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Research article

Examining new integral inequalities with applications to the sine integral

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France.

*Corresponding Author: Christophe Chesneau. Email: christophe.chesneau@gmail.com
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Abstract: This paper studies integral inequalities involving a function and its derivative, with the aim of establishing sharp lower bounds under general assumptions. The analysis uses elementary techniques to produce clear and transparent results. Several examples demonstrate the effectiveness of the inequalities, paying particular attention to applications involving the sine integral.

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1 Introduction

Integral inequalities form the basis of mathematical analysis. They are essential for deriving lower or upper bounds in a wide range of functional and differential problems. In particular, integral inequalities involving a function and its derivative often arise in approximation theory, convex analysis and the theory of special functions. See, for example, the books [1, 2, 8, 10, 19, 20]. Recent advances in the study of integral inequalities can be found in [3–7, 9, 11–18].

This paper is devoted to the study of a particular integral of the form

$$\int_0^x \frac{f(t)f'(t)}{t}dt, \tag{1.1}$$

where $f : [0, \infty) \to \mathbb{R}$ is a differentiable function. Integrals of this type naturally arise in the analysis of function growth properties, energy-type estimates for differential equations, and investigations into moment or transform inequalities. The primary objective of this paper is to establish several precise lower bounds for the integral in Equation (1.1), based on mild and general assumptions on the function f. Our approach uses elementary analytical arguments to avoid advanced techniques while maintaining simplicity and transparency. Furthermore, we provide several illustrative examples to demonstrate the effectiveness and applicability of the derived results. In particular, we present a class of sharp integral inequalities involving the sine integral defined by

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt, \tag{1.2}$$

for any $x \geq 0$.

The remainder of the paper is organized as follows: In Section 2, we present an initial approach that yields a precise inequality for the integral in Equation (1.1). Section 3 presents a more refined method that leads to a different lower bound. Section 4 contains concluding remarks and possible directions for future research.

2 Initial approach

2.1 Result

The initial approach involves determining a lower bound for the integral in Equation (1.1), assuming that f is non-negative and increasing. The details are provided in the theorem below.

Theorem 2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a non-negative, differentiable and increasing function such that $f(0) = 0$. Then, for any $x > 0$, we have*

$$\int_0^x \frac{f(t) f'(t)}{t} dt \geq \frac{[f(x)]^2}{2x}.$$

Proof: Since f is a differentiable increasing function, we have $f'(t) \geq 0$ for any $t \geq 0$. Using this, the fact that f is non-negative, a standard composed-power primitive and $f(0) = 0$, we obtain

$$\int_0^x \frac{f(t) f'(t)}{t} dt \geq \frac{1}{x} \int_0^x f(t) f'(t) dt = \frac{1}{x} \left[\frac{1}{2} [f(t)]^2 \right]_0^x$$
$$= \frac{1}{x} \left(\frac{1}{2} [f(x)]^2 - \frac{1}{2} [f(0)]^2 \right) = \frac{[f(x)]^2}{2x}.$$

This completes the proof. □

In Theorem 2.1, the function f directly determines the lower bound. However, this bound is obtained using the basic inequality $1/t \geq 1/x$ for any $t \in [0, x]$, which, under the restrictive assumptions imposed on f, can be considered a suboptimal approach. A more refined method will be developed in Section 3.

Before proceeding, the subsection below presents several illustrative examples of Theorem 2.1, considering specific functions f.

2.2 Examples

Several examples of Theorem 2.1 are given below. All the considered functions f satisfy the required assumptions of this theorem.

1. Setting $f(x) = \log(1+x)$, for any $x > 0$, we obtain

$$\int_0^x \frac{\log(1+t)}{t(1+t)} dt \geq \frac{[\log(1+x)]^2}{2x}.$$

2. Setting $f(x) = e^x - 1$, for any $x > 0$, we get

$$\int_0^x \frac{(e^t-1)e^t}{t} dt \geq \frac{(e^x-1)^2}{2x}.$$

3. Setting $f(x) = \arctan(x)$, for any $x > 0$, we obtain

$$\int_0^x \frac{\arctan(t)}{t(1+t^2)} dt \geq \frac{[\arctan(x)]^2}{2x}.$$

4. Setting $f(x) = \sin(x)$, for any $x \in (0, \pi/2]$, we get

$$\int_0^x \frac{\sin(t)\cos(t)}{t} dt \geq \frac{[\sin(x)]^2}{2x}.$$

Using the trigonometric formula $\sin(2z) = 2\sin(z)\cos(z)$ for any $z \in \mathbb{R}$, we have

$$\int_0^x \frac{\sin(2t)}{2t} dt \geq \frac{[\sin(x)]^2}{2x}.$$

By the change of variables $y = 2t$, we obtain

$$\int_0^{2x} \frac{\sin(y)}{y} dy \geq \frac{[\sin(x)]^2}{x}.$$

Using the sine integral as given in Equation (1.2), we can write

$$\mathrm{Si}(2x) \geq \frac{[\sin(x)]^2}{x}.$$

To the best of our knowledge, this is a new sine integral inequality in the literature. In order to visualize this inequality, let us introduce the difference function

$$h(x) = \mathrm{Si}(2x) - \frac{[\sin(x)]^2}{x}. \tag{2.1}$$

Figure 1 plots $h(x)$ for $x \in (0, \pi/2]$.

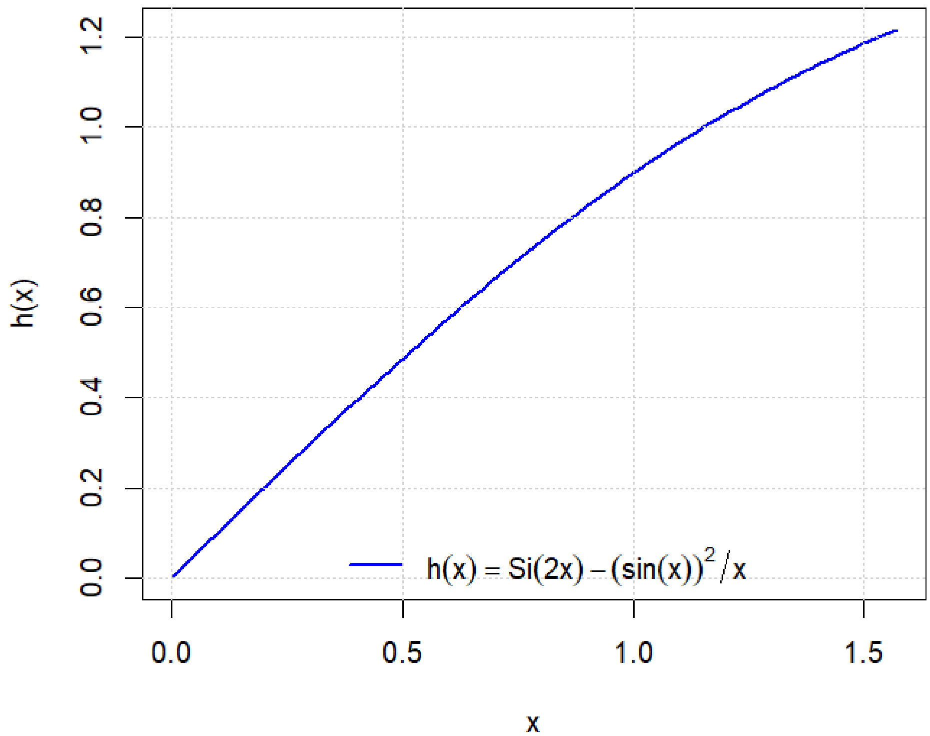

Figure 1: Plot of $h(x)$ in Equation (2.1) for $x \in (0, \pi/2]$.

We see that $h(x) \geq 0$ for any $x \in (0, \pi/2]$. The obtained lower bound is thus clearly illustrated.

5. Setting $f(x) = \tan(x)$, for any $x \in (0, \pi/2]$, we get

$$\int_0^x \frac{\tan(t)}{t[\cos(t)]^2} dt \geq \frac{[\tan(x)]^2}{2x}.$$

6. Setting $f(x) = \sinh(x)$, for any $x > 0$, we obtain

$$\int_0^x \frac{\sinh(t)\cosh(t)}{t} dt \geq \frac{[\sinh(x)]^2}{2x}.$$

Using the hyperbolic formula $\sinh(2z) = 2\sinh(z)\cosh(z)$ for any $z \in \mathbb{R}$, this inequality becomes

$$\int_0^x \frac{\sinh(2t)}{2t} dt \geq \frac{[\sinh(x)]^2}{2x}.$$

By the change of variables $y = 2t$, we get

$$\int_0^{2x} \frac{\sinh(y)}{y} dy \geq \frac{[\sinh(x)]^2}{x}.$$

By introducing the hyperbolic sine integral as

$$\mathrm{Sih}(x) = \int_0^x \frac{\sinh(t)}{t} dt,$$

we have

$$\mathrm{Sih}(2x) \geq \frac{[\sinh(x)]^2}{x}.$$

Further examples may be presented, particularly those incorporating special functions.

3 Second approach

3.1 Results

Our second approach aims to provide an alternative lower bound based on different assumptions to those of Theorem 2.1. We begin with the theorem below, which establishes an interesting integral lower bound that will be used as a target in a subsequent result.

Theorem 3.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable convex function such that $f(0) = 0$. Then, for any $x > 0$, we have*

$$\int_0^x \frac{f(t)f'(t)}{t} dt \geq \frac{f(x)}{x} \int_0^x \frac{f(t)}{t} dt.$$

Proof: Since f is a twice differentiable convex function with $f(0) = 0$, then f' and $f(t)/t$ are non-decreasing. It follows from the Chebyshev integral inequality applied to these two functions on the interval $(0, x)$ that

$$\int_0^x \frac{f(t)f'(t)}{t} dt \geq \frac{1}{x-0}\left(\int_0^x f'(t)dt\right)\left(\int_0^x \frac{f(t)}{t} dt\right)$$
$$= \frac{1}{x}(f(x) - f(0)) \int_0^x \frac{f(t)}{t} dt = \frac{f(x)}{x} \int_0^x \frac{f(t)}{t} dt.$$

This completes the proof. □

Theorem 3.1 establishes a specific integral inequality involving a function f, its derivative f', and associated primitives. However, this result relies on the function being convex. This naturally raises the question of whether this assumption is essential or whether it can be relaxed. The following theorem addresses this question. Furthermore, it provides a comprehensive proof that does not rely on any intermediary known results.

Theorem 3.2. *Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function with $f(0) = 0$ and such that $f'(0)$ exists. Then, for any $x > 0$, we have*

$$\int_0^x \frac{f(t)f'(t)}{t}dt \geq \frac{f(x)}{x}\int_0^x \frac{f(t)}{t}dt.$$

Proof: Let us examine the left-hand side term. An integration by parts combined with the limit result $\lim_{t\to 0}[f(t)]^2/t = f(0)f'(0) = 0$ (since $f(0) = 0$ and $f'(0)$ exists) gives

$$\int_0^x \frac{f(t)f'(t)}{t}dt = \left[\frac{1}{2}[f(t)]^2\frac{1}{t}\right]_0^x - \int_0^x \frac{1}{2}[f(t)]^2\left(-\frac{1}{t^2}\right)dt$$
$$= \left(\frac{1}{2}\frac{[f(x)]^2}{x} - \frac{1}{2}\lim_{t\to 0}\frac{[f(t)]^2}{t}\right) + \frac{1}{2}\int_0^x \frac{[f(t)]^2}{t^2}dt$$
$$= \frac{1}{2}\left(\frac{[f(x)]^2}{x} + \int_0^x \frac{[f(t)]^2}{t^2}dt\right).$$

Therefore, the desired inequality can be reformulated as

$$\frac{1}{2}\left(\frac{[f(x)]^2}{x} + \int_0^x \frac{[f(t)]^2}{t^2}dt\right) \geq \frac{f(x)}{x}\int_0^x \frac{f(t)}{t}dt,$$

so that

$$\frac{[f(x)]^2}{x} + \int_0^x \frac{[f(t)]^2}{t^2}dt - 2\frac{f(x)}{x}\int_0^x \frac{f(t)}{t}dt \geq 0.$$

This is equivalent to

$$\int_0^x \left(\frac{[f(x)]^2}{x^2} + \frac{[f(t)]^2}{t^2} - 2\frac{f(x)}{x}\frac{f(t)}{t}\right)dt \geq 0,$$

which can be rewritten as

$$\int_0^x \left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)^2 dt \geq 0.$$

This inequality is always true, completing the proof. □

The convexity assumption that is essential for Theorem 3.1 is completely relaxed in Theorem 3.2. Moreover, the proof of Theorem 3.2 relies solely on basic integral techniques and manipulations, eliminating the need for well-known inequalities, such as the Chebyshev integral inequality, which were employed in the proof of Theorem 3.1.

The subsection below illustrates Theorem 3.2 with several examples, focusing particularly on the sine integral.

3.2 Examples

Several examples illustrating Theorem 3.2 are presented below, each yielding a new integral inequality. All the considered functions f satisfy the required assumptions of this theorem.

1. Setting $f(x) = \log(1 + x)$, for any $x > 0$, we get

$$\int_0^x \frac{\log(1+t)}{t(1+t)} dt \geq \frac{\log(1+x)}{x} \int_0^x \frac{\log(1+t)}{t} dt.$$

2. Setting $f(x) = e^x - 1$, for any $x > 0$, we obtain

$$\int_0^x \frac{(e^t - 1)e^t}{t} dt \geq \frac{e^x - 1}{x} \int_0^x \frac{e^t - 1}{t} dt.$$

3. Setting $f(x) = \arctan(x)$, for any $x > 0$, we get

$$\int_0^x \frac{\arctan(t)}{t(1+t^2)} dt \geq \frac{\arctan(x)}{x} \int_0^x \frac{\arctan(t)}{t} dt.$$

4. Setting $f(x) = \sin(x)$, for any $x > 0$, we obtain

$$\int_0^x \frac{\sin(t)\cos(t)}{t} dt \geq \frac{\sin(x)}{x} \int_0^x \frac{\sin(t)}{t} dt.$$

By a standard trigonometric inequality, we have

$$\int_0^x \frac{\sin(2t)}{2t} dt \geq \frac{\sin(x)}{x} \int_0^x \frac{\sin(t)}{t} dt,$$

and, by the change of variables $y = 2t$,

$$\int_0^{2x} \frac{\sin(y)}{y} dy \geq 2\frac{\sin(x)}{x} \int_0^x \frac{\sin(t)}{t} dt.$$

Therefore, the sine integral satisfies

$$\mathrm{Si}(2x) \geq 2\frac{\sin(x)}{x} \mathrm{Si}(x).$$

Let us illustrate this inequality by introducing the function

$$k(x) = \mathrm{Si}(2x) - 2\frac{\sin(x)}{x}\mathrm{Si}(x). \tag{3.1}$$

Figures 2 and 3 plot $k(x)$ for any $x \in (0, 1]$ and $x \in (0, 30]$, respectively.

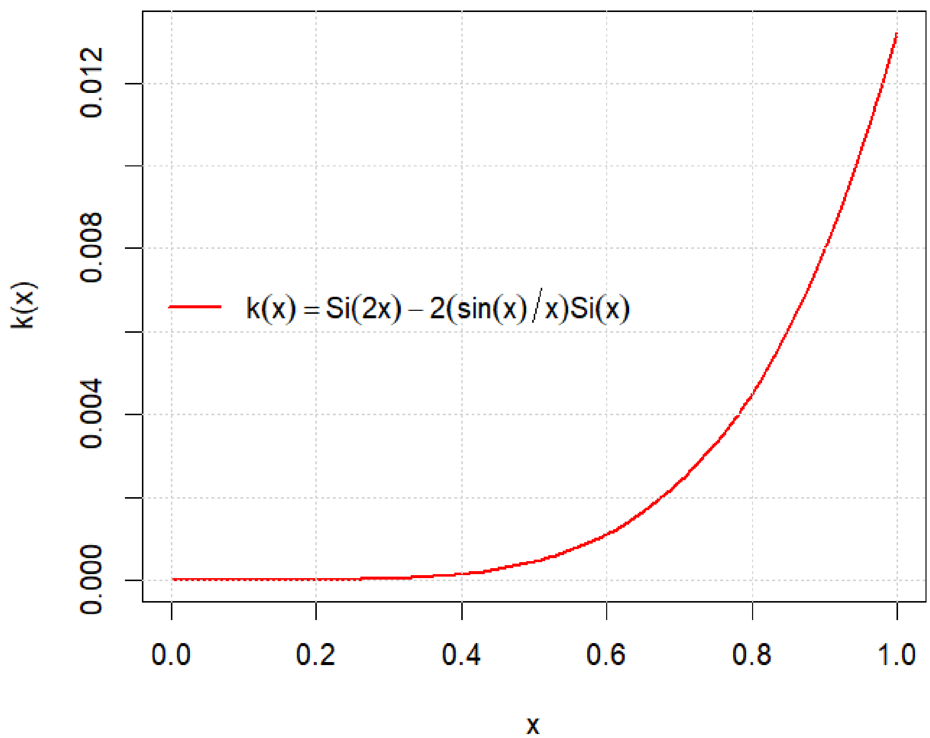

Figure 2: Plot of $k(x)$ in Equation (3.1) for $x \in (0, 1]$.

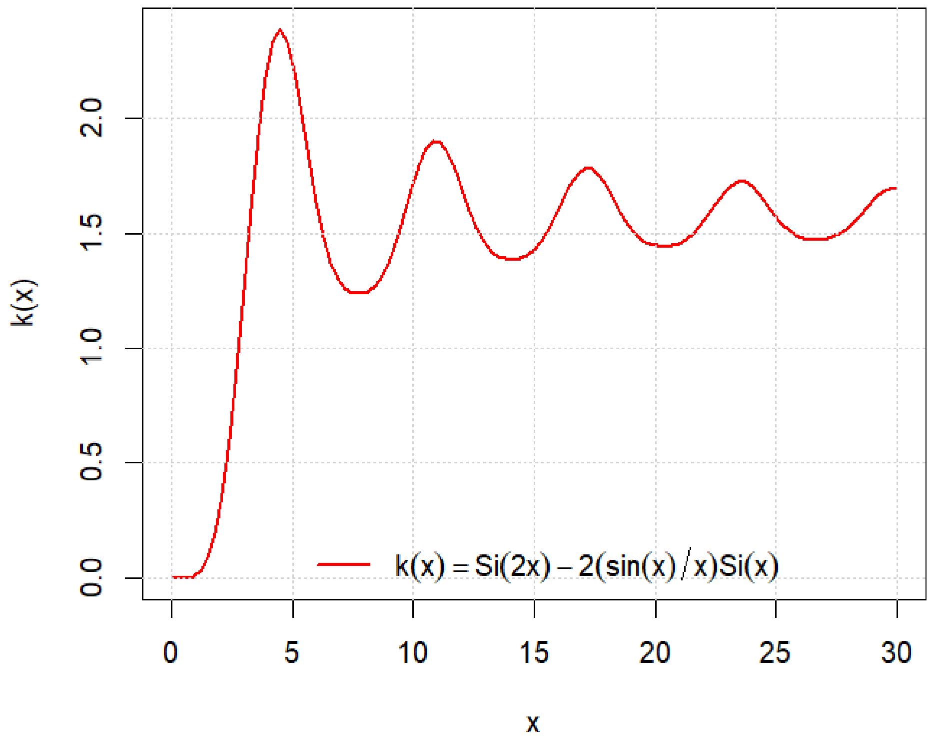

Figure 3: Plot of $k(x)$ in Equation (3.1) for $x \in (0, 30]$.

From Figure 2, we see that the inequality is very sharp near $x = 0$. Figure 3 shows that it is valid for large values of x, as theoretically demonstrated.

5. Setting $f(x) = \tan(x)$, for any $x > 0$, we get

$$\int_0^x \frac{\tan(t)}{t[\cos(t)]^2} dt \geq \frac{\tan(x)}{x} \int_0^x \frac{\tan(t)}{t} dt.$$

6. Setting $f(x) = \sinh(x)$, for any $x > 0$, we obtain

$$\int_0^x \frac{\sinh(t)\cosh(t)}{t} dt \geq \frac{\sinh(x)}{x} \int_0^x \frac{\sinh(t)}{t} dt.$$

A basic hyperbolic formula gives

$$\int_0^x \frac{\sinh(2t)}{2t} dt \geq \frac{\sinh(x)}{x} \int_0^x \frac{\sinh(t)}{t} dt,$$

and, by the change of variables $y = 2t$,

$$\int_0^{2x} \frac{\sinh(y)}{y} dy \geq 2\frac{\sinh(x)}{x} \int_0^x \frac{\sinh(t)}{t} dt.$$

The hyperbolic sine integral satisfies the following inequality

$$\mathrm{Sih}(2x) \geq 2\frac{\sinh(x)}{x}\,\mathrm{Sih}(x).$$

4 Conclusion

This paper established several lower bounds for integrals involving a function and its derivative, offering transparent and accessible analytical methods. These results are demonstrated using concrete examples and applications to special functions, such as the sine integral. Future work could involve extending these techniques to broader classes of functions and investigating upper bounds, which could then be applied to more complex differential and integral inequalities.

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