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Research article

Fixed Point Results and Convergence Analysis for G-Path-Averaged Contractions in G-Metric Space

Abraham Augustine Ahiaba[1], **Cornelius Pelumi Olawoore**[1]*, **Moses Obinna Francis**[1]

[1]Department of Mathematical Sciences, Faculty of Science and Technology, Bingham University, Km 26 Abuja–Keffi Expressway, Kodape (Karu), Nasarawa State 961105, Nigeria.

*Corresponding Author: Cornelius. Pelumi Olawoore. Email: olabipelumi@gmail.com

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Abstract: In this paper we introduce a new orbit-based contractive framework in the setting of G-metric spaces, called (m, α) *G-path-averaged (G-PA) contractions* with $m \geq 2$. This notion extends Fabiano's path-averaged contractions to the triadic geometry of Mustafa–Sims G-metrics and is designed to avoid collapse to pointwise contractivity. For a G-continuous self-map on a complete G-metric space, we establish existence and uniqueness of a fixed point and prove that the Picard iteration converges to it in the sense of G-convergence. Moreover, we derive explicit quantitative estimates, including a posteriori and a priori geometric error bounds for the iterates. We also relate the new class to the induced metric d_G, showing that every G-PA contraction yields a path-averaged contraction on (X, d_G), and we provide examples demonstrating that the G-PA class can be strictly larger than the Banach-type contraction class. Finally, we obtain multi-step (t-point) fixed point and convergence results by embedding the recursion into a shift map on the product space (X^t, G^t) and applying the single-valued theory.

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Keywords: G-metric space; G-path-averaged contraction; fixed point; induced metric; t-point iteration

1 Introduction

Fixed point theory is a fundamental tool for establishing existence, uniqueness, and approximation of solutions to nonlinear problems in analysis and applied mathematics. In the metric setting, Banach's contraction principle guarantees a unique fixed point and convergence of Picard iterates in complete metric spaces [1]. Many extensions relax pointwise contractivity while preserving existence and convergence; classical examples include Kannan-, Chatterjea-, Reich-, and Čirić-type contractions and several nonlinear variants [2–6], as well as integral-type contractions and completeness characterizations [7,8]. Fixed point methods have also been developed in generalized distance settings. In particular, Mustafa and Sims introduced *G-metric spaces*, where a triadic distance $G(x, y, z)$ induces notions of G-convergence,

G-Cauchy sequences, and completeness [9,10]. A related and active direction concerns *modular G-metric spaces* and their applications [11–13]. We also note recent related contributions on common fixed points and generalized contractive conditions in b-metric and related settings [16–18].

Motivated by orbit-based behavior, where contractivity may hold only in an averaged sense along iterates, Fabiano introduced *path-averaged (PA) contractions* [14] and extended the theory to complete b-metric spaces [15]. The aim of the present paper is to develop an analogous framework in the triadic setting by introducing (m, α) *G-path-averaged (G-PA) contractions* on G-metric spaces. We establish existence and uniqueness of fixed points for suitable self-maps on complete G-metric spaces, prove G-convergence of Picard iterates, and derive explicit a priori and a posteriori error bounds. We also treat a t-point extension by embedding multi-step recursions into a shift map on a product G-metric space.

2 Preliminaries

Throughout this paper, $\mathbb{N} = \{1, 2, 3, \dots\}$ and (X, G) denotes a G-metric space. For a self-map $T : X \to X$, we write T^n for the n-fold iterate ($T^0 = \mathrm{Id}_X$).

Definition 1 (Mustafa–Sims [10]): *Let X be a nonempty set. A function $G : X^3 \to [0, \infty)$ is called a* G-metric *if for all $x, y, z, a \in X$ the following hold:*

(G1) $G(x, y, z) = 0$ *if and only if* $x = y = z$;
(G2) $0 < G(x, x, y)$ *for all* $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ *whenever* $y \neq z$;
(G4) G *is symmetric in all variables;*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality).

Then (X, G) is called a G-metric space.

Definition 2 ([10]): *Let (X, G) be a G-metric space and $\{x_n\} \subset X$.*

(i) $\{x_n\}$ G-converges *to $x \in X$, written* $x_n \xrightarrow{G} x$, *if* $\lim_{n\to\infty} G(x_n, x, x) = 0$.
(ii) $\{x_n\}$ *is* G-Cauchy *if* $\lim_{n,m,\ell\to\infty} G(x_n, x_m, x_\ell) = 0$.
(iii) (X, G) *is* complete *if every G-Cauchy sequence is G-convergent in X.*

Definition 3: *A mapping $T : X \to X$ is called* G-continuous *at $x \in X$ if $x_n \xrightarrow{G} x$ implies $Tx_n \xrightarrow{G} Tx$. If T is G-continuous at every point of X, we simply say that T is* G-continuous.

Lemma 1: *Let (X, G) be a G-metric space and let $\{x_n\} \subset X$. For integers $p > n$,*

$$G(x_p, x_n, x_n) \leq \sum_{k=n}^{p-1} G(x_{k+1}, x_k, x_k).$$

Proof: Apply (G5) to get

$$G(x_p, x_n, x_n) \leq G(x_p, x_{p-1}, x_{p-1}) + G(x_{p-1}, x_n, x_n).$$

Iterating this estimate yields the stated inequality. □

Remark. Lemma 1 is the key tool for passing from decay of successive increments $G(x_{k+1}, x_k, x_k)$ to the G-Cauchy property.

The induced metric. Define, for $x, y \in X$,

$$d_G(x, y) := G(x, y, y) + G(y, x, x). \tag{2.1}$$

Proposition 1: *If (X, G) is a G-metric space, then d_G defined by (2.1) is a metric on X.*

Proof: Clearly $d_G(x, y) \geq 0$ and $d_G(x, y) = d_G(y, x)$. If $d_G(x, y) = 0$, then $G(x, y, y) = 0$, and by (G1) we get $x = y$.

For the triangle inequality, fix $x, y, z \in X$. By (G5) with $a = z$,

$$G(x, y, y) \leq G(x, z, z) + G(z, y, y), \qquad G(y, x, x) \leq G(y, z, z) + G(z, x, x).$$

Adding these gives $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$. □

Lemma 2: *Let (X, G) be a G-metric space and let $\{x_n\} \subset X$, $x \in X$. Then $x_n \xrightarrow{G} x$ if and only if $d_G(x_n, x) \to 0$.*

Proof: Assume $x_n \xrightarrow{G} x$, i.e. $G(x_n, x, x) \to 0$. Using symmetry and (G5) with $a = x$,

$$G(x, x_n, x_n) = G(x_n, x_n, x) \leq G(x_n, x, x) + G(x, x_n, x).$$

By symmetry, $G(x, x_n, x) = G(x_n, x, x)$, hence

$$G(x, x_n, x_n) \leq 2\, G(x_n, x, x) \to 0.$$

Therefore

$$d_G(x_n, x) = G(x_n, x, x) + G(x, x_n, x_n) \to 0.$$

Conversely, if $d_G(x_n, x) \to 0$, then $G(x_n, x, x) \leq d_G(x_n, x) \to 0$, so $x_n \xrightarrow{G} x$. □

Lemma 3: *A sequence $\{x_n\}$ is G-Cauchy if and only if it is Cauchy in the metric d_G. Consequently, (X, G) is complete if and only if (X, d_G) is complete.*

Proof: Assume $\{x_n\}$ is G-Cauchy. Given $\varepsilon > 0$, choose N such that $G(x_n, x_m, x_\ell) < \varepsilon/2$ for all $n, m, \ell \geq N$. Taking $\ell = m$ and $\ell = n$ gives, for all $m, n \geq N$,

$$G(x_n, x_m, x_m) < \varepsilon/2, \qquad G(x_m, x_n, x_n) < \varepsilon/2,$$

hence $d_G(x_n, x_m) < \varepsilon$. Thus $\{x_n\}$ is d_G-Cauchy.

Conversely, assume $\{x_n\}$ is d_G-Cauchy. Given $\varepsilon > 0$, choose N such that $d_G(x_i, x_j) < \varepsilon/2$ for all $i, j \geq N$. For $n, m, \ell \geq N$, apply (G5) with $a = x_m$:

$$G(x_n, x_m, x_\ell) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_\ell).$$

By symmetry, $G(x_m, x_m, x_\ell) = G(x_\ell, x_m, x_m)$. Therefore,

$$G(x_n, x_m, x_\ell) \leq G(x_n, x_m, x_m) + G(x_\ell, x_m, x_m) \leq d_G(x_n, x_m) + d_G(x_\ell, x_m) < \varepsilon.$$

Hence $\{x_n\}$ is G-Cauchy. The completeness equivalence follows from Lemma 2. □

Remark. If one allowed the averaging length $m = 1$ in a path-averaged inequality, then the case $n = 1$ would immediately reduce to a pointwise contractive condition. To keep the notion genuinely orbit-based, we always assume $m \geq 2$.

Definition 4 ([14])**:** *Let (X, d) be a metric (or b-metric) space and let $T : X \to X$. We say that T is an (m, α)* path-averaged contraction *if there exist $m \in \mathbb{N}$ with $m \geq 2$ and $\alpha \in (0, 1)$ such that for all $x, y \in X$ and all $n \geq m$,*

$$\sum_{k=0}^{n-1} d(T^{k+1}x,\ T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} d(T^k x,\ T^k y).$$

Definition 5: *Let (X, G) be a G-metric space and let $T : X \to X$. We say that T is an (m, α)* G-path-averaged contraction *if there exist $m \in \mathbb{N}$ with $m \geq 2$ and $\alpha \in (0, 1)$ such that for all $x, y, z \in X$ and all $n \geq m$,*

$$\sum_{k=0}^{n-1} G(T^{k+1}x,\ T^{k+1}y,\ T^{k+1}z) \leq \alpha \sum_{k=0}^{n-1} G(T^k x,\ T^k y,\ T^k z).$$

Remark. In the fixed point arguments below, it will be enough to use Definition 5 with the single choice $n = m$, which yields a recursive decay of suitable block sums along the orbit.

Example 1: *Let (X, d) be a metric space and define*

$$G(x, y, z) := \max\{d(x, y),\ d(y, z),\ d(z, x)\}, \qquad x, y, z \in X.$$

Then G is a G-metric on X (see, e.g., [10]).

Example 2: *Let $X = \mathbb{R}$ with $d(x, y) = |x - y|$ and let G be induced by d as in Example 1. Let $T(x) = \lambda x$ with $|\lambda| < 1$. Then T is a Banach contraction, and consequently it is an (m, α) G-PA contraction for every $m \geq 2$ with $\alpha = |\lambda|$.*

Example 3: *Let $X = \{0,1,2\}$ with the discrete metric $d(u,v) = 1$ if $u \neq v$ and 0 otherwise, and let G be as in Example 1. Define $T : X \to X$ by*

$$T(0) = 1, \qquad T(1) = 2, \qquad T(2) = 2.$$

Then T is an (m,α) G-PA contraction with $m = 2$ and $\alpha = \frac{1}{2}$. Indeed, since T^2 is the constant map 2, for any $x, y, z \in X$ and any $n \geq 2$ we have

$$\sum_{k=0}^{n-1} G(T^{k+1}x, T^{k+1}y, T^{k+1}z) = G(Tx, Ty, Tz),$$

and

$$\sum_{k=0}^{n-1} G(T^{k}x, T^{k}y, T^{k}z) = G(x,y,z) + G(Tx, Ty, Tz).$$

If $G(Tx,Ty,Tz) = 0$ the inequality holds trivially; if $G(Tx,Ty,Tz) = 1$ then necessarily $G(x,y,z) = 1$, so $1 \leq \frac{1}{2} \cdot 2$. On the other hand, T is not a pointwise contraction on (X,d) with constant < 1.

Remark. Example 3 shows that the G-PA class can be strictly larger than the class of Banach-type (pointwise) contractions.

t-point iterations.

Definition 6: *Let $t \geq 2$ and let $f : X^t \to X$. Define $F : X^t \to X^t$ by*

$$F(u_0, u_1, \dots, u_{t-1}) = \big(f(u_{t-1}, u_{t-2}, \dots, u_0),\ u_0,\ u_1, \dots, u_{t-2}\big).$$

A point $x^* \in X$ is called a fixed point of f *if $f(x^*, x^*, \dots, x^*) = x^*$. Equivalently, $(x^*, \dots, x^*)$ is a fixed point of F.*

Definition 7: *Let (X,G) be a G-metric space and let $t \geq 2$. For $\mathbf{x} = (x_0, \dots, x_{t-1}) \in X^t$, define*

$$G_t(\mathbf{x}, \mathbf{y}, \mathbf{z}) := \max_{0 \leq i \leq t-1} G(x_i, y_i, z_i), \qquad \mathbf{y}, \mathbf{z} \in X^t.$$

Then (X^t, G_t) is a G-metric space; moreover, if (X,G) is complete, then (X^t, G_t) is complete.

Remark. With Definitions 6–7, the t-point recursion

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-t+1})$$

can be written as the Picard iteration of F on X^t.

3 Results

We now establish fixed point and convergence results for (m, α) G-path-averaged contractions, together with quantitative error estimates and a t-point extension via a shift map on a product G-metric space.

Lemma 4: *Let (X, G) be a complete G-metric space and let $t \geq 2$. Define $G_t : (X^t)^3 \to [0, \infty)$ by*

$$G_t(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \max_{0 \leq i \leq t-1} G(x_i, y_i, z_i), \qquad \mathbf{x} = (x_0, \dots, x_{t-1}),\ \mathbf{y} = (y_0, \dots, y_{t-1}),\ \mathbf{z} = (z_0, \dots, z_{t-1}).$$

Then (X^t, G_t) is a complete G-metric space.

Proof: *G_t is a G-metric (see Definition 7).* is immediate since each $G(x_i, y_i, z_i)$ is symmetric and max preserves symmetry. For (G1), if $G_t(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$, then $G(x_i, y_i, z_i) = 0$ for every i, hence $x_i = y_i = z_i$ for every i by (G1) of G, i.e. $\mathbf{x} = \mathbf{y} = \mathbf{z}$. Conversely, if $\mathbf{x} = \mathbf{y} = \mathbf{z}$ then each coordinate term is 0, hence $G_t = 0$. Properties (G2) and (G3) follow coordinatewise and are preserved under taking a maximum. For (G5), fix $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a} \in X^t$. For each coordinate i,

$$G(x_i, y_i, z_i) \leq G(x_i, a_i, a_i) + G(a_i, y_i, z_i).$$

Taking maxima over i gives

$$G_t(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq G_t(\mathbf{x}, \mathbf{a}, \mathbf{a}) + G_t(\mathbf{a}, \mathbf{y}, \mathbf{z}),$$

which is (G5) for G_t.

Completeness. Let $\{\mathbf{x}_n\} \subset X^t$ be G_t-Cauchy, where $\mathbf{x}_n = (x_n^{(0)}, \dots, x_n^{(t-1)})$. Then for each i,

$$G(x_n^{(i)}, x_m^{(i)}, x_\ell^{(i)}) \leq G_t(\mathbf{x}_n, \mathbf{x}_m, \mathbf{x}_\ell) \to 0,$$

so $\{x_n^{(i)}\}$ is G-Cauchy in X. By completeness of (X, G), there exists $x^{(i)} \in X$ such that $x_n^{(i)} \xrightarrow{G} x^{(i)}$. Set $\mathbf{x} := (x^{(0)}, \dots, x^{(t-1)})$. Then

$$G_t(\mathbf{x}_n, \mathbf{x}, \mathbf{x}) = \max_i G(x_n^{(i)}, x^{(i)}, x^{(i)}) \to 0,$$

so $\mathbf{x}_n \xrightarrow{G_t} \mathbf{x}$. Hence (X^t, G_t) is complete. □

Lemma 5: *Let $t \geq 2$, let $f : X^t \to X$, and define $F : X^t \to X^t$ by*

$$F(u_0, u_1, \dots, u_{t-1}) = \big(f(u_{t-1}, u_{t-2}, \dots, u_0),\ u_0,\ u_1, \dots, u_{t-2}\big).$$

Given $x_0, \dots, x_{t-1} \in X$*, set* $\mathbf{u}_0 := (x_{t-1}, x_{t-2}, \dots, x_0)$ *and define* $\mathbf{u}_{n+1} := F(\mathbf{u}_n)$ *for* $n \geq 0$*. Then the first coordinate of* $\mathbf{u}_n$ *equals* x_{t-1+n}*, where* $\{x_n\}$ *satisfies the* t*-point recursion*

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-t+1}) \qquad (n \geq t-1).$$

Moreover, x^* *solves* $f(x^*, \dots, x^*) = x^*$ *if and only if* $(x^*, \dots, x^*)$ *is a fixed point of* F*.*

Proof: Write $\mathbf{u}_n = (u_n^{(0)}, u_n^{(1)}, \dots, u_n^{(t-1)})$. From the definition of F, one has

$$u_{n+1}^{(0)} = f\left(u_n^{(t-1)}, u_n^{(t-2)}, \dots, u_n^{(0)}\right), \qquad u_{n+1}^{(j)} = u_n^{(j-1)} \ (1 \leq j \leq t-1).$$

A straightforward induction shows $u_n^{(0)} = x_{t-1+n}$, and then the displayed t-point recursion follows. The fixed point equivalence is immediate from

$$F(x^*, \dots, x^*) = (f(x^*, \dots, x^*), x^*, \dots, x^*).$$

□

Theorem 1: *Let* (X, G) *be a complete G-metric space and let* $T : X \to X$ *be G-continuous. Assume that* T *is an* (m, α) *G-path-averaged contraction for some* $m \geq 2$ *and* $\alpha \in (0, 1)$*, i.e., for all* $x, y, z \in X$ *and all* $n \geq m$*,*

$$\sum_{k=0}^{n-1} G(T^{k+1}x,\ T^{k+1}y,\ T^{k+1}z) \leq \alpha \sum_{k=0}^{n-1} G(T^k x,\ T^k y,\ T^k z).$$

Then for every $x_0 \in X$*, the Picard sequence* $x_{n+1} = Tx_n$ *G-converges to a fixed point* $x^* \in X$*.*

Moreover, defining

$$s_n := G(x_{n+1}, x_n, x_n), \qquad B_n := \sum_{k=0}^{m-1} s_{n+k} = \sum_{k=0}^{m-1} G(x_{n+k+1}, x_{n+k}, x_{n+k}) \quad (n \geq 0),$$

one has

$$B_{n+1} \leq \alpha B_n \qquad (n \geq 0), \tag{3.1}$$

and consequently the following error bounds hold for all $n \geq 0$*:*

$$G(x^*, x_n, x_n) \leq \frac{B_n}{1-\alpha}, \tag{3.2}$$

$$G(x^*, x_n, x_n) \leq \frac{B_0}{1-\alpha}\, \alpha^n. \tag{3.3}$$

Proof: Fix $x_0 \in X$ and set $x_n := T^n x_0$.

Step 1: block-sum decay. Apply the G-PA inequality with $n = m$ to the triple $(x, y, z) = (x_n, x_n, x_{n+1})$. Using $T^k x_n = x_{n+k}$ and $T^k x_{n+1} = x_{n+k+1}$, we obtain

$$\sum_{k=0}^{m-1} G(x_{n+k+1}, x_{n+k+1}, x_{n+k+2}) \le \alpha \sum_{k=0}^{m-1} G(x_{n+k}, x_{n+k}, x_{n+k+1}).$$

By symmetry of G, $G(x_{n+k}, x_{n+k}, x_{n+k+1}) = G(x_{n+k+1}, x_{n+k}, x_{n+k}) = s_{n+k}$ and $G(x_{n+k+1}, x_{n+k+1}, x_{n+k+2}) = G(x_{n+k+2}, x_{n+k+1}, x_{n+k+1}) = s_{n+k+1}$. Hence the preceding inequality is exactly (3.1), i.e. $B_{n+1} \le \alpha B_n$. Iterating gives $B_n \le \alpha^n B_0$ for all $n \ge 0$.

Step 2: tail estimate for increments. For each $j \ge 0$, since $s_{n+j} \le B_{n+j} \le \alpha^j B_n$, we have

$$\sum_{k=n}^{\infty} s_k = \sum_{j=0}^{\infty} s_{n+j} \le \sum_{j=0}^{\infty} \alpha^j B_n = \frac{B_n}{1-\alpha}.$$

Step 3: $\{x_n\}$ is G-Cauchy. By Lemma 1, for any $p \ge 1$,

$$G(x_{n+p}, x_n, x_n) \le \sum_{k=n}^{n+p-1} s_k \le \sum_{k=n}^{\infty} s_k \le \frac{B_n}{1-\alpha}.$$

Hence $G(x_{n+p}, x_n, x_n) \to 0$ as $n \to \infty$ uniformly in $p \ge 1$, so $\{x_n\}$ is G-Cauchy. Since (X, G) is complete, there exists $x^* \in X$ such that $x_n \xrightarrow{G} x^*$.

Step 4: x^* is a fixed point. By G-continuity, $Tx_n = x_{n+1} \xrightarrow{G} Tx^*$. But also $x_{n+1} \xrightarrow{G} x^*$. Limits in a G-metric space are unique (e.g. via Lemma 2 and the induced metric d_G), hence $Tx^* = x^*$.

Step 5: error bounds. From Step 3, for each fixed n and all $p \ge 1$,

$$G(x_{n+p}, x_n, x_n) \le \frac{B_n}{1-\alpha}.$$

Letting $p \to \infty$ and using $x_{n+p} \xrightarrow{G} x^*$ yields $G(x^*, x_n, x_n) \le \frac{B_n}{1-\alpha}$, which is (3.2). Finally, since $B_n \le \alpha^n B_0$, (3.3) follows. □

Theorem 2: *Let (X, G) be a G-metric space and let $T : X \to X$ be an (m, α) G-path-averaged contraction for some $m \ge 2$ and $\alpha \in (0, 1)$. If T has a fixed point in X, then it is unique.*

Proof: Assume $x^*, y^* \in X$ are fixed points: $Tx^* = x^*$ and $Ty^* = y^*$. Apply the G-PA inequality with $n = m$ to the triple $(x, y, z) = (x^*, y^*, y^*)$:

$$\sum_{k=0}^{m-1} G(T^{k+1}x^*, T^{k+1}y^*, T^{k+1}y^*) \le \alpha \sum_{k=0}^{m-1} G(T^k x^*, T^k y^*, T^k y^*).$$

Since $T^k x^* = x^*$ and $T^k y^* = y^*$ for all k, this becomes

$$m\, G(x^*, y^*, y^*) \leq \alpha m\, G(x^*, y^*, y^*).$$

Thus $(1-\alpha)m\, G(x^*, y^*, y^*) \leq 0$, and since $(1-\alpha)m > 0$ we get $G(x^*, y^*, y^*) = 0$. By (G1), $x^* = y^*$. □

Corollary 1: *Let (X, G) be a complete G-metric space and let $T : X \to X$ be G-continuous. Assume that T is an (m, α) G-path-averaged contraction for some $m \geq 2$ and $\alpha \in (0,1)$. Then T has a* unique *fixed point $x^* \in X$, and for every $x_0 \in X$ the Picard sequence $x_{n+1} = Tx_n$ G-converges to x^*. Moreover, with $x_n := T^n x_0$ and*

$$B_n := \sum_{k=0}^{m-1} G(x_{n+k+1}, x_{n+k}, x_{n+k}),$$

one has for all $n \geq 0$,

$$G(x^*, x_n, x_n) \leq \frac{B_n}{1-\alpha} \leq \frac{B_0}{1-\alpha}\, \alpha^n.$$

Proof: Existence, convergence, and the bounds follow from Theorem 1. Uniqueness follows from Theorem 2. □

Theorem 3: *Let (X, G) be a complete G-metric space and let $t \geq 2$. Let $f : X^t \to X$ and define $F : X^t \to X^t$ by*

$$F(u_0, u_1, \dots, u_{t-1}) = \big(f(u_{t-1}, u_{t-2}, \dots, u_0),\ u_0,\ u_1, \dots, u_{t-2}\big).$$

On X^t, let G_t be the product G-metric from Definition 7. Assume that F is G_t-continuous and that F is an (m, α) G_t-path-averaged contraction for some $m \geq 2$ and $\alpha \in (0,1)$. Then there exists a unique *$x^* \in X$ such that*

$$f(x^*, x^*, \dots, x^*) = x^*.$$

Moreover, for any initial points $x_0, \dots, x_{t-1} \in X$, the sequence defined by

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-t+1}) \qquad (n \geq t-1)$$

G-converges to x^*. In addition, if $\mathbf{u}_0 := (x_{t-1}, \dots, x_0)$ and $\mathbf{u}_{n+1} := F(\mathbf{u}_n)$, then with

$$S_n := G_t(\mathbf{u}_{n+1}, \mathbf{u}_n, \mathbf{u}_n), \qquad B_n^{(t)} := \sum_{k=0}^{m-1} S_{n+k},$$

one has for all $n \geq 0$,

$$G(x^*, x_{t-1+n}, x_{t-1+n}) \leq G_t\big((x^*, \dots, x^*), \mathbf{u}_n, \mathbf{u}_n\big) \leq \frac{B_n^{(t)}}{1-\alpha} \leq \frac{B_0^{(t)}}{1-\alpha}\, \alpha^n.$$

Proof: By Lemma 4, (X^t, G_t) is complete. Applying Theorem 1 and Theorem 2 to the self-map F on (X^t, G_t) yields a unique fixed point $\mathbf{u}^* \in X^t$ and G_t-convergence $\mathbf{u}_n \xrightarrow{G_t} \mathbf{u}^*$. Writing $\mathbf{u}^* = (u_0, \dots, u_{t-1})$ and using $F(\mathbf{u}^*) = \mathbf{u}^*$ gives $u_1 = u_0, \dots, u_{t-1} = u_{t-2}$ and $u_0 = f(u_{t-1}, \dots, u_0)$, hence $u_0 = \cdots = u_{t-1} =: x^*$ and $f(x^*, \dots, x^*) = x^*$; uniqueness of $\mathbf{u}^*$ implies uniqueness of x^*.

Finally, Lemma 5 identifies the first coordinate of $\mathbf{u}_n$ with x_{t-1+n}, and $\mathbf{u}_n \xrightarrow{G_t} (x^*, \dots, x^*)$ implies in particular $x_{t-1+n} \xrightarrow{G} x^*$. The stated quantitative estimate follows by applying the bounds in Theorem 1 to F on (X^t, G_t). □

Theorem 4: *Let (X, G) be a complete G-metric space and let $T : X \to X$ be G-continuous. Assume that T is an (m, α) G-path-averaged contraction for some $m \geq 2$ and $\alpha \in (0, 1)$. Fix $x_0 \in X$ and define $x_{n+1} = Tx_n$. Then $\{x_n\}$ G-converges to the unique fixed point x^* of T, and for every $n \geq 0$:*

(a) ***(A posteriori bound)***

$$G(x^*, x_n, x_n) \leq \frac{1}{1-\alpha} B_n, \qquad B_n := \sum_{k=0}^{m-1} G(x_{n+k+1}, x_{n+k}, x_{n+k}).$$

(b) ***(A priori geometric bound)***

$$G(x^*, x_n, x_n) \leq \frac{B_0}{1-\alpha} \alpha^n.$$

(c) ***(Max-form bound)***

$$G(x^*, x_n, x_n) \leq \frac{m}{1-\alpha} \max_{0 \leq k \leq m-1} G(x_{n+k+1}, x_{n+k}, x_{n+k}).$$

Proof: This is precisely the quantitative part of Theorem 1 (and Corollary 1), together with the elementary estimate $B_n \leq m \max_{0 \leq k \leq m-1} G(x_{n+k+1}, x_{n+k}, x_{n+k})$. □

Theorem 5: *Let (X, G) be a G-metric space and let d_G be defined by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x).$$

If $T : X \to X$ is an (m, α) G-path-averaged contraction (Definition 5), then T is an (m, α) path-averaged contraction on the metric space (X, d_G), i.e., for all $x, y \in X$ and all $n \geq m$,

$$\sum_{k=0}^{n-1} d_G(T^{k+1}x, T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} d_G(T^k x, T^k y).$$

Proof: Fix $x, y \in X$ and $n \geq m$. Apply Definition 5 to the triple (x, y, y) to get

$$\sum_{k=0}^{n-1} G(T^{k+1}x, T^{k+1}y, T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} G(T^k x, T^k y, T^k y).$$

Apply the same definition to the triple (y, x, x) to obtain

$$\sum_{k=0}^{n-1} G(T^{k+1}y, T^{k+1}x, T^{k+1}x) \leq \alpha \sum_{k=0}^{n-1} G(T^k y, T^k x, T^k x).$$

Adding the two inequalities yields the desired estimate for d_G. □

Lemma 6: *Let (X, G) be a G-metric space and let d_G be defined by* (2.1)*. Then for all $x, y \in X$,*

$$G(x, y, y) \leq d_G(x, y) \leq 3\, G(x, y, y), \qquad G(y, x, x) \leq d_G(x, y) \leq 3\, G(y, x, x). \tag{3.4}$$

In particular, d_G and the one-sided quantities $G(x, y, y)$ and $G(y, x, x)$ are equivalent up to constants.

Proof: The lower bounds $G(x, y, y) \leq d_G(x, y)$ and $G(y, x, x) \leq d_G(x, y)$ are immediate from $d_G(x, y) = G(x, y, y) + G(y, x, x)$.

For the upper bound, note that by symmetry $G(y, x, x) = G(x, x, y)$. By (G5) with $a = y$,

$$G(x, x, y) \leq G(x, y, y) + G(y, x, y).$$

By symmetry, $G(y, x, y) = G(x, y, y)$, hence $G(y, x, x) = G(x, x, y) \leq 2\, G(x, y, y)$. Therefore,

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \leq G(x, y, y) + 2\, G(x, y, y) = 3\, G(x, y, y).$$

The second upper bound follows by swapping x and y. □

Lemma 7: *Limits in a G-metric space are unique: if $x_n \xrightarrow{G} x$ and $x_n \xrightarrow{G} y$, then $x = y$.*

Proof: By Lemma 2, $x_n \xrightarrow{G} x$ implies $d_G(x_n, x) \to 0$, and similarly $d_G(x_n, y) \to 0$. Since (X, d_G) is a metric space, metric limits are unique, hence $x = y$. □

Proposition 2: *Let (X, G) be a G-metric space and let d_G be given by* (2.1)*. A self-map $T : X \to X$ is G-continuous if and only if it is continuous on the metric space (X, d_G).*

Proof: Assume T is continuous on (X, d_G). If $x_n \xrightarrow{G} x$, then by Lemma 2 we have $d_G(x_n, x) \to 0$, hence $d_G(Tx_n, Tx) \to 0$, and again by Lemma 2 we obtain $Tx_n \xrightarrow{G} Tx$. Thus T is G-continuous.

Conversely, assume T is G-continuous. If $d_G(x_n, x) \to 0$, then $x_n \xrightarrow{G} x$ by Lemma 2, hence $Tx_n \xrightarrow{G} Tx$, and therefore $d_G(Tx_n, Tx) \to 0$ again by Lemma 2. Thus T is d_G-continuous. □

Corollary 2: *Under the assumptions of Corollary 1, the Picard iterates satisfy the metric estimate*

$$d_G(x^*, x_n) \leq \frac{3B_n}{1-\alpha} \leq \frac{3B_0}{1-\alpha}\,\alpha^n, \qquad B_n = \sum_{k=0}^{m-1} G(x_{n+k+1}, x_{n+k}, x_{n+k}),$$

where x^* is the unique fixed point and $x_{n+1} = Tx_n$.

Proof: By Theorem 1, $G(x^*, x_n, x_n) \leq \frac{B_n}{1-\alpha}$. Applying Lemma 6 with $(x, y) = (x^*, x_n)$ yields

$$d_G(x^*, x_n) \leq 3\,G(x^*, x_n, x_n) \leq \frac{3}{1-\alpha}\,B_n.$$

The geometric bound follows from $B_n \leq \alpha^n B_0$. □

Lemma 8: *Let (X, G) be a G-metric space and let $T : X \to X$ be an (m, α) G-path-averaged contraction. For fixed $x, y, z \in X$, define*

$$D_n(x, y, z) := \sum_{k=0}^{m-1} G(T^{n+k}x,\ T^{n+k}y,\ T^{n+k}z) \qquad (n \geq 0).$$

Then

$$D_{n+1}(x, y, z) \leq \alpha\, D_n(x, y, z) \qquad (n \geq 0),$$

and hence for all $n \geq 0$,

$$G(T^n x,\ T^n y,\ T^n z) \leq D_n(x, y, z) \leq \alpha^n D_0(x, y, z). \tag{3.5}$$

Proof: Apply the G-PA inequality with the choice $n = m$ to the triple $(T^n x,\ T^n y,\ T^n z)$. The left-hand side becomes

$$\sum_{k=0}^{m-1} G(T^{k+1}(T^n x),\ T^{k+1}(T^n y),\ T^{k+1}(T^n z)) = \sum_{k=0}^{m-1} G(T^{n+k+1}x,\ T^{n+k+1}y,\ T^{n+k+1}z) = D_{n+1}(x, y, z),$$

and the right-hand side is exactly $D_n(x, y, z)$. Thus $D_{n+1} \leq \alpha D_n$. Iterating gives $D_n \leq \alpha^n D_0$, and since $G(T^n x, T^n y, T^n z)$ is one term in the sum D_n, we also have $G(T^n x, T^n y, T^n z) \leq D_n$. □

Corollary 3: *Let (X, G) be a G-metric space and let $T : X \to X$ be an (m, α) G-path-averaged contraction. Then for all $x, y \in X$ and all $n \geq 0$,*

$$d_G(T^n x, T^n y) \leq \alpha^n \sum_{k=0}^{m-1} d_G(T^k x, T^k y).$$

Proof: By Theorem 5, T is an (m, α) path-averaged contraction on (X, d_G). Applying the path-averaged inequality with $n = m$ to the pair $(T^n x, T^n y)$ yields

$$\sum_{k=0}^{m-1} d_G(T^{n+k+1}x, T^{n+k+1}y) \leq \alpha \sum_{k=0}^{m-1} d_G(T^{n+k}x, T^{n+k}y).$$

Defining $\Delta_n := \sum_{k=0}^{m-1} d_G(T^{n+k}x, T^{n+k}y)$, we get $\Delta_{n+1} \leq \alpha\Delta_n$, hence $\Delta_n \leq \alpha^n\Delta_0$. Since $d_G(T^n x, T^n y) \leq \Delta_n$, the claim follows. □

Theorem 6: *Let (X, G) be a G-metric space and let $T : X \to X$ be an (m, α) G-path-averaged contraction for some $m \geq 2$ and $\alpha \in (0, 1)$. Assume that T has a fixed point $x^* \in X$. Then for every $x \in X$ the Picard orbit $T^n x$ G-converges to x^*, and one has the quantitative bound*

$$G(T^n x,\ x^*,\ x^*) \leq \alpha^n \sum_{k=0}^{m-1} G(T^k x,\ x^*,\ x^*) \qquad (n \geq 0). \tag{3.6}$$

Consequently,

$$d_G(T^n x, x^*) \leq 3\,\alpha^n \sum_{k=0}^{m-1} G(T^k x,\ x^*,\ x^*) \qquad (n \geq 0).$$

Proof: Fix $x \in X$ and define

$$E_n := \sum_{k=0}^{m-1} G(T^{n+k}x,\ x^*,\ x^*) \qquad (n \geq 0).$$

Apply the G-PA inequality with $n = m$ to the triple $(T^n x,\ x^*,\ x^*)$. Since $T^k x^* = x^*$ for all k, we obtain

$$\sum_{k=0}^{m-1} G(T^{n+k+1}x,\ x^*,\ x^*) \leq \alpha \sum_{k=0}^{m-1} G(T^{n+k}x,\ x^*,\ x^*),$$

i.e. $E_{n+1} \leq \alpha E_n$. Hence $E_n \leq \alpha^n E_0$, and since $G(T^n x, x^*, x^*)$ is a term of E_n, we get (3.6). This implies $G(T^n x, x^*, x^*) \to 0$, i.e. $T^n x \xrightarrow{G} x^*$.

Finally, by Lemma 6 (with $(x, y) = (T^n x, x^*)$), $d_G(T^n x, x^*) \leq 3\,G(T^n x, x^*, x^*)$, and the displayed metric bound follows. □

Theorem 7: *Let (X, G) be a complete G-metric space. Let $T, S : X \to X$ be G-continuous self-maps such that each of T and S is an (m, α) G-path-averaged contraction (possibly with different parameters), so that each has a unique fixed point (by Corollary 1). If, in addition, T and S commute (i.e. $TS = ST$), then they have the* same *fixed point. In particular, T and S have a unique common fixed point in X.*

Proof: Let p be the unique fixed point of T and q the unique fixed point of S. Since $Tp = p$ and $TS = ST$, we have

$$T(Sp) = S(Tp) = Sp,$$

so Sp is a fixed point of T. By uniqueness of the fixed point of T, it follows that $Sp = p$, hence p is a fixed point of S, so $p = q$ by uniqueness of the fixed point of S. □

Proposition 3: *Let (X, d) be a metric space and let G be the induced G-metric*

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}.$$

If $T : X \to X$ is an (m, α) path-averaged contraction on (X, d) and $\alpha \in (0, \frac{1}{3})$, then T is an $(m, 3\alpha)$ G-path-averaged contraction on (X, G).

Proof: Fix $x, y, z \in X$ and $n \geq m$. For each k,

$$G(T^{k+1}x, T^{k+1}y, T^{k+1}z) = \max\{d(T^{k+1}x, T^{k+1}y),\ d(T^{k+1}y, T^{k+1}z),\ d(T^{k+1}z, T^{k+1}x)\}$$

$$\leq d(T^{k+1}x, T^{k+1}y) + d(T^{k+1}y, T^{k+1}z) + d(T^{k+1}z, T^{k+1}x).$$

Summing over $k = 0, \dots, n-1$ and applying the (m, α) PA inequality (in d) to each of the pairs (x, y), (y, z) and (z, x) gives

$$\sum_{k=0}^{n-1} G(T^{k+1}x, T^{k+1}y, T^{k+1}z) \leq \alpha \sum_{k=0}^{n-1} \Big(d(T^k x, T^k y) + d(T^k y, T^k z) + d(T^k z, T^k x) \Big).$$

For each k, the sum of the three pairwise distances is bounded by 3 times their maximum, i.e.

$$d(T^k x, T^k y) + d(T^k y, T^k z) + d(T^k z, T^k x) \leq 3\, G(T^k x, T^k y, T^k z).$$

Hence,

$$\sum_{k=0}^{n-1} G(T^{k+1}x, T^{k+1}y, T^{k+1}z) \leq 3\alpha \sum_{k=0}^{n-1} G(T^k x, T^k y, T^k z).$$

Since $\alpha < \frac{1}{3}$, we have $3\alpha \in (0, 1)$, so T is an $(m, 3\alpha)$ G-PA contraction. □

Theorem 8: *Let (X, G) be a complete G-metric space and let $T : X \to X$ be G-continuous. Assume that T is an (m, α) G-path-averaged contraction for some $m \geq 2$ and $\alpha \in (0, 1)$, and let $x^* \in X$ be its (unique) fixed point. Let $\{y_n\} \subset X$ be any sequence (an* inexact Picard iteration*) satisfying*

$$y_{n+1} = Ty_n + e_n \quad \textit{in the sense that} \quad \delta_n := G(y_{n+1}, Ty_n, Ty_n) \geq 0 \qquad (n \geq 0).$$

Then for every $n \geq 0$,

$$G(y_n, x^*, x^*) \leq \alpha^n \sum_{k=0}^{m-1} G(T^k y_0, x^*, x^*) + \sum_{j=0}^{n-1} \alpha^{n-1-j}\, \delta_j,$$

and in particular, if $\delta_n \to 0$ (or $\sum_{n=0}^{\infty} \delta_n < \infty$), then $y_n \xrightarrow{G} x^*$.

Proof: Define $E_n := \sum_{k=0}^{m-1} G(T^{n+k} y_0, x^*, x^*)$ as in Theorem 6. Since x^* is a fixed point, the G-PA inequality with $n = m$ applied to $(T^n y_0, x^*, x^*)$ yields $E_{n+1} \leq \alpha E_n$, hence $E_n \leq \alpha^n E_0$.

For the inexact orbit, use the rectangle inequality (G5) with $a = Ty_n$:

$$G(y_{n+1}, x^*, x^*) \leq G(y_{n+1}, Ty_n, Ty_n) + G(Ty_n, x^*, x^*) = \delta_n + G(Ty_n, x^*, x^*).$$

By Theorem 6 applied to the exact Picard orbit starting at y_n,

$$G(Ty_n, x^*, x^*) = G(T^1 y_n, x^*, x^*) \leq \alpha \sum_{k=0}^{m-1} G(T^k y_n, x^*, x^*).$$

Iterating this estimate along n and unrolling the resulting recursion gives

$$G(y_n, x^*, x^*) \leq \alpha^n \sum_{k=0}^{m-1} G(T^k y_0, x^*, x^*) + \sum_{j=0}^{n-1} \alpha^{n-1-j}\, \delta_j.$$

If $\delta_n \to 0$, then the convolution term tends to 0; if $\sum \delta_n < \infty$, it also tends to 0. Hence $G(y_n, x^*, x^*) \to 0$, i.e. $y_n \xrightarrow{G} x^*$. □

Theorem 9: *Let (X, G) be a complete G-metric space and let $T, S : X \to X$ be G-continuous. Assume that both T and S are (m, α) G-path-averaged contractions for the same parameters $m \geq 2$ and $\alpha \in (0, 1)$, and let x^* and y^* be their unique fixed points, respectively. Suppose there exists $\varepsilon \geq 0$ such that*

$$G(Tx, Sx, Sx) \leq \varepsilon \qquad \text{for all } x \in X.$$

Then

$$G(x^*, y^*, y^*) \leq \frac{\varepsilon}{1-\alpha} \qquad \text{and} \qquad d_G(x^*, y^*) \leq \frac{3\varepsilon}{1-\alpha}.$$

Proof: Using (G5) with $a = Sx^*$ and the assumption,

$$G(x^*, y^*, y^*) = G(Tx^*, Sy^*, Sy^*) \leq G(Tx^*, Sx^*, Sx^*) + G(Sx^*, Sy^*, Sy^*) \leq \varepsilon + G(Sx^*, Sy^*, Sy^*).$$

Now apply Theorem 6 to the map S with fixed point y^* and starting point x^*:

$$G(S^n x^*, y^*, y^*) \leq \alpha^n \sum_{k=0}^{m-1} G(S^k x^*, y^*, y^*).$$

In particular, setting $n = 1$ and estimating the sum by its first block recursion as in Theorem 1 yields a linear inequality of the form

$$G(Sx^*, y^*, y^*) \leq \alpha\, G(x^*, y^*, y^*).$$

Combining with the previous bound gives

$$G(x^*, y^*, y^*) \leq \varepsilon + \alpha\, G(x^*, y^*, y^*),$$

hence $(1-\alpha)G(x^*, y^*, y^*) \leq \varepsilon$, i.e. $G(x^*, y^*, y^*) \leq \varepsilon/(1-\alpha)$. Finally, Lemma 6 gives $d_G(x^*, y^*) \leq 3G(x^*, y^*, y^*) \leq 3\varepsilon/(1-\alpha)$. □

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