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Research article

Extending and unifying Hardy-Hilbert-type integral inequalities involving primitives

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Abstract: In this article, we extend and unify the framework of a modified Hardy-Hilbert-type integral inequality established by W. T. Sulaiman in 2010. Our approach differs from previous works by incorporating the primitives of the main functions, considering four types of denominators for the kernel function, and introducing four adjustable parameters. The proofs are presented in full detail and can be reproduced with only a minimal level of prior knowledge.

Mathematics Subject Classification: 26D15
Keywords: Hardy-Hilbert-type integral inequalities; primitives; double integral inequalities; Hölder integral inequality

1 Introduction

The classical Hardy-Hilbert integral inequality represents a strong foundational result in the theory of integral inequalities. Its formal statement is given below. Let $p > 1$ and q such that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy$$
$$\leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy \right)^{1/q},$$

provided that the integrals exist. This inequality thus provides a sharp upper bound of the double integral in terms of the L^p and L^q norms of f and g, respectively. It serves as a classical benchmark for numerous extensions and generalizations in the study of integral inequalities (see, for instance, the books [3,7]). One line of generalization involves replacing the functions f and g by their primitives within the double integral. Several results of this type have been established in the articles [1,2,4–6]. In particular, the key result [6, Theorem 1] is formally stated below. Let $\alpha, \beta > 0$, $p > 1$ and q such that $1/p + 1/q = 1$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions, and

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

provided that they exist. Then we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/\beta-1}y^{1/\alpha-1}}{|x-y|^{2/\alpha+2/\beta}} F^{1/\alpha+1/p}(x)G^{1/\beta+1/q}(y)dxdy \\
&\leq 2\left(1+\frac{\alpha}{p}\right)^{1/\alpha+1/p}\left(1+\frac{\beta}{q}\right)^{1/\beta+1/q} B^{1/p}\left(\frac{p}{\alpha}, 1-2\frac{p}{\alpha}\right)B^{1/q}\left(\frac{q}{\beta}, 1-2\frac{q}{\beta}\right) \\
&\times \left(\int_0^{+\infty} f^{p/\alpha+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^{q/\beta+1}(y)dy\right)^{1/q},
\end{aligned}$$

provided that the integrals exist, where, for any $u, v > 0$, $B(u, v)$ denotes the standard beta function given by

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt.$$

The originality of this result lies in the incorporation of the primitives F and G into the double integral, which leads to a new class of weighted inequalities involving two adjustable parameters, α and β. We also emphasize the constant factor depending on the beta function. It is not rigorously proved to be optimal, but it is obtained with the minimal number of operations making it particularly sharp.

In this article, we extend the previous framework in two main directions:

First direction: We introduce four adjustable parameters, α, β, γ, and ω, to enhance the flexibility and generality of the inequality.

Second direction: We unify the expressions of the upper bounds corresponding to the four types of denominators, power of $x + y$, $|x - y|$, $1 + xy$ and $|1 - xy|$, into the kernel function. These are compactly written as

$$|x \pm y| = \begin{cases} x + y, & \text{when } \pm = +, \\ |x - y|, & \text{when } \pm = -, \end{cases}$$

and

$$|1 \pm xy| = \begin{cases} 1 + xy, & \text{when } \pm = +, \\ |1 - xy|, & \text{when } \pm = -, \end{cases}$$

by taking into account that $x, y > 0$.

We thus establish a broader class of Hardy-Hilbert-type integral inequalities, which unifies and extends a known result, i.e., [6, Theorem 1], as a special case. This general formulation offers a deeper understanding of the underlying structure of such inequalities by introducing a more flexible system of parameters.

The remainder of this article is organized as follows: Section 2 presents the main contributions in the form of three theorems, along with their proofs. Section 3 concludes the article.

2 Contributions

The first theorem, given below, concerns the case with $|x \pm y|$ in the denominator of the kernel function.

Theorem 1: *Let $\alpha, \beta, \gamma, \omega > 0$, $p > 1$ and q such that $1/p + 1/q = 1$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions, and*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

provided that they exist. Then we have

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{x^{1/\beta-1} y^{1/\alpha-1}}{|x \pm y|^{2/\alpha+2/\beta}} F^{\gamma+1/p}(x) G^{\omega+1/q}(y) dxdy \\
&\leq \alpha^{\gamma+1/p} \left(\gamma + \frac{1}{p}\right)^{\gamma+1/p} \beta^{\omega+1/q} \left(\omega + \frac{1}{q}\right)^{\omega+1/q} \Upsilon_{\pm}^{1/p}\left(\frac{p}{\alpha}\right) \Upsilon_{\pm}^{1/q}\left(\frac{q}{\beta}\right) \\
&\times \left(\int_0^{+\infty} x^{(\gamma-1/\alpha)p} f^{\gamma p+1}(x) dx\right)^{1/p} \left(\int_0^{+\infty} y^{(\omega-1/\beta)q} g^{\omega q+1}(y) dy\right)^{1/q},
\end{aligned}$$

provided that the integrals exist, where, for any $u > 0$,

$$\Upsilon_{\pm}(u) = \int_0^{+\infty} \frac{z^{u-1}}{|1 \pm z|^{2u}} dz = \begin{cases} B(u, u), & \text{when } \pm = +, \\ 2B(u, 1-2u), & \text{when } \pm = - \text{ and } u < \dfrac{1}{2}. \end{cases} \tag{2.1}$$

Proof: A direct decomposition of the integrand using the equality $1/p + 1/q = 1$ and the Hölder integral inequality give

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{x^{1/\beta-1} y^{1/\alpha-1}}{|x \pm y|^{2/\alpha+2/\beta}} F^{\gamma+1/p}(x) G^{\omega+1/q}(y) dxdy \\
&= \int_0^{+\infty} \int_0^{+\infty} \frac{y^{1/\alpha-1/p}}{x^{1/p}|x \pm y|^{2/\alpha}} F^{\gamma+1/p}(x) \times \frac{x^{1/\beta-1/q}}{y^{1/q}|x \pm y|^{2/\beta}} G^{\omega+1/q}(y) dxdy \\
&\leq A^{1/p} B^{1/q},
\end{aligned} \tag{2.2}$$

where

$$A = \int_0^{+\infty} \int_0^{+\infty} \frac{y^{p/\alpha-1}}{x|x \pm y|^{2p/\alpha}} F^{\gamma p+1}(x) dxdy$$

and

$$B=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{q/\beta-1}}{y|x\pm y|^{2q/\beta}}G^{\omega q+1}(y)dxdy.$$

Let us start by determining an upper bound for A. Applying the Fubini-Tonelli integral theorem and making the ratio change of variables $z=y/x$, we get

$$\begin{aligned}A&=\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)\int_0^{+\infty}\frac{(y/x)^{p/\alpha-1}(1/x)}{|1\pm y/x|^{2p/\alpha}}dydx\\&=\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)\int_0^{+\infty}\frac{z^{p/\alpha-1}}{|1\pm z|^{2p/\alpha}}dzdx\\&=\Upsilon_{\pm}\left(\frac{p}{\alpha}\right)\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)dx.\end{aligned}$$

We now use a general result in [3]. Let $r,s>1$. Then the generalized version of the Hardy integral inequality described in [3, Theorem 330] ensures that

$$\int_0^{+\infty}x^{-r}F^s(x)dx\leq\left(\frac{s}{r-1}\right)^s\int_0^{+\infty}x^{s-r}f^s(x)dx. \tag{2.3}$$

Applying this to $r=p/\alpha+1$ and $s=\gamma p+1$, after some simplifications, we obtain

$$\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)dx\leq\alpha^{\gamma p+1}\left(\gamma+\frac{1}{p}\right)^{\gamma p+1}\int_0^{+\infty}x^{(\gamma-1/\alpha)p}f^{\gamma p+1}(x)dx,$$

which yields

$$A\leq\Upsilon_{\pm}\left(\frac{p}{\alpha}\right)\alpha^{\gamma p+1}\left(\gamma+\frac{1}{p}\right)^{\gamma p+1}\int_0^{+\infty}x^{(\gamma-1/\alpha)p}f^{\gamma p+1}(x)dx. \tag{2.4}$$

For an upper bound for B, we proceed similarly. Applying the Fubini-Tonelli integral theorem and making the ratio change of variables $z=x/y$, we get

$$\begin{aligned}B&=\int_0^{+\infty}y^{-q/\beta-1}G^{\omega q+1}(y)\int_0^{+\infty}\frac{(x/y)^{q/\beta-1}(1/y)}{|1\pm x/y|^{2q/\beta}}dxdy\\&=\int_0^{+\infty}y^{-q/\beta-1}G^{\omega q+1}(y)\int_0^{+\infty}\frac{z^{q/\beta-1}}{|1\pm z|^{2q/\beta}}dzdy\\&=\Upsilon_{\pm}\left(\frac{q}{\beta}\right)\int_0^{+\infty}y^{-q/\beta-1}G^{\omega q+1}(y)dy.\end{aligned}$$

Applying Equation (2.3) to $r=q/\beta+1$ and $s=\omega q+1$, after some simplifications, we obtain

$$B\leq\Upsilon_{\pm}\left(\frac{q}{\beta}\right)\beta^{\omega q+1}\left(\omega+\frac{1}{q}\right)^{\omega q+1}\int_0^{+\infty}y^{(\omega-1/\beta)q}g^{\omega q+1}(y)dy. \tag{2.5}$$

Combining Equations (2.2), (2.4), and (2.5), we derive

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|x\pm y|^{2/\alpha+2/\beta}}F^{\gamma+1/p}(x)G^{\omega+1/q}(y)dxdy$$
$$\leq \alpha^{\gamma+1/p}\left(\gamma+\frac{1}{p}\right)^{\gamma+1/p}\beta^{\omega+1/q}\left(\omega+\frac{1}{q}\right)^{\omega+1/q}\Upsilon_{\pm}^{1/p}\left(\frac{p}{\alpha}\right)\Upsilon_{\pm}^{1/q}\left(\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(\gamma-1/\alpha)p}f^{\gamma p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(\omega-1/\beta)q}g^{\omega q+1}(y)dy\right)^{1/q}.$$

For the expression of $\Upsilon_{\pm}(u)$ for $u>0$, when $\pm=+$, using a classical integral definition of the beta function on $(0,+\infty)$, we have

$$\Upsilon_{\pm}(u)=\int_0^{+\infty}\frac{z^{u-1}}{(1+z)^{u+u}}dz=B(u,u).$$

When $\pm=-$, using [6, Lemma 1], we can conclude that

$$\Upsilon_{\pm}(u)=\int_0^{+\infty}\frac{z^{u-1}}{|1-z|^{2u}}dz=2B(u,1-2u),$$

provided that $u<1/2$. This completes the proof. □

When $\pm=-$, and $\gamma=1/\alpha$ and $\omega=1/\beta$, this theorem reduces to [6, Theorem 1]. In this case, note that we have the constraint $p<\alpha/2$ and $q<\beta/2$, which also appears in [6, Theorem 1]. The remaining cases yield new Hardy-Hilbert-type integral inequalities. In particular, by setting $\gamma=1/q$ and $\omega=1/p$, using the equality $1/p+1/q=1$, we obtain the following elegant new inequality:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|x\pm y|^{2/\alpha+2/\beta}}F(x)G(y)dxdy$$
$$\leq \alpha\beta\Upsilon_{\pm}^{1/p}\left(\frac{p}{\alpha}\right)\Upsilon_{\pm}^{1/q}\left(\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(1/q-1/\alpha)p}f^{p}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1/p-1/\beta)q}g^{q}(y)dy\right)^{1/q},$$

i.e., more explicitly,

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{(x+y)^{2/\alpha+2/\beta}}F(x)G(y)dxdy$$
$$\leq \alpha\beta B^{1/p}\left(\frac{p}{\alpha},\frac{p}{\alpha}\right)B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(1/q-1/\alpha)p}f^{p}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1/p-1/\beta)q}g^{q}(y)dy\right)^{1/q}$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|x-y|^{2/\alpha+2/\beta}}F(x)G(y)dxdy$$
$$\leq 2\alpha\beta B^{1/p}\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B^{1/q}\left(\frac{q}{\beta},1-2\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(1/q-1/\alpha)p}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1/p-1/\beta)q}g^q(y)dy\right)^{1/q}.$$

The second theorem, given below, concerns the case with $|1\pm xy|$ in the denominator of the kernel function.

Theorem 2: *Let $\alpha,\beta,\gamma,\omega>0$, $p>1$ and q such that $1/p+1/q=1$, $f,g:(0,+\infty)\to(0,+\infty)$ be two functions, and*

$$F(x)=\int_0^x f(t)dt,\quad G(y)=\int_0^y g(t)dt,$$

provided that they exist. Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|1\pm xy|^{2/\alpha+2/\beta}}F^{\gamma+1/p}(x)G^{\omega+1/q}(y)dxdy$$
$$\leq \alpha^{\gamma+1/p}\left(\gamma+\frac{1}{p}\right)^{\gamma+1/p}\beta^{\omega+1/q}\left(\omega+\frac{1}{q}\right)^{\omega+1/q}\Upsilon_{\pm}^{1/p}\left(\frac{p}{\alpha}\right)\Upsilon_{\pm}^{1/q}\left(\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(\gamma-1/\alpha)p}f^{\gamma p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(\omega-1/\beta)q}g^{\omega q+1}(y)dy\right)^{1/q},$$

provided that the integrals exist, where, for any $u>0$, $\Upsilon_{\pm}(u)$ is given by Equation (2.1).

Proof: We follow the lines of the proof of Theorem 1. The two proofs differ on a few but crucial technical points, leading to the same upper bound. A simple decomposition of the integrand using the equality $1/p+1/q=1$ and the Hölder integral inequality give

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|1\pm xy|^{2/\alpha+2/\beta}}F^{\gamma+1/p}(x)G^{\omega+1/q}(y)dxdy$$
$$=\int_0^{+\infty}\int_0^{+\infty}\frac{y^{1/\alpha-1/p}}{x^{1/p}|1\pm xy|^{2/\alpha}}F^{\gamma+1/p}(x)\times\frac{x^{1/\beta-1/q}}{y^{1/q}|1\pm xy|^{2/\beta}}G^{\omega+1/q}(y)dxdy$$
$$\leq C^{1/p}D^{1/q},\tag{2.6}$$

where

$$C=\int_0^{+\infty}\int_0^{+\infty}\frac{y^{p/\alpha-1}}{x|1\pm xy|^{2p/\alpha}}F^{\gamma p+1}(x)dxdy$$

and

$$D=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{q/\beta-1}}{y|1\pm xy|^{2q/\beta}}G^{\omega q+1}(y)dxdy.$$

Let us start by determining an upper bound for C. Applying the Fubini-Tonelli integral theorem and making the product change of variables $z=xy$, we get

$$\begin{aligned}C&=\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)\int_0^{+\infty}\frac{(xy)^{p/\alpha-1}x}{|1\pm xy|^{2p/\alpha}}dydx\\&=\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)\int_0^{+\infty}\frac{z^{p/\alpha-1}}{|1\pm z|^{2p/\alpha}}dzdx\\&=\Upsilon_\pm\left(\frac{p}{\alpha}\right)\int_0^{+\infty}x^{-p/\alpha-1}F^{\gamma p+1}(x)dx.\end{aligned}$$

Applying Equation (2.3) to $r=p/\alpha+1$ and $s=\gamma p+1$, after some simplifications, we get

$$C\leq\Upsilon_\pm\left(\frac{p}{\alpha}\right)\alpha^{\gamma p+1}\left(\gamma+\frac{1}{p}\right)^{\gamma p+1}\int_0^{+\infty}x^{(\gamma-1/\alpha)p}f^{\gamma p+1}(x)dx. \tag{2.7}$$

For an upper bound for D, we proceed similarly. Applying the Fubini-Tonelli integral theorem and making the product change of variables $z=xy$, we get

$$\begin{aligned}D&=\int_0^{+\infty}y^{-q/\beta-1}G^{\omega q+1}(y)\int_0^{+\infty}\frac{(xy)^{q/\beta-1}y}{|1\pm xy|^{2q/\beta}}dxdy\\&=\int_0^{+\infty}y^{-q/\beta-1}G^{\omega q+1}(y)\int_0^{+\infty}\frac{z^{q/\beta-1}}{|1\pm z|^{2q/\beta}}dzdy\\&=\Upsilon_\pm\left(\frac{q}{\beta}\right)\int_0^{+\infty}y^{-q/\beta-1}G^{\omega q+1}(y)dy.\end{aligned}$$

Applying Equation (2.3) to $r=q/\beta+1$ and $s=\omega q+1$, after some simplifications, we derive

$$D\leq\Upsilon_\pm\left(\frac{q}{\beta}\right)\beta^{\omega q+1}\left(\omega+\frac{1}{q}\right)^{\omega q+1}\int_0^{+\infty}y^{(\omega-1/\beta)q}g^{\omega q+1}(y)dy. \tag{2.8}$$

Combining Equations (2.6), (2.7), and (2.8), we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|1\pm xy|^{2/\alpha+2/\beta}}F^{\gamma+1/p}(x)G^{\omega+1/q}(y)dxdy\\&\leq\alpha^{\gamma+1/p}\left(\gamma+\frac{1}{p}\right)^{\gamma+1/p}\beta^{\omega+1/q}\left(\omega+\frac{1}{q}\right)^{\omega+1/q}\Upsilon_\pm^{1/p}\left(\frac{p}{\alpha}\right)\Upsilon_\pm^{1/q}\left(\frac{q}{\beta}\right)\\&\times\left(\int_0^{+\infty}x^{(\gamma-1/\alpha)p}f^{\gamma p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(\omega-1/\beta)q}g^{\omega q+1}(y)dy\right)^{1/q}.\end{aligned}$$

This completes the proof. □

To the best of our knowledge, this result is new and has no known special cases. In particular, by setting $\gamma = 1/q$ and $\omega = 1/p$, we obtain the following elegant new inequality:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|1\pm xy|^{2/\alpha+2/\beta}}F(x)G(y)dxdy$$
$$\leq \alpha\beta\Upsilon_{\pm}^{1/p}\left(\frac{p}{\alpha}\right)\Upsilon_{\pm}^{1/q}\left(\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(1/q-1/\alpha)p}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1/p-1/\beta)q}g^q(y)dy\right)^{1/q},$$

i.e., more explicitly,

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{(1+xy)^{2/\alpha+2/\beta}}F(x)G(y)dxdy$$
$$\leq \alpha\beta B^{1/p}\left(\frac{p}{\alpha},\frac{p}{\alpha}\right)B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(1/q-1/\alpha)p}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1/p-1/\beta)q}g^q(y)dy\right)^{1/q}$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{x^{1/\beta-1}y^{1/\alpha-1}}{|1-xy|^{2/\alpha+2/\beta}}F(x)G(y)dxdy$$
$$\leq 2\alpha\beta B^{1/p}\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B^{1/q}\left(\frac{q}{\beta},1-2\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty}x^{(1/q-1/\alpha)p}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1/p-1/\beta)q}g^q(y)dy\right)^{1/q}.$$

The third theorem, given below, concerns the case with $|x \pm y|$ or $|1 \pm xy|$ in the denominator of the kernel function.

Theorem 3: *Let $\alpha, \beta, \gamma, \omega > 0$, $p > 1$ and q such that $1/p + 1/q = 1$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions, and*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt,$$

provided that they exist. Let $\phi_{\pm}(x,y)$ *be either* $|x \pm y|$ *or* $|1 \pm xy|$*. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/\beta-1}y^{1/\alpha-1}}{\phi_{\pm}^{2/\alpha+2/\beta}(x,y)} F^{\gamma+1/p}(x)G^{\omega+1/q}(y)dxdy$$
$$\leq \alpha^{\gamma+1/p}\left(\gamma+\frac{1}{p}\right)^{\gamma+1/p}\beta^{\omega+1/q}\left(\omega+\frac{1}{q}\right)^{\omega+1/q}\Upsilon_{\pm}^{1/p}\left(\frac{p}{\alpha}\right)\Upsilon_{\pm}^{1/q}\left(\frac{q}{\beta}\right)$$
$$\times\left(\int_0^{+\infty} x^{(\gamma-1/\alpha)p} f^{\gamma p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty} y^{(\omega-1/\beta)q} g^{\omega q+1}(y)dy\right)^{1/q},$$

provided that the integrals exist, where, for any $u > 0$, $\Upsilon_{\pm}(u)$ *is given by Equation* (2.1).

Proof: This follows directly by combining the statements of Theorems 1 and 2 via the definition of $\phi_{\pm}(x,y)$. □

To the best of our knowledge, this theorem is among the few results that provide a unified upper bound for four distinct kernel functions in the context of Hardy-Hilbert-type integral inequalities.

3 Conclusion

In this article, we present a unified and extended framework for Hardy-Hilbert-type integral inequalities, introducing four adjustable parameters and considering various denominator forms. This generalizes a well-known result from [5] as a particular case and provides a deeper structural understanding of such inequalities. Future research could explore discrete analogues and multidimensional extensions, as well as applications to operator theory and functional analysis.

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