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Research article

t-secure hop domination in graphs

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Abstract: Hop domination was introduced as a distance-two analogue of domination and has been studied extensively in recent years. A secure hop dominating set, recently introduced, models a single adversarial attack at an unoccupied vertex (a vertex not in the current guard set) that can be defended by relocating one guard at distance two while preserving hop domination. Motivated by finite-order (multi-step) protection in classical secure domination, we introduce *t-secure hop dominating sets* ($t \in \mathbb{N}_0$), in which an adversary may launch a sequence of at most t attacks, each at a currently unoccupied vertex, and the defender responds by sequentially relocating one guard at distance two after each attack while maintaining hop domination throughout. Our main contribution is an exact correspondence: t-secure hop domination in a graph G is equivalent to smart t-secure domination in the hop graph $H(G)$. This yields structural properties (monotonicity and additivity over components) and exact values for several graph families, including complete multipartite graphs, stars, paths, and cycles. In particular, we obtain closed formulas for $\gamma_{sh,t}(P_n)$ and $\gamma_{sh,t}(C_n)$ for all $t \in \mathbb{N}_0$, with explicit small-n exceptions in the cycle case.

Mathematics Subject Classification: 05C69, 05C12
Keywords: hop domination; secure hop domination; finite-order domination; hop graph.

1 Introduction

Hop domination is a distance-two variant of domination in graphs introduced by Natarajan and Ayyaswamy [7] and further developed in subsequent work, including probabilistic and extremal aspects [5]. Recently, the notion of *secure hop domination* was introduced to model a single attack at a vertex not occupied by a guard, where a nearby guard (at distance two) can be relocated to defend the attack while preserving the hop domination property [1]. This extends the classical concept of secure domination (and related protection parameters) that has been studied broadly (see, for instance, [4,6]). Related hop-based protection variants have also been explored recently; see, for example, [3].

In many protection settings, it is natural to consider not only one attack, but a sequence of attacks over time. In classical domination theory, Burger *et al.* introduced *finite-order* (smart k-secure) domination and determined exact values for paths and cycles in their setting [2]. In this paper, we adapt this finite-order viewpoint to hop domination, under the standing convention that attacks occur only at vertices not currently occupied by a guard.

We introduce *t-secure hop dominating sets* and the associated parameter $\gamma_{sh,t}(G)$. Our main results are as follows.

- We prove that t-secure hop domination in G is precisely smart t-secure domination in the hop graph $H(G)$, hence $\gamma_{sh,t}(G) = \gamma_{s,t}(H(G))$.
- We establish basic properties: $\gamma_{sh,0}(G) = \gamma_h(G)$, monotonicity in t, and additivity over components.
- We compute $\gamma_{sh,t}$ exactly for complete multipartite graphs and stars.
- Using the structure of hop graphs of paths and cycles together with finite-order domination formulas from the literature, we derive explicit closed forms for $\gamma_{sh,t}(P_n)$ for all $n \geq 1$ and for $\gamma_{sh,t}(C_n)$ with explicit small-n exceptions.

2 Preliminaries and notation

All graphs considered are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets. For vertices $u, v \in V(G)$, let $0em_G(u, v)$ denote their graph distance. The open neighborhood of v is $N_G(v)$, and the *distance-two neighborhood* of v is

$$N^2_G(v) = \{u \in V(G) : 0em_G(u, v) = 2\}.$$

We write $n = |V(G)|$. For an arbitrary graph X, we write $N_X(v)$ for the open neighborhood of v in X.

2.1 Hop graph and hop domination

Definition 1 (Hop graph)**:** *The* hop graph *of G, denoted $H(G)$, is the graph with vertex set $V(H(G)) = V(G)$ and*

$$uv \in E(H(G)) \iff 0em_G(u, v) = 2.$$

Definition 2 (Hop dominating set [7])**:** *A set $S \subseteq V(G)$ is a* hop dominating set *of G if for every $v \in V(G) \setminus S$ there exists $u \in S$ such that $0em_G(u, v) = 2$. The minimum cardinality of a hop dominating set is the* hop domination number *$\gamma_h(G)$.*

Lemma 1: *A set $S \subseteq V(G)$ is a hop dominating set of G if and only if S is a dominating set of the hop graph $H(G)$. Consequently,*

$$\gamma_h(G) = \gamma(H(G)),$$

where $\gamma(\cdot)$ denotes the classical domination number.

2.2 Secure hop domination and finite-order security

Definition 3 (Secure hop dominating set [1])**:** *A hop dominating set S of G is* secure hop dominating *if for every unoccupied vertex $v \in V(G) \setminus S$ there exists a vertex $w \in S \cap N_G^2(v)$ such that $(S \setminus \{w\}) \cup \{v\}$ is a hop dominating set of G. The minimum cardinality of a secure hop dominating set is denoted $\gamma_{sh,1}(G)$ (often written $\gamma_{sh}(G)$ in [1]).*

Definition 4 (t-secure hop dominating set)**:** *Let $t \in \mathbb{N}_0$. A set $S \subseteq V(G)$ is* t-secure hop dominating *if S is hop dominating and the following holds.*

Start with $S_0 = S$. For every integer r with $0 \le r \le t$ and every sequence of attacks $(v_1, \ldots, v_r)$ satisfying $v_i \in V(G) \setminus S_{i-1}$ for each $i = 1, \ldots, r$, the defender can choose vertices $w_i \in S_{i-1} \cap N_G^2(v_i)$ and update

$$S_i = (S_{i-1} \setminus \{w_i\}) \cup \{v_i\} \qquad (i = 1, \ldots, r),$$

so that each S_i is a hop dominating set of G. The minimum cardinality of a t-secure hop dominating set of G is denoted $\gamma_{sh,t}(G)$.

Remark 2. By definition, $\gamma_{sh,0}(G) = \gamma_h(G)$ and $\gamma_{sh,1}(G)$ is the secure hop domination number introduced in [1]. In all security notions used in this paper, attacks are allowed only at unoccupied vertices (vertices not currently in the guard set).

3 Core correspondence with finite-order secure domination

Definition 5 (Smart t-secure domination)**:** *Let X be a graph and let $t \in \mathbb{N}_0$. A dominating set $D \subseteq V(X)$ is* smart t-secure dominating *if, starting from $D_0 = D$, the following holds: for every integer r with $0 \le r \le t$ and every sequence $(x_1, \ldots, x_r)$ with $x_i \in V(X) \setminus D_{i-1}$, there exist defenders $y_i \in D_{i-1} \cap N_X(x_i)$ such that*

$$D_i = (D_{i-1} \setminus \{y_i\}) \cup \{x_i\} \qquad (i = 1, \ldots, r)$$

and D_i is a dominating set of X for all $i = 1, \ldots, r$. The minimum size of such a set is denoted $\gamma_{s,t}(X)$.

Remark 3. Different sources phrase finite-order domination games with slightly different conventions (for instance, whether one explicitly restricts attacks to currently unoccupied vertices, or whether an attack at an occupied vertex is treated as vacuous). In this paper we fix the unoccupied attack convention as part of the definition (Definition 5). When we invoke closed formulas from the literature (notably [2]), we interpret them in the standard discrete sense that the right-hand side represents the unique integer forced by the stated ratio; see Lemma 15 below.

Theorem 4 (Hop-graph equivalence)**:** *Let G be a graph and $t \in \mathbb{N}_0$. A set $S \subseteq V(G)$ is t-secure hop dominating in G if and only if S is smart t-secure dominating in the hop graph $H(G)$. Consequently,*

$$\gamma_{sh,t}(G) = \gamma_{s,t}(H(G)).$$

Proof: Fix $t \in \mathbb{N}_0$ and let $S \subseteq V(G)$.

By Lemma 1, S is hop dominating in G if and only if S is dominating in $H(G)$.

Assume first that S is t-secure hop dominating in G. Let r satisfy $0 \le r \le t$ and let $(x_1, \dots, x_r)$ be any attack sequence in $H(G)$ with $x_i \in V(H(G)) \setminus S_{i-1}$. Interpreting the same vertices in G, we have $x_i \notin S_{i-1}$ as well. Since S is t-secure hop dominating, there exist defenders $w_i \in S_{i-1} \cap N_G^2(x_i)$ such that $S_i = (S_{i-1} \setminus \{w_i\}) \cup \{x_i\}$ remains hop dominating in G for each $i = 1, \dots, r$. The condition $w_i \in N_G^2(x_i)$ is equivalent to $w_i \in N_{H(G)}(x_i)$, and S_i hop dominates G if and only if S_i dominates $H(G)$. Thus the same defender choices certify that S is smart t-secure dominating in $H(G)$.

Conversely, assume that S is smart t-secure dominating in $H(G)$. Let r satisfy $0 \le r \le t$ and let $(v_1, \dots, v_r)$ be any attack sequence in G with $v_i \in V(G) \setminus S_{i-1}$. In $H(G)$, each v_i is a vertex not in S_{i-1} as well. Since S is smart t-secure dominating in $H(G)$, there exist defenders $w_i \in S_{i-1} \cap N_{H(G)}(v_i)$ such that $S_i = (S_{i-1} \setminus \{w_i\}) \cup \{v_i\}$ dominates $H(G)$ for each $i = 1, \dots, r$. The condition $w_i \in N_{H(G)}(v_i)$ is equivalent to $0em_G(w_i, v_i) = 2$, that is, $w_i \in N_G^2(v_i)$ in G. Finally, S_i dominates $H(G)$ if and only if S_i hop dominates G. Hence S is t-secure hop dominating in G.

Taking minima over S yields $\gamma_{sh,t}(G) = \gamma_{s,t}(H(G))$. □

4 Basic properties

Proposition 5 (Monotonicity and bounds)**:** *Let G be a graph and $t \in \mathbb{N}_0$.*

(a) $\gamma_{sh,0}(G) = \gamma_h(G)$.

(b) $\gamma_{sh,t}(G) \le \gamma_{sh,t+1}(G) \le |V(G)|$.

(c) $\gamma_h(G) \le \gamma_{sh,t}(G)$ *for all* $t \in \mathbb{N}_0$.

Proof: Part (a) is immediate from Definition 4. For (b), any $(t+1)$-secure hop dominating set must defend every attack sequence of length at most $t+1$, and hence in particular every attack sequence of length at most t; thus it is also t-secure, yielding the first inequality. The second inequality holds since $V(G)$ is always hop dominating. Part (c) follows because every t-secure hop dominating set is, in particular, hop dominating. □

Proposition 6 (Additivity over components)**:** *Let G and F be graphs and let $t \in \mathbb{N}_0$. Then*

$$\gamma_{sh,t}(G \cup F) = \gamma_{sh,t}(G) + \gamma_{sh,t}(F),$$

where $G \cup F$ denotes the disjoint union.

Proof: Let $X = G \cup F$ and assume $V(G) \cap V(F) = \emptyset$. Then no path joins a vertex of G to a vertex of F, so for $u \in V(G)$ and $v \in V(F)$ one has $0em_X(u, v) = \infty$ and in particular $0em_X(u, v) \neq 2$. Hence for any vertex z the distance-two neighborhood $N_G^2(z)$ in X is contained in the component of z. In particular, a set hop dominates X if and only if it hop dominates each component.

For $\le$, let S_G and S_F be t-secure hop dominating sets of minimum sizes in G and F, and set $S = S_G \cup S_F \subseteq V(X)$. Then S hop dominates X. Consider any attack sequence in X of length $r \le t$ at unoccupied vertices.

Each attack occurs in exactly one component, and every admissible defender move must occur in that same component. Defend componentwise using the corresponding strategy in G or F, leaving the other component unchanged. After each defense, hop domination is preserved within each component and hence in X, so S is t-secure hop dominating in X. Therefore $\gamma_{sh,t}(X) \leq |S| = |S_G| + |S_F|$.

For $\geq$, let S be any t-secure hop dominating set of X, and put $S_G = S \cap V(G)$ and $S_F = S \cap V(F)$. Consider an arbitrary attack sequence of length $r \leq t$ in G at vertices unoccupied relative to the evolving set in G, and view the same sequence as an attack sequence in X. Since S is t-secure hop dominating in X, each such attack can be defended by moving a guard from distance two in X, and every such defender must lie in $V(G)$. Hence the induced evolution on $V(G)$ gives a valid defense in G and preserves hop domination of G after each step. Thus S_G is t-secure hop dominating in G, so $|S_G| \geq \gamma_{sh,t}(G)$; similarly $|S_F| \geq \gamma_{sh,t}(F)$. Therefore

$$|S| = |S_G| + |S_F| \geq \gamma_{sh,t}(G) + \gamma_{sh,t}(F),$$

and minimizing over S yields the result. □

Proposition 7 (Additivity for smart t-secure domination)**:** *Let X and Y be graphs and let $t \in \mathbb{N}_0$. Then*

$$\gamma_{s,t}(X \cup Y) = \gamma_{s,t}(X) + \gamma_{s,t}(Y),$$

where $X \cup Y$ denotes the disjoint union.

Proof: Let $Z = X \cup Y$ and assume $V(X) \cap V(Y) = \emptyset$. Then no edge of Z joins X to Y, so for each $z \in V(Z)$ the neighborhood $N_Z(z)$ is contained in the component of z. Hence a set dominates Z if and only if it dominates each component.

For $\leq$, let D_X and D_Y be smart t-secure dominating sets of minimum sizes in X and Y, and let $D = D_X \cup D_Y \subseteq V(Z)$. Then D dominates Z. Consider any attack sequence in Z of length $r \leq t$ at vertices unoccupied relative to the evolving set in Z. Each attack occurs in exactly one component, and every admissible defender move must occur in that component. Defend componentwise using the corresponding strategy in X or Y, leaving the other component unchanged. After each defense, domination is preserved in the attacked component and hence in Z. Thus D is smart t-secure dominating in Z, and $\gamma_{s,t}(Z) \leq |D| = |D_X| + |D_Y|$.

For $\geq$, let D be any smart t-secure dominating set of Z, and put $D_X = D \cap V(X)$ and $D_Y = D \cap V(Y)$. Consider any attack sequence in X of length $r \leq t$ at vertices unoccupied relative to the evolving set in X, and view it as an attack sequence in Z. Since D is smart t-secure dominating in Z, each attack admits a defender in $D \cap N_Z(\cdot)$, which must lie in $V(X)$. Hence the induced evolution on $V(X)$ gives a valid defense in X and preserves domination of X after each step. Thus D_X is smart t-secure dominating in X, so $|D_X| \geq \gamma_{s,t}(X)$; similarly $|D_Y| \geq \gamma_{s,t}(Y)$. Therefore

$$|D| = |D_X| + |D_Y| \geq \gamma_{s,t}(X) + \gamma_{s,t}(Y),$$

and minimizing over D yields the result. □

5 Exact values for complete multipartite graphs and stars

Theorem 8: *Let $K_{n_1,n_2,\ldots,n_r}$ be a complete multipartite graph with $r \geq 2$ parts, where each $n_i \geq 1$. Then for every $t \in \mathbb{N}_0$,*

$$\gamma_{sh,t}\left(K_{n_1,n_2,\ldots,n_r}\right) = r.$$

In particular, $\gamma_{sh,t}(K_{m,n}) = 2$ for all $m, n \geq 1$ and $t \in \mathbb{N}_0$.

Proof: In a complete multipartite graph, two vertices are at distance 2 if and only if they lie in the same part. Hence the hop graph $H\left(K_{n_1,\ldots,n_r}\right)$ is the disjoint union of cliques:

$$H\left(K_{n_1,\ldots,n_r}\right) = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}.$$

In a clique K_m, a single vertex dominates the graph and is smart t-secure dominating for every $t \in \mathbb{N}_0$: if the adversary attacks any unoccupied vertex, the unique guard can move to the attacked vertex along an edge and the resulting singleton set still dominates the clique. Thus $\gamma_{s,t}(K_m) = 1$ for all $m \geq 1$ and $t \in \mathbb{N}_0$.

Since components are disjoint, a dominating set in $K_{n_1} \cup \cdots \cup K_{n_r}$ must contain at least one vertex from each clique; conversely, choosing one vertex in each clique yields a smart t-secure dominating set by defending attacks inside each clique independently. Hence

$$\gamma_{s,t}\left(K_{n_1} \cup \cdots \cup K_{n_r}\right) = \sum_{i=1}^{r} \gamma_{s,t}(K_{n_i}) = \sum_{i=1}^{r} 1 = r.$$

Finally, Theorem 4 yields $\gamma_{sh,t}(K_{n_1,\ldots,n_r}) = r$. □

Corollary 9: *Let $S_n = K_{1,n-1}$ be the star on $n \geq 2$ vertices. Then for every $t \in \mathbb{N}_0$,*

$$\gamma_{sh,t}(S_n) = 2.$$

Proof: This is Theorem 8 with $r = 2$, $n_1 = 1$, $n_2 = n - 1$. □

6 Hop graphs of paths and cycles

We label the vertices of the path P_n as $1, 2, \ldots, n$ in order, and the vertices of the cycle C_n as $1, 2, \ldots, n$ modulo n. We use the convention that P_0 denotes the empty graph (with no vertices); in particular, $\gamma_{s,t}(P_0) = 0$ for all $t \in \mathbb{N}_0$.

Lemma 10: *For $n \geq 1$,*

$$H(P_n) \cong P_{\lceil n/2 \rceil} \ \cup \ P_{\lfloor n/2 \rfloor},$$

where P_0 is interpreted as the empty graph.

Proof: In P_n, $0em_{P_n}(i,j) = |i-j|$, so $0em_{P_n}(i,j) = 2$ if and only if $|i-j| = 2$. Hence in $H(P_n)$, the edges are exactly $i(i+2)$ for $1 \le i \le n-2$. This preserves parity: odd vertices connect only to odd vertices, and even vertices connect only to even vertices. The induced subgraph on odd indices $(1, 3, 5, \dots)$ is a path, and similarly for even indices $(2, 4, 6, \dots)$. The number of odd indices is $\lceil n/2 \rceil$ and the number of even indices is $\lfloor n/2 \rfloor$. Therefore $H(P_n)$ is the disjoint union of these two paths (and when $n = 1$, the even-index component is P_0). □

Lemma 11: *Let $n \ge 3$. Then*

$$H(C_n) \cong \begin{cases} 3K_1, & \text{if } n = 3, \\ 2K_2, & \text{if } n = 4, \\ C_n, & \text{if } n \ge 5 \text{ is odd}, \\ C_{n/2} \ \cup \ C_{n/2}, & \text{if } n \ge 5 \text{ is even}. \end{cases}$$

Proof: For $n = 3$, every pair of distinct vertices in C_3 is adjacent, so there are no pairs at distance 2; hence $H(C_3)$ has no edges and is $3K_1$.

For $n = 4$, the only pairs at distance 2 are opposite vertices, so $H(C_4)$ consists of the two disjoint edges joining opposite pairs, that is, $2K_2$.

Assume now that $n \ge 5$. In C_n, we have $0em_{C_n}(i,j) = 2$ if and only if $j \equiv i \pm 2 \pmod{n}$. Thus every vertex i is adjacent in $H(C_n)$ to $i+2$ and $i-2 \pmod{n}$, so $H(C_n)$ is 2-regular.

If n is odd, then $\gcd(2,n) = 1$, so adding 2 generates all residues mod n. Hence the step-size-2 graph is connected and 2-regular, and therefore isomorphic to a single cycle of length n, namely C_n.

If n is even, then $\gcd(2,n) = 2$, so the step-size-2 action splits $\mathbb{Z}_n$ into two parity classes. Each class induces a 2-regular connected subgraph on $n/2$ vertices, hence a cycle $C_{n/2}$. Therefore $H(C_n) \cong C_{n/2} \cup C_{n/2}$ for even $n \ge 5$. □

Lemma 12: *For every $t \in \mathbb{N}_0$, one has $\gamma_{s,t}(C_3) = 1$ under Definition 5.*

Proof: Let C_3 have vertices $\{1,2,3\}$. Any singleton $D = \{i\}$ dominates C_3. For any attack at an unoccupied vertex $x \in \{1,2,3\} \setminus D$, the unique occupied vertex i is adjacent to x, so the defender moves from i to x, yielding the singleton set $\{x\}$, which again dominates C_3. This can be repeated for any number of steps, so D is smart t-secure dominating for all $t \in \mathbb{N}_0$, and $\gamma_{s,t}(C_3) = 1$. □

Remark 13. Burger *et al.* [2] determine exact values for finite-order domination on paths and cycles under a formulation in which an "attack" at an already occupied vertex is treated as having no effect. To transfer their closed forms to the conventions used here, we introduce an "extended" variant encoding vacuous occupied-vertex steps and then show it agrees with $\gamma_{s,t}$. In addition, some sources state the final answers as rational multiples of n without explicitly writing a ceiling. Since the left-hand side is an integer parameter, this is understood as the least integer meeting the corresponding inequality; Lemma 15 records the equivalence with a ceiling.

Definition 6 (Extended smart t-secure domination)**:** *Let X be a graph and let $t \in \mathbb{N}_0$. A dominating set $D \subseteq V(X)$ is* extended smart t-secure dominating *if, starting from $D_0 = D$, for every sequence of vertices $(x_1, \dots, x_t)$ with $x_i \in V(X)$, one can define sets D_i recursively as follows. If $x_i \in D_{i-1}$, then the ith step is declared* vacuous *and we set $D_i = D_{i-1}$. If $x_i \notin D_{i-1}$, then the defender chooses a vertex $y_i \in D_{i-1} \cap N_X(x_i)$ and updates*

$$D_i = (D_{i-1} \setminus \{y_i\}) \cup \{x_i\},$$

such that D_i is a dominating set of X. The minimum size of an extended smart t-secure dominating set of X is denoted $\gamma^{\text{ext}}_{s,t}(X)$.

Lemma 14: *For every graph X and every $t \in \mathbb{N}_0$,*

$$\gamma^{\text{ext}}_{s,t}(X) = \gamma_{s,t}(X).$$

Proof: Fix a graph X and $t \in \mathbb{N}_0$.

Assume first that D is smart t-secure dominating in the sense of Definition 5. Let $(x_1, \dots, x_t)$ be an arbitrary sequence of vertices in $V(X)$ for the extended game, and set $D_0 = D$. We define a defense inductively in the extended game. At step i, if $x_i \in D_{i-1}$ we set $D_i = D_{i-1}$ (vacuous step). If $x_i \notin D_{i-1}$, then this is an unoccupied attack. Let j be the number of nonvacuous steps among $1, \dots, i$; the attacked vertices at those nonvacuous steps form a sequence $(u_1, \dots, u_j)$ of length $j \leq t$ in which each u_k is unoccupied at the moment it is attacked. By the smart t-secure property, D can be defended against every such unoccupied attack sequence of length at most t, and in particular against $(u_1, \dots, u_j)$. Hence there exists a neighbor of $x_i = u_j$ in the current set D_{i-1} whose relocation to x_i preserves domination. Choose such a defender and perform the required update to obtain D_i. This yields a valid extended defense for the full sequence $(x_1, \dots, x_t)$. Therefore D is extended smart t-secure dominating, and so $\gamma^{\text{ext}}_{s,t}(X) \leq \gamma_{s,t}(X)$.

Conversely, assume that D is extended smart t-secure dominating. Let r satisfy $0 \leq r \leq t$, and let $(u_1, \dots, u_r)$ be any unoccupied attack sequence in the sense of Definition 5, starting from $D_0 = D$. If $r = 0$, there is nothing to prove. If $r \geq 1$, consider the length-t sequence

$$(u_1, u_2, \dots, u_r, \underbrace{u_r, \dots, u_r}_{t-r \text{ times}}).$$

By extended t-security, this sequence admits a defense. After the rth step, the update rule forces $u_r \in D_r$, regardless of the chosen defender. Hence each of the subsequent occurrences of u_r is an occupied-vertex attack and therefore vacuous under the extended rules, so the defended steps $1, \dots, r$ constitute a valid defense for the original unoccupied attack sequence $(u_1, \dots, u_r)$. Since this holds for every $r \leq t$, the set D is smart t-secure dominating. Therefore $\gamma_{s,t}(X) \leq \gamma^{\text{ext}}_{s,t}(X)$.

Combining the two inequalities yields $\gamma^{\text{ext}}_{s,t}(X) = \gamma_{s,t}(X)$. □

Lemma 15: *Let $a, b \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. Then*

$$\min\{m \in \mathbb{N}_0 \ : \ bm \geq an\} \ = \ \left\lceil \frac{a}{b}\, n \right\rceil.$$

Proof: Let $x = \frac{a}{b}n$. By definition of the ceiling function, $\lceil x \rceil$ is the least integer m such that $m \geq x$, which is equivalent to $bm \geq an$. Hence $\lceil x \rceil$ is exactly the minimum on the left-hand side. □

Theorem 16 (Finite-order domination on paths and cycles [2])**:** *Let $t \in \mathbb{N}_0$.*

(a) *For $n \geq 1$,*

$$\gamma_{s,t}^{\mathrm{ext}}(P_n) = \min\{m \in \mathbb{N} \ : \ (4t+3)m \geq (2t+1)n\} = \left\lceil \frac{(2t+1)n}{4t+3} \right\rceil.$$

(b) *For $n \geq 4$,*

$$\gamma_{s,t}^{\mathrm{ext}}(C_n) = \min\{m \in \mathbb{N} \ : \ (4t+3)m \geq (2t+1)n\} = \left\lceil \frac{(2t+1)n}{4t+3} \right\rceil.$$

Proof: The stated values are given in [2] for the finite-order (smart) secure domination parameters on paths and cycles. The ceiling form follows from Lemma 15. □

By Lemma 14, the same closed forms in Theorem 16 hold for $\gamma_{s,t}$ as defined in Definition 5.

Theorem 17: *Let $n \geq 1$ and $t \in \mathbb{N}_0$. Then*

$$\gamma_{sh,t}(P_n) = \left\lceil \frac{2t+1}{4t+3} \left\lceil \frac{n}{2} \right\rceil \right\rceil + \left\lceil \frac{2t+1}{4t+3} \left\lfloor \frac{n}{2} \right\rfloor \right\rceil,$$

where the second term is 0 *when* $\lfloor n/2 \rfloor = 0$.

Proof: By Lemma 10 and Theorem 4,

$$\gamma_{sh,t}(P_n) = \gamma_{s,t}(H(P_n)) = \gamma_{s,t}\big(P_{\lceil n/2 \rceil} \cup P_{\lfloor n/2 \rfloor}\big).$$

Since $P_{\lceil n/2 \rceil}$ and $P_{\lfloor n/2 \rfloor}$ are disjoint components (with P_0 empty when $\lfloor n/2 \rfloor = 0$), Proposition 7 gives

$$\gamma_{s,t}\big(P_{\lceil n/2 \rceil} \cup P_{\lfloor n/2 \rfloor}\big) = \gamma_{s,t}\big(P_{\lceil n/2 \rceil}\big) + \gamma_{s,t}\big(P_{\lfloor n/2 \rfloor}\big).$$

Apply Theorem 16(a) together with Lemma 14 to each component (and use $\gamma_{s,t}(P_0) = 0$). □

Theorem 18: *Let $t \in \mathbb{N}_0$ and $n \geq 3$. Then*

$$\gamma_{sh,t}(C_n) = \begin{cases} 3, & \text{if } n = 3, \\ 2, & \text{if } n = 4, \\ 2, & \text{if } n = 6, \\ \left\lceil \frac{2t+1}{4t+3}\, n \right\rceil, & \text{if } n \geq 5 \text{ is odd}, \\ 2\left\lceil \frac{2t+1}{4t+3}\, \frac{n}{2} \right\rceil, & \text{if } n \geq 8 \text{ is even}. \end{cases}$$

Proof: By Lemma 11 and Theorem 4,

$$\gamma_{sh,t}(C_n) = \gamma_{s,t}(H(C_n)).$$

If $n = 3$, then $H(C_3) \cong 3K_1$, so $\gamma_{sh,t}(C_3) = \gamma_{s,t}(3K_1) = 3$ for all $t \in \mathbb{N}_0$, since domination in an edgeless graph requires all vertices and there are no admissible attacks once all vertices are occupied.

If $n = 4$, then $H(C_4) \cong 2K_2$. In each K_2 a single occupied vertex is smart t-secure dominating for all $t \in \mathbb{N}_0$ (the unique defender can always move along the edge to the attacked unoccupied vertex, preserving domination). Since the two edges are disjoint components, one needs one vertex per component, and hence $\gamma_{sh,t}(C_4) = \gamma_{s,t}(2K_2) = 2$.

If $n = 6$, then Lemma 11 gives $H(C_6) \cong C_3 \cup C_3$. By Lemma 12, $\gamma_{s,t}(C_3) = 1$ for all $t \in \mathbb{N}_0$, and Proposition 7 yields

$$\gamma_{sh,t}(C_6) = \gamma_{s,t}(C_3 \cup C_3) = \gamma_{s,t}(C_3) + \gamma_{s,t}(C_3) = 2.$$

Now assume $n \geq 5$. If n is odd, then $H(C_n) \cong C_n$ by Lemma 11, and Theorem 16(b) together with Lemma 14 yields

$$\gamma_{sh,t}(C_n) = \gamma_{s,t}(C_n) = \left\lceil \frac{2t+1}{4t+3}\, n \right\rceil.$$

If $n \geq 8$ is even, then $H(C_n) \cong C_{n/2} \cup C_{n/2}$ with $n/2 \geq 4$, so by Proposition 7 and Theorem 16(b) together with Lemma 14,

$$\gamma_{sh,t}(C_n) = \gamma_{s,t}(C_{n/2} \cup C_{n/2}) = 2\,\gamma_{s,t}(C_{n/2}) = 2\left\lceil \frac{2t+1}{4t+3}\, \frac{n}{2} \right\rceil.$$

This completes the proof. □

7 A separation result

Secure hop domination (the case $t = 1$) can exceed hop domination by an arbitrarily large amount [1]. The same phenomenon holds for every fixed $t \geq 1$.

Theorem 19: *Fix $t \geq 1$. Then there exists a sequence of graphs $\{G_k\}_{k\geq 1}$ such that*

$$\gamma_{sh,t}(G_k) - \gamma_h(G_k) \to \infty \quad \text{as } k \to \infty.$$

Proof: Let $t \geq 1$ be fixed. Choose integers $q \geq 1$ and set

$$m = 3(4t + 3)q, \qquad n = 2m.$$

Consider $G = P_n$. By Theorem 17 and the choice of $n = 2m$, we have $\lceil n/2 \rceil = \lfloor n/2 \rfloor = m$ and $(2t+1)m/(4t+3)$ is an integer, hence

$$\gamma_{sh,t}(P_n) = 2 \cdot \frac{2t + 1}{4t + 3}\, m = \frac{2t + 1}{4t + 3}\, n.$$

On the other hand, $\gamma_h(P_n) = \gamma(H(P_n))$ by Lemma 1. Since $H(P_n) \cong P_m \cup P_m$ by Lemma 10, and the domination number of a path satisfies $\gamma(P_m) = \lceil m/3 \rceil$, we obtain

$$\gamma_h(P_n) = 2\left\lceil \frac{m}{3} \right\rceil = 2 \cdot \frac{m}{3} = \frac{n}{3},$$

using that m is a multiple of 3. Therefore,

$$\gamma_{sh,t}(P_n) - \gamma_h(P_n) = n\left(\frac{2t + 1}{4t + 3} - \frac{1}{3}\right) = n\left(\frac{6t + 3 - (4t + 3)}{3(4t + 3)}\right) = n \cdot \frac{2t}{12t + 9}.$$

As $q \to \infty$, we have $n = 2m \to \infty$, so the difference tends to infinity. This proves the claim. □

8 Conclusions

We introduced t-secure hop dominating sets and established a direct equivalence with smart t-secure domination in the hop graph. This transfer principle yields immediate structural properties and exact values for several graph families, including complete multipartite graphs, stars, paths, and cycles. The formulas for paths and cycles follow from explicit hop-graph decompositions together with finite-order domination values for paths and cycles, interpreted via Theorem 16 and Lemma 14, with explicit small-n exceptions in the cycle case.

Several directions remain open. It would be natural to (i) determine $\gamma_{sh,t}(G)$ for broader graph classes such as trees beyond paths, (ii) study algorithmic complexity and approximation, in the spirit of work on hop domination [5], and (iii) investigate behavior under graph products and standard graph operations, extending the operation-based characterizations obtained for secure hop domination in [1].

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