}{|2|_v^m} \varphi(2^m x_1, \cdots, 2^m x_n) = 0 \tag{3.2}$$

for all vectors $x_1, \cdots, x_n \in X$, and

$$\widetilde{\Phi_1}(x, x) = \lim_{m\to\infty} \max\Big\{ \frac{1}{|2|_v^i} \widetilde{\varphi}(2^i x, 2^i x) \,:\, 0 \le i < m \Big\} < \infty \tag{3.3}$$

converges for all $x \in X$, where $\widetilde{\varphi}$ is defined as in (3.6). Then there exists an Euler–Lagrange additive mapping $T_1 : X \to Y$ defined by $T_1(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}$ such that

$$\|f(x) - T_1(x)\|_v \le \frac{1}{|2|_v} \widetilde{\Phi_1}(x, x) \tag{3.4}$$

for all $x \in X$. Moreover, if

$$\lim_{l\to\infty} \lim_{m\to\infty} \max\Big\{ \frac{1}{|2|_v^i} \widetilde{\varphi}(2^i x, 2^i x) \,:\, l \le i < l + m \Big\} = 0 \tag{3.5}$$

for all $x \in X$, then the additive mapping T_1 is uniquely determined with approximation (3.4).

Proof: Applying the inequalities (2.4) through (2.7) to non-Archimedean Banach space Y, we have

$$\|f(x) + f(y) - f(x + y)\|_v \le \widetilde{\varphi}(x, y) \tag{3.6}$$

for all $x, y \in X$, where

$$\begin{aligned}\widetilde{\varphi}(x,y) \quad := \quad & \frac{1}{|\lambda_3|_v} \max\Big\{\varphi\Big(-\frac{x}{\lambda_1}, -\frac{x}{\lambda_1}, (1-\frac{1}{\lambda_1})x, \cdots, (1-\frac{1}{\lambda_1})x\Big) \\ & , \varphi\Big(0,\ -\frac{y}{\lambda_2},\ (1-\frac{1}{\lambda_2})y, \cdots,\ (1-\frac{1}{\lambda_2})y\Big) \\ & , \varphi\Big(-\frac{x}{\lambda_1}, -\frac{x}{\lambda_1}-\frac{y}{\lambda_2},\ (1-\frac{1}{\lambda_1})x+(1-\frac{1}{\lambda_2})y, \\ & \cdots,\ (1-\frac{1}{\lambda_1})x+(1-\frac{1}{\lambda_2})y\Big)\Big\}\end{aligned}$$

for all $x, y \in X$. It follows from (3.6) that

$$\Big\|f(x) - \frac{f(2^m x)}{2^m}\Big\|_v \le \frac{1}{|2|_v} \max_{0\le i<m} \Big\{\frac{\widetilde{\varphi}(2^i x, 2^i x)}{|2|_v^i}\Big\} \tag{3.7}$$

for all positive integers $m \in \mathbb{N}$ and all $x \in X$. Then it follows from (3.2) and (3.3) that a sequence $\{\frac{f(2^m x)}{2^m}\}_{m=1}^{\infty}$ is Cauchy for all $x \in X$, and therefore, we may define a mapping $T_1 : X \to Y$ by

$$T_1(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^m}, \quad x \in X.$$

By approaching m to infinity in (3.7) and using (3.3), one leads to the approximation (3.4). By (3.1) and (3.2), we obtain

$$\begin{aligned}\Big\|\sum_{j=1}^{n}\Big[\Big(\sum_{k=j}^{n}\lambda_k\Big)\frac{f(2^m(x_j - x_{j-1}))}{2^m}\Big]\Big\|_v \quad \le \quad & \Big\|\frac{1}{2^m} f\Big(2^m \sum_{j=1}^{n}\lambda_j x_j\Big)\Big\|_v \\ + \quad & \frac{1}{|2|_v^m}\varphi(2^m x_1, \cdots, 2^m x_n),\end{aligned}$$

for all $x_1, \cdots, x_n \in X$, of which the last term tends to zero as $m \to \infty$. Thus, the mapping T_1 satisfies the functional inequality (1.1) and so it is additive.

To prove the uniqueness, we assume that $T' : X \to Y$ is an additive mapping satisfying the approximation (3.4). Then, we figure out that for any $x \in X$ and all natural numbers l,

$$\begin{aligned}\|T_1(x) - T'(x)\|_v &= \frac{1}{|2|_v^l}\|T_1(2^l x) - T'(2^l x)\|_v \\ &\le \frac{1}{|2|_v^l}\max\{\|T_1(2^l x) - f(2^l x)\|_v,\ \|f(2^l x) - T'(2^l x)\|_v\} \\ &\le \frac{1}{|2|_v}\lim_{m\to\infty}\max_{l\le i<l+m}\Big\{\frac{1}{|2|_v^i}\widetilde{\varphi}(2^i x, 2^i x)\Big\},\end{aligned}$$

which tends to zero as $l \to \infty$. This completes the proof. □

Now, we investigate the following stability result of the functional inequality (1.1), which is dual and alternative stability theorem in the non-Archimedean complete normed space Y.

Theorem 3.2. *Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional inequality (3.1) and that the perturbing function $\varphi : X^n \to [0,\infty)$ satisfies*

$$\lim_{m\to\infty} |2|_v^m \varphi(2^{-m}x_1, \cdots, 2^{-m}x_n) = 0 \tag{3.8}$$

for all vectors $x_1, \cdots, x_n \in X$, and

$$\widetilde{\Phi_2}(x,x) = \lim_{m\to\infty} \max\left\{ |2|_v^i \widetilde{\varphi}(2^{-i}x, 2^{-i}x) \,:\, 0 < i \le m \right\} < \infty \tag{3.9}$$

converges for all $x \in X$, where $\widetilde{\varphi}$ is defined as in (3.6). Then there exists an Euler–Lagrange additive mapping $T_2 : X \to Y$ defined by $T_2(x) = \lim_{m\to\infty} 2^m f(2^{-m}x)$ such that

$$\|f(x) - T_2(x)\|_v \le \frac{1}{|2|_v}\widetilde{\Phi_2}(x,x) \tag{3.10}$$

for all $x \in X$. Moreover, if

$$\lim_{l\to\infty}\lim_{m\to\infty} \max\left\{ |2|_v^i \widetilde{\varphi}(2^{-i}x, 2^{-i}x) \,:\, l < i \le l+m \right\} = 0 \tag{3.11}$$

for all $x \in X$, then the additive mapping T_2 is uniquely determined with approximation (3.10).

Proof: It follows from (3.6) that

$$\begin{aligned} \left\| 2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m}) \right\|_v &= \left\| \sum_{i=l+1}^{m} 2^{i-1} f(\frac{x}{2^{i-1}}) - 2^i f(\frac{x}{2^i}) \right\|_v \\ &\le \frac{1}{|2|_v} \max_{l+1\le i\le m} \left\{ |2|_v^i \widetilde{\varphi}\big(\frac{x}{2^i}, \frac{x}{2^i}\big) \right\} \end{aligned} \tag{3.12}$$

for all $x \in X$. Then, it follows from (3.9) and (3.12) that a sequence $\{2^m f(\frac{x}{2^m})\}$ is Cauchy for all $x \in X$, and thus, we can define a mapping $T_2 : X \to Y$ as

$$T_2(x) = \lim_{m\to\infty} 2^m f\big(\frac{x}{2^m}\big), \quad x \in X.$$

By taking m to approach infinity in (3.12) together with $l = 0$, one leads to the approximation (3.10).

Applying the similar argument to the corresponding proof of Theorem 3.1, we obtain the desired results. □

As a corollary of Theorem 3.2, we obtain the following stability result of the functional inequality (1.1) in the non-Archimedean Banach space Y equipped with $|2|_v < 1$.

Corollary 3.3. *Assume that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the functional inequality*

$$\Big\| \sum_{j=1}^{n} \Big[\Big(\sum_{k=j}^{n} \lambda_k \Big) f(x_j - x_{j-1}) \Big] \Big\|_v \le \Big\| f\Big(\sum_{j=1}^{n} \lambda_j x_j \Big) \Big\|_v + \delta$$

for all $x_1, \cdots, x_n \in X$, *where* $\delta \ge 0$ *is a fixed constant, and* $|2|_v < 1$. *Then, there exists a unique Euler–Lagrange additive mapping* $T_2 : X \to Y$ *defined by* $T_2(x) = \lim_{m\to\infty} 2^m f(2^{-m}x)$ *such that the mapping* T_2 *satisfies the approximation*

$$\|f(x) - T_2(x)\|_v \le \frac{\delta}{|\overline{\lambda_3}|_v}$$

for all $x \in X$.

Author Contributions: *M. H. Han:* Validation, Investigation, Formal analysis, Writing—Review and Editing. *H. M. Kim:* Investigation, Results analysis, Writing—Review and Editing, Visualization. *J. M. Rassias:* Conceptualization, Formal analysis, Supervision. *E. Son:* Methodology, Validation, Investigation, Writing—Review and Editing. All authors have read and approved the final version of the manuscript for publication.

Acknowledgement: The authors sincerely appreciate the anonymous referees for their detailed review and helpful suggestions that improved the overall quality of this manuscript.

Funding Statement: The author(s) received no specific funding for this study.

Data Availability Statement: Not applicable.

Ethics Approval: Not applicable

Use of Generative-AI tools declaration: The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflicts of Interest: The authors have no competing interests to disclose.

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