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Research article

Some Peculiarities of Bounded BF-algebras

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Abstract: In this paper, bounded BF-algebras are introduced and studied. Their properties and characterizations are investigated. Some important results and examples are given.

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Keywords: BF-algebra, bounded BF-algebra, direct product of bounded BF-algebras, (incomplete) sub-algebra, ideal, filter.

1 Introduction

In 1966, K. Iséki [7] introduced a new notion called a BCK algebra. It is an algebraic formulation of the BCK-propositional calculus. In [12], as a generalization of BCK algebras, H. S. Kim and Y. H. Kim defined BE-algebras. The notion of BCI-algebras has been introduced by K. Iséki in 1966 (see, for example, [11]). BCI-algebras are algebraic formulation of the BCI-system in combinatory logic which has application in the language of functional programming. In [6], Q. P. Hu and X.Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Then some new versions of logical algebras began to appear, such as, for example, BI-algebras (2017, [5]) and QI-algebra (2017, [1]). This author also participated in examining the properties of these two last noted logical algebras ([14], [19]).

Considering the properties of BCK-algebras in 1979, K. Iseki ([10]) raised the question of the existence of non-commutative BCK-algebras that satisfy the so-called double negation condition. Such logical algebras, i.e. bounded logical algebras that, in addition, satisfy the double negation condition, are called involutive algebras. The study of various bounded (and involutive) algebras has been the focus of several researchers. So, for example, bounded BCK-algebras are studied in [8,9] (1975) by K. Iséki. The idea of a bounded BCK-algebra, that is, a BCK-algebra $\mathfrak{A} =: (A, *, 0)$ with the unit 1 which, in addition, satisfies the condition (F): $(\forall x \in A)(x * 1 = 0)$ was the subject of research in [10] (1997) by K. Iséki and in [20] (2018) vby S. K. Shoar et al.

Then, research into the properties of other logical algebras with the added condition of boundedness began. The boundedness of BE-algebra considered in [4] (2013) by Z. Çiloğlu and Y. Çeven and in [3] (2014) by R. Borzooei et al. Bounded GE-algebras were discussed in [2] (2024) by R. K. Bandaru et al. The internal architecture of involutive WE-algebras was the focus of a paper [25] (2025) written by A. Walendziak. This author also participated in examining the properties of involutive WE-algebras ([15]). Further on, this author also introduced and analyzed the boundedness condition on QI/BH/BRK-algebras in [16–18]. The above sources should justify our interest in observing some other bounded logical algebras. One of the peculiarities of these studies is the discovery of the existence of some specific substructures in bounded algebras that are not otherwise observed in algebras of a general type.

BF-algebra was introduced in 2007 by A. Walendziak ([23], Definition 2.1).

In this article we introduce and analyze the concept of bounded BF-algebra. This introduction is not realized by simply adding the condition (F) to the axioms system that determine the concept of BF-algebra, since an inconsistency appears in one specific case. That system of axioms must be transformed in order for the condition (F) to be valid in the new circumstances. It is shown that every BF-algebra can be extended to a bounded BF-algebra. Also, in this article, substructures of (incomplete) sub-algebras in bounded BF-algebras are observed. Furthermore, it is shown that families of ideals and filters in bounded BF-algebras have a very modest number of elements, which is another specificity of this class of logical algebras. Namely, in a bounded BF-algebra $\mathfrak{A}$ there is only zero ideal $\{0\}$ as a non-trivial ideal in $\mathfrak{A}$. Further on, when considering the determination of three types of filters in the bounded BF-algebra $\mathfrak{A}$, it was found that only in the case of filters of type **0** in $\mathfrak{A}$ there is only one non-trivial filter. In the remaining two cases, there is only a trivial filter in $\mathfrak{A}$.

2 Preliminaries

This section introduces the necessary concepts and processes with them so that the potential reader can follow the presentation of the material in the third section - the main part of this article - with as little difficulty as possible.

Definition 1: *([23], Definition 2.1) A BF-algebra is an algebra* $\mathfrak{A} =: (A, *, 0)$ *of type* $(2, 0)$ *satisfying the following axioms*

(Re) $(\forall x \in A)(x * x = 0)$.

(M) $(\forall x \in A)(x * 0 = x)$.

(BF) $(\forall x, y, \in A)(0 * (x * y) = y * x)$.

Proposition 1 ([23], Proposition 2.5)**:** *If* $\mathfrak{A} =: (A, *, 0)$ *is a BF-algebra, then:*

(1) $(\forall x \in A)(0 * (0 * x) = x)$.

(2) $(\forall x, y \in A)(0 * x = 0 * y \implies x = y)$.

(3) $(\forall x, y \in A)(x * y = 0 \implies y * x = 0)$.

The concept of B-algebras was introduced by J. Neggers and H. S. Kim ([13], pp. 22). They defined a B-algebra as an algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ (i.e., a nonempty set A with a binary operation $*$ and a constant 0) satisfying (Re), (M) and the following axiom:

(B) $(\forall x, y, z \in A)((x * y) * z = x * (z * (0 * y)))$.

Proposition 2 ([22], Proposition 1.5)**:** *If $\mathfrak{A} =: (A, *, 0)$ is a B-algebra, then it is a BF-algebra.*

Example 1. Let $A = \{0, a, b, c\}$ a set and the operation $*$ given by the following table

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	0
b	b	c	0	b
c	c	0	b	0

$*_2$	0	a	b	c
0	0	a	b	c
a	a	0	a	a
b	b	a	0	a
c	c	a	a	0

$*_3$	0	a	b	c
0	0	a	b	c
a	a	0	0	0
b	b	0	0	0
c	c	0	0	0

Then $\mathfrak{A} =: (A, *, 0)$ ([23], Example 3.2), $\mathfrak{B} =: (A, *_2, 0)$ ([23], Example 3.6) and $\mathfrak{C} =: (A, *_3, 0)$ ([24], Proposition 3.7) are BF-algebras. □

3 The main results

3.1 Bounded BF-algebras

The specificity of the determination of the concept of bounded BF-algebras is reflected in the fact that definition of its is not realized by simply adding the conditions (F) to the system axioms $\{(Re), (M), (BF)\}$ that define the BF-algebras. There is a valid reason for this specificity. (See Remark 3.1)

Definition 2: *An algebra $\mathfrak{A} =: (A, *, 0)$ is a bounded BF-algebra if there is an element $1 \in A$ such that*

(F) $(\forall x \in A)(x * 1 = 0)$

and satisfying (Re), (M) and the following axiom

(bBF) $(\forall x, y \in A)((x \neq 0 \lor y \neq 1) \implies 0 * (x * y) = y * x))$.

We call the element $1 \in A$, that satisfies the condition (F), the unit in $\mathfrak{A}$. To indicate that the algebra $\mathfrak{A}$ is a bounded BF-algebra, we write $\mathfrak{A} =: (A, *, 0, 1)$.

The following proposition states some of the important properties of bounded BF-algebras.

Proposition 3: *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BF-algebra. Then:*

(4) $0 * 1 = 0$

(5) $1 * 0 = 1$.

(6) $1 * 1 = 0$.

(7) $(\forall y \in A \setminus \{1\})(0 * (0 * y) = y)$.

(8) $(\forall x, y \in A)(((x \neq 0 \lor y \neq 1) \land x * y = 0) \implies y * x = 0)$.

(9) $(\forall x \in A \setminus \{0\})(1 * x = 0)$.

Proof: (4) is a direct consequence of (F).

(5) is a direct consequence of (M).

(6) is a direct consequence of (Re).

(7): Let $y \in A \setminus \{1\}$ be an arbitrary element. Then by (bBF) and (M) we obtain $0 * (0 * y) = y * 0 = y$. This means that (7) is valid.

(8) Let's take $x, y \in A$ such that $x \neq 0 \lor y \neq 1$ and $x * y = 0$. Then $0 = 0 * 0 = 0 * (x * y) = y * x$ according to (Re) and (bBF). This gives (8).

(9) Let $x \in A$ be an arbitrary element such that $x \neq 0$. Then $0 = 0 * 0 = 0 * (x * 1) = 1 * x$ in accordance with (Re), (F) and (bBF). □

Remark. If we put $x = 0$ and $y = 1$ in (BF), we would get $0 = 0 * 0 = 0 * (0 * 1) = 1 * 0 = 1$ according to (Re), (F) which is a contradiction. This conclusion justifies the presence of the condition $x \neq 0 \lor y \neq 1$ in (dBF).

Theorem 2: *Every BF-algebra can be extended to a bounded BF-algebra.*

Proof: Let $(A, *, 0)$ be a BF-algebra and let $1 \notin A$. Let us put $A^b = A \cup \{1\}$ and define the operation $\star$ on A^b as follows:

$$x \star y = \begin{cases} x * y & x \in A \land y \in A, \\ 0 & x = y = 1, \\ 0 & x \in A \cup \{1\} \land y = 1, \\ 0 & x = 1 \land y \in A \setminus \{0\}, \\ 1 & x = 1 \land y = 0. \end{cases}$$

With a little effort it can be verified that the structure $(A^b, \star, 0, 1)$ is a bounded BF-algebra.

The conditions (Re), (M), (F) and (bBF) in the algebra $(A^b, \star, 0, 1)$ are satisfied for every $x, y \in A$, according to the determination of the operation '$\star$'. Further on, we have $1 \star 1 = 0$ by (6), $1 \star 0 = 1$ by (5) and $0 \star 1 = 0$ by (4). So (Re) and (M) are valid formulas in $(A^b, \star, 0, 1)$. Consider the validity of the formula (bBF) for $x, y \in \{0, 1\}$. For arbitrary $y \in A \setminus \{0\}$, we have $0 \star (1 \star y) = 0 * 0 = 0 = y \star 1$ according to (9), (Re) and (F). Also, for the arbitrary element $x \in A \setminus \{0\}$, we have $0 \star (x \star 1) = 0 * 0 = 0 = 1 * x$ according to (F), (Re) and (9). Finally, for $x = y = 1$, we have $0 \star (1 \star 1) = 0 * 0 = 0 = 1 \star 1$ according to (6) and (Re). We conclude that (bBF) is a valid formula in $(A^b, \star, 0, 1)$.

Therefore, $(A^b, \star, 0, 1)$ is a bounded BF-algebra. □

Example 3. Let $A = \{0, a, b, c, 1\}$ a set and the operation $*$ given by the following table

$*$	0	a	b	c	1
0	0	a	b	c	0
a	a	0	c	0	0
b	b	c	0	b	0
c	c	0	b	0	0
1	1	0	0	0	0

$*_2$	0	a	b	c	1
0	0	a	b	c	0
a	a	0	a	a	0
b	b	a	0	a	0
c	c	a	a	0	0
1	1	0	0	0	0

$*_3$	0	a	b	c	1
0	0	a	b	c	0
a	a	0	0	0	0
b	b	0	0	0	0
c	c	0	0	0	0
1	1	0	0	0	0

Then $\mathfrak{A} =: (A, *, 0, 1)$, $\mathfrak{B} =: (A, *_2, 0, 1)$ and $\mathfrak{C} =: (A, *_3, 0, 1)$ are bounded BF-algebras. □

Remark. We note here that when checking whether these algebraic structures are bounded BF-algebras, we should consider the condition (dBF) and not the condition (BF) since in each case above, we have $0 * (0 * 1) = 0 * 0 = 0 \neq 1 = 1 * 0$. Therefore, the above algebraic structures are not BF-algebras since they do not satisfy the condition (BF).

In what follows, we introduce the concept of the direct product of two bounded BF-algebras. Let $\mathfrak{A} =: (A, *_A, 0_A, 1_A)$ and $\mathfrak{B} =: (B, *_B, 0_B, 1_B)$ be two bounded BF-algebras. Let us define the operation $\odot$ on $A \times B$ in the usual way as it is done in determining the concept of direct product of logical algebras:

$$(\forall x, y \in A)(\forall y, v \in B)((x, u) \odot (y, v) =: (x *_A y, u *_B v))$$

and consider the structure $(A \times B, \odot, (0_A, 0_B), (1_A, 1_B))$ as a direct product $\mathfrak{A} \times \mathfrak{B}$ of bounded BF-algebras algebras $\mathfrak{A}$ and $\mathfrak{B}$.

It can be concluded without difficulty that the formulas (Re), (M) and (F) are valid formulas in the system $\mathfrak{A} \times \mathfrak{B}$. However, we still have:

Theorem 4: *The direct product of two bounded BF-algebras need not be a bounded BF-algebra.*

Proof: To prove this theorem, it is enough to find at least one example that confirms it. With this in mind, let us take the bounded BF-algebras $\mathfrak{A}$ and $\mathfrak{B}$ from Example 3 and check the validity of the condition (dBF) in the product $\mathfrak{A} \times \mathfrak{B}$. For elements $(0, 1)$ and $(1, 0)$, we have $(1, 0) \neq (0, 0)$ and $(1, 0) \neq (1, 1)$ and

$$\begin{aligned}
&(0, 0) \odot ((0, 1) \odot (1, 0)) = (0, 0) \odot (0 * 1, 1 *_2 0) \\
&= (0, 0) \odot (0, 1) = (0 * 0, 0 *_2 1) = (0, 0) \\
&\neq (1, 0) = (1 * 0, 0 *_2 1) = (1, 0) \odot (0, 1).
\end{aligned}$$

Therefore, the structure $\mathfrak{A} \times \mathfrak{B}$, determined as above, does not satisfy the condition (bBF). □

To see a specificity of bounded BF-algebras, the previous theorem should be compared with Theorem 2.4 in [21].

For an arbitrary element y in the bounded BF-algebra $\mathfrak{A}$ we put $1 * y = y^-$. Then $0^- = 1 * 0 = 1$ and $y^- = 1 * y = 0$ for each $y \in A \setminus \{0\}$.

3.2 Subalgebras in bounded BF-algebras

When trying to determine the substructure of sub-algebras in bounded BF-algebras, one can notice another specificity of this class of abstract algebras.

Definition 3: *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded BF-algebra. A non-empty subset* K *in* A *is called a sub-algebra in* $\mathfrak{A}$ *if it satisfies the conditions*

(K0) $1 \in K$.

(S1) $(\forall x, y \in A)((x \in K \wedge y \in K) \implies x * y \in K)$.

We denote the family of all sub-algebras of the bounded BF-algebra $\mathfrak{A}$ *by* $\mathfrak{K}(A)$.

Definition 4: *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded BF-algebra. A non-empty subset* S *in* A *is called an incomplete sub-algebra in* $\mathfrak{A}$ *if it satisfies the condition (S1) and*

(K1) $1 \notin K$.

We denote the family of all incomplete sub-algebras of the bounded BF-algebra $\mathfrak{A}$ *by* $\mathfrak{S}(A)$.

Proposition 4: *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded BF-algebra. Every (incomplete) sub-algebra* Z *in* $\mathfrak{A}$ *satisfies the condition*

(S0) $0 \in Z$.

Proof: Since Z is not empty, there exists at least some $x \in A$ such that $x \in Z$. Now, according to (S1), we have $0 = x * x \in Z$ with respect to (Re). □

Example 5. Let $\mathfrak{A} = (A, *, 0, 1)$ be a bounded BF-algebra as in Example 3. The subsets $K_0 =: \{0, 1\}$, $K_1 =: \{0, 1, a\}$, $K_2 =: \{0, 1, b\}$, $K_3 =: \{0, 1, c\}$, $K_5 =: \{0, 1, a, c\}$ and $K_6 =: \{0, 1, b, c\}$ are sub-algebras in $\mathfrak{A}$. The subset $K_4 =: \{0, 1, a, b\}$ is not a sub-algebra because, for example, we have $a * b = c \notin K_4$. So, $\mathfrak{K}(A) = \{K_0, K_1, K_2, K_3, K_5, K_6, A\}$.

Subsets $S_0 =: \{0\}$, $S_1 =: \{0, a\}$, $S_2 =: \{0, b\}$, $S_3 =: \{0, c\}$, $S_5 =: \{0, a, c\}$, $S_6 =: \{0, b, c\}$ and $S_7 =: \{0, a, b, c\}$ are incomplete sub-algebras in $\mathfrak{A}$. So, $\mathfrak{S}(A) = \{S_0, S_1, S_2, S_3, S_5, S_6, S_7\}$. □

Since the families $\mathfrak{K}(A)$ and $\mathfrak{S}(A)$ of any bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ are not empty because $K_0, \mathfrak{A} \in \mathfrak{K}(A)$ and $S_0 \in \mathfrak{S}(A)$, the following theorem can be proved:

Theorem 6: *Let* $\mathfrak{A} =: (A. *, 0, 1)$ *be a bounded BF-algebra. Families* $\mathfrak{K}(A)$ *and* $\mathfrak{S}(A)$ *form complete lattices.*

Proof: We will demonstrate the proof of this statement for the family $\mathfrak{K}(A)$. The proof of this statement for the family $\mathfrak{S}(A)$ can be shown analogously to the previous one.

Let $\{K_i\}_{i \in I}$ be a family of sub-algebras in $\mathfrak{A}$. Then:

(i) It is obvious that $0 \in \cap_{i \in I} K_i$ and $1 \in \cap_{i \in I} K_i$ because $0 \in K_i$ and $1 \in K_i$ for each $i \in I$. Let $x, y \in A$ be such that $x \in \cap_{i \in I} K_i$ and $y \in \cap_{i \in I} K_i$. This means $x \in K_i$ and $y \in K_i$ for each $i \in I$. Thus $x * y \in K_I$ for each $i \in I$ by (S1). Hence, $x * y \in \cap_{i \in I} K_i$. This shows that $\cap_{i \in I} K_i$ is a sub-algebra in $\mathfrak{A}$.

(ii) Let $\mathcal{Z}$ be the family of all sub-algebras in the bounded BF-algebra $\mathfrak{A}$ that contains the set $\cup_{i\in I}K_i$. Then $\cap\mathcal{Z}$ is a sub-algebra in $\mathfrak{A}$ according to the first part of this proof.

(iii) If we put $\sqcap_{i\in I}X_i = \cap_{i\in I}K_i$ and $\sqcup_{i\in I}K_i = \cap\mathcal{Z}$, then $(\mathfrak{K}(A), \sqcap, \sqcup)$ is a complete lattice. □

Corollary 1: *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BF-algebra and $x \in A$. Then there is a smallest (incomplete) sub-algebra S_x in $\mathfrak{A}$ containing x.*

Proof: Let $\mathcal{Z}$ be the family of all (incomplete) sub-algebras in $\mathfrak{A}$ that contain the element x. Then, by the previous theorem, $S_x =: \cap\mathcal{Z}$ is (an incomplete) a sub-algebra in $\mathfrak{A}$ containing x.

Let Y be (an incomplete) a sub-algebra in $\mathfrak{A}$ which contains x. Then $Y \in \mathcal{Z}$, so, therefore, $S_x \subseteq Y$. Therefore, S_x is the smallest (incomplete) subalgebra in $\mathfrak{A}$ containing x. □

The validity of the following proposition can be proven without difficulty:

Proposition 5: *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BF-algebra. Then*

(a) $(\forall K \in \mathfrak{K}(A))(\forall y \in A)(y \in K \implies y^- \in K)$.

(b) $(\forall S \in \mathfrak{S}(A))(\forall y \in A \setminus \{0\})(y \in S \implies y^- \in S)$.

3.3 Ideals in bounded BF-algebras

Similarly as in BCK/BCI/BH/BF-algebras, we define the notion of ideals in bounded BF-algebras.

Definition 5: *Let $\mathfrak{A} =: (A. *, 0, 1)$ be a bounded BF-algebra. A non-empty subset J of A is called an ideal in $\mathfrak{A}$ if it satisfies:*

(J0) $0 \in J$.

(J1) $(\forall x, y \in A)((x * y \in J \land y \in J) \implies x \in J)$.

We say that an ideal J in $\mathfrak{A}$ is a non-trivial ideal in $\mathfrak{A}$ if $J \neq A$ holds. The family of all ideals in the bounded BF-algebra $\mathfrak{A}$ is denoted by $\mathfrak{J}(A)$.

Proposition 6: *For every non-trivial ideal J in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ holds*

(J01) $1 \notin J$.

Proof: Suppose that $1 \in J$ for some ideal J in $\mathfrak{A}$. Then, for an arbitrary $x \in A$, we would have $x * 1 = 0 \in J$ and $1 \in J$ implies $x \in J$, which would give $A = J$. We get a contradiction since J is a non-trivial ideal in $\mathfrak{A}$. Therefore, it must be $1 \notin J$. □

Proposition 7: *The only non-trivial ideal in a bounded BF-algebra is the subset* $\{0\}$.

Proof: First, it is obvious that the subset $\{0\}$ is an ideal in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ because we have $0 \in \{0\}$ and from $x * 0 = 0$ it follows that $x = 0$ according to (M).

Suppose that the ideal J contains the element $a \neq 0$. According to (9), we have $1 * a = 0 \in J$, from where, according to (J1), it follows $1 \in J$ which contradicts (J01). So, such an element $a \in J$ does not exist. Therefore, for a non-trivial ideal J in a bounded BF-algebra $\mathfrak{A}$, we have $J = \{0\}$. □

Taking into account Proposition 7, we have $\mathfrak{J}(A) =: \{\{0\}, A\}$.

3.4 Filters in bounded BF-algebras

In this subsection we will consider several candidates for the position of filters in bounded BF-algebras.

3.4.1 Filter of type 0

A filter of class **0** in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ is determined by the standard way:

Definition 6: *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BF-algebra. We call a non-empty subset F in A a filter of type **0** in $\mathfrak{A}$ if it holds*

(F1) $(\forall x, y \in A)((x \in F \wedge x * y \in F) \implies y \in A)$.

*For a filter F of type **0** in $\mathfrak{A}$ we say it is nontrivial if $F \neq A$. We denote the family of all filters of type **0** in the bounded BF-algebra $\mathfrak{A}$ by $\mathfrak{F}_0(A)$.*

Proposition 8: *For every filter F of type **0** in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ holds*

(F0) $F_0 =: \{0, 1\} \subseteq F$.

Proof: Since F is a non-empty subset in A, there exists at least some $x \in A$ such that $x \in F$. Now, from $x \in F$ and $x * 0 = x \in F$, in accordance with (M), it follows $0 \in F$ by (F1). Further on, from $x \in F$ and $x * 1 = 0 \in F$ (according to (F)), we have $1 \in F$ by (F1). □

Proposition 9: *Only non-trivial filter of type **0** in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ is just F_0.*

Proof: Let F be a filter of type **0** in a bounded BF-algebra $\mathfrak{A}$. For arbitrary $x \in A \setminus \{0\}$, from $1 \in F$ and $1 * x = 0 \in F$, we get $x \in F$ in accordance with (F1), Therefore, $A \subseteq F$ since $0 \in F$ according to (F0). Since F is a non-trivial filter of type **0** in $\mathfrak{A}$, it must be $F = F_0$. □

3.4.2 Filter of type 1

A filter of class **1** in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ is determined by the following way:

Definition 7: *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BF-algebra. We call a non-empty subset F in A a filter of type **1** in $\mathfrak{A}$ if it holds*

(F2) $(\forall x, y \in A)((x \in F \wedge x \in F) \implies (x * (x * y) \in F \wedge y * (y * x) \in F))$.

(F02) $(\forall x, y \in A)((x \in F \wedge x * y = 0) \implies y \in F)$.

*For a filter F of type **1** in $\mathfrak{A}$ we say it is non-trivial if $F \neq A$. We denote the family of all filters of type **1** in the bounded BF-algebra $\mathfrak{A}$ by $\mathfrak{F}_1(A)$.*

Before we make any conclusions about a filter F of type **1** in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$, let's do a little analysis:

Suppose that $0 \in F$. Then, for any $y \in A \setminus \{0\}$ it should be: $0 * (0 * 0) = 0 * 0 = 0$ and $0 * y = 0 \implies y \in F$. The option $0 * y = 0$ is possible, among other things, for $y = 0$ and, in accordance with (F), $y = 1$. Therefore, there should also be $1 \in F$. Thus, the element 1 should satisfy the condition (F02). This means that there should be $1 * y = 0 \wedge y \neq 0 \implies y \in F$. However, since, according to (9), $1 * y = 0$ holds for every $y \in A \setminus \{0\}$, we conclude that $F = A$. We have a contradiction, since F is not a trivial filter in $\mathfrak{A}$. Therefore, $0 \notin F$. On the other hand, since F is a nonempty subset of A, there exists at least some $x \in A$ such that $x \in F$. In this case, according to (F02), we have that $x \in F$ and $x * 1 = 0$ implies $1 \in F$. Then, again according to (F02), we would have that $1 \in F$ and $1 * 0 = 1 \in F$ gives $0 \in F$. Again we have a contradiction.

Summarizing the previous analysis, we conclude:

Proposition 10: *A bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ does not have a single non-trivial filter F of type **1**. This means $\mathfrak{F}_1(A) = \{A\}$.*

3.4.3 ***Filter of type 2***

A filter of class **2** in a bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ is determined by the following way:

Definition 8: *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BF-algebra. We call a non-empty subset F in A a filter of type **2** in $\mathfrak{A}$ if it holds*

(F3) $(\forall x, y \in A)((x \in F \wedge x \in F) \implies ((x * y) * y \in F \wedge (y * x) * x \in F))$.

(F02) $(\forall x, y \in A)((x \in F \wedge x * y = 0) \implies y \in F)$.

*For a filter F of type **2** in $\mathfrak{A}$ we say it is non-trivial if $F \neq A$. We denote the family of all filters of type **2** in the bounded BF-algebra $\mathfrak{A}$ by $\mathfrak{F}_2(A)$.*

Analogous to the case of filters of type **1** in a bounded BF-algebra $\mathfrak{A}$, it can be proved that:

Proposition 11: *A bounded BF-algebra $\mathfrak{A} =: (A, *, 0, 1)$ does not have a single non-trivial filter F of type **2**. This means $\mathfrak{F}_2(A) = \{A\}$.*

Final comments

Logical BF-algebra, as a generalization of B-algebra, was introduced in 2007 by A. Walendziak by means of the standard recognizable axioms (Re) and (M) with the addition of a special axiom (BF). In this article, we introduce the concept of bounded BF-algebras not only by adding the axiom (F), which normally determines the boundedness condition, but also by using the axiom (bBF) instead of the axiom (BF). This described procedure is not the only specificity of bounded BF-algebras. The internal architecture of bounded

BF-algebras is very modest since we can recognize in them only a richer family of (incomplete) sub-algebras. The number of substructures in families of ideals and filters in them is almost negligible.

Since this article is the first of its kind, the author is convinced that it opens up at least a small space for the research of these logical algebras.

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References

1. R. K. Bandaru, On QI-algebras, *Discuss. Math. Gen. Algebra Appl.* 37 **(2017)**, 137–137. https://doi.org/10.7151/dmgaa.1269.
2. R. K. Bandaru, M. A. Öztürk, and Y. B. Jun, Bordered GE-algebras, *J. Algebraic Syst.* 12(1) **(2024)**, 43–58. https://doi.org/10.22044/JAS.2022.11184.1558.
3. R. Borzooei, A. Borumand Saeid, R. Ameri, and A. Rezaei, Involutory BE-algebras, *J. Math. Appl.* 37 **(2014)**, 13–26.
4. Çiloğlu, Z. and Çeven, Y., Commutative and bounded BE-algebras, *Algebra* 2013 **(2013)**, Article ID 473714. https://doi.org/10.1155/2013/473714.
5. S. Arsham Borumand, K. H. Sik, and R. Akbar, On BI-algebras, *An. St. Univ. Ovidius Constanta Ser. Math.* 25(1) **(2017)**, 177–194. https://doi.org/10.1515/auom-2017-0014.
6. Q. P. Hu and X. Li, On BCH-algebras, *Math. Semin. Notes Kobe Univ.* 11(2) **(1983)**, 313–320.
7. K. Iséki, An algebra related with a propositional calculus, *Proc. Japan Acad.* 42(1) **(1966)**, 26–29. https://doi.org/10.3792/pja/1195522171.
8. K. Iséki, On bounded BCK-algebras, *Math. Semin. Notes Kobe Univ.* 3 **(1976)**, 23–33.
9. K. Iséki, On bounded BCK-algebras, II, *Math. Semin. Notes Kobe Univ.* 3 **(1976)**, 34–36.
10. K. Iséki, On some problems on BCK-algebras, *Math. Semin. Notes Kobe Univ.* 7 **(1979)**, 173–178.
11. K. Iséki, On BCI-algebras, *Math. Semin. Notes Kobe Univ.* 8(1) **(1980)**, 125–130.
12. H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.* 66(1) **(2007)**, 113–128. https://doi.org/10.32219/isms.66.1-113.
13. J. Neggers and H. S. Kim, On B-algebras, *Mat. Vesnik* 54(1) **(2002)**, 21–29.
14. D. A. Romano, Right distributive BI-algebras, *Ann. Commun. Math.* 7(3) **(2024)**, 254–263. https://doi.org/10.6202/acm.2024.070304.
15. D. A. Romano, Internal architecture of involutive WE-algebras, *Asian-Eur. J. Math.* **(Year)**. https://doi.org/10.1142/S179355712550127X.
16. D. A. Romano, Bounded and involutive QI-algebras, *Asian-Eur. J. Math.* **(Year)**. https://doi.org/10.1142/S1793557125501554.
17. D. A. Romano, On bounded BH-algebras, *Asian-Eur. J. Math.* **(Year)**. https://doi.org/10.1142/S1793557125501505.

18. D. A. Romano, On bounded and involutive BRK-algebras, *Ann. Univ. Mariae Curie-Sklodowska Sect. A Math.* 79(2) **(2025)**, 69–87. https://doi.org/10.17951/a.2025.79.2.
19. D. A. Romano, Some classes of implicative ideals in BI-algebras, *Kragujevac J. Math.* 51(1) **(2027)**, 23–33.
20. S. K. Shoar, R. A. Borzooei, R. Moradian, and A. Radfar, PC-lattices: A class of bounded BCK-algebras, *Bull. Sect. Logic* 47(1) **(2018)**, 33–44. https://doi.org/10.18778/0138-0680.47.1.03.
21. R. C. Teves and J. C. Endam, Direct product of BF-algebras, *Int. J. Algebra* 10(3) **(2016)**, 125–132. https://doi.org/10.12988/ija.2016.614.
22. A. Walendziak, A note on normal subalgebras in B-algebras, *Sci. Math. Jpn.* 62 **(2005)**, 49–53.
23. A. Walendziak, On BF-algebras, *Math. Slovaca* 57(2) **(2007)**, 119–128. https://doi.org/10.2478/sl2175-007-0003-x.
24. A. Walendziak, BG/BF_1/B/BM-algebras are congruence permutable, *Math. Aeterna* 5(2) **(2015)**, 351–356.
25. A. Walendziak, On involutive weak exchange algebras, *Bull. Sect. Logic* 54(3) **(2025)**, 383–406. https://doi.org/10.18778/0138-0680.2025.13.