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Research article

Asymptotic behavior of a vector-host disease model with piecewise-smooth treatment

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Abstract: In this paper, we analyze a vector-host epidemic model with a piecewise-smooth treatment rate. The use of piecewise-smooth treatment depicts the limited medical resource situation in the community. The treatment increases linearly with infective population until the treatment capacity is reached, after which constant treatment(i.e maximum treatment) is applied. The analysis indicates that there exists a critical value $I_{h0}^c = \frac{b_h}{\mu_h}$ for the infective human population level I_{h0} at which the health care system reaches its capacity. We derive that when $I_{h0} \geq I_{h0}^c$, the dynamics of the model is completely determined by the basic reproduction number $\mathcal{R}_0$. When $I_{h0} < I_{h0}^c$, the model exhibits multiple endemic equilibria.

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1 Introduction

Vector-borne diseases constitue a significant risk to populations due to their widespread transmission worldwide. An estimated of 80% of the world's population is at risk of developing at least one vector-borne disease [9]. These diseases can have detrimental, debilitating, and life-threatening impacts, with more than 700, 000 deaths caused by vector-borne diseases each year worldwide [9]. According to the World Health Organization (WHO), vector-borne diseases account for more than 17% of all infectious diseases. Furthermore, the burden of these diseases is highest in tropical and subtropical areas, and they disproportionately affect the poorest populations. Since 2014, major epidemics of dengue fever, malaria, chikungunya, yellow fever, and Zika have affected populations, claimed lives, and overwhelmed health systems in many countries. Changes at the environmental, demographic, and societal levels also contribute to the proliferation and spread of vectors and, consequently, the diseases they carry and transmit. Pathogens are not limited by national borders, and local and international movements of people can rapidly spread infections. Increasing urbanization results in large and dense populations, increasing the likelihood of transmission and occurrence of infectious diseases. In addition, climate change can expand the habitats of some vectors to new regions, thereby exposing new populations to the disease they transmit, and it can alter the patterns and intensity of seasonal diseases. Detection, prevention, and response to infectious disease threats are therefore essential to global health security. In addition to climate-related risk factors, vector-borne diseases tend to affect a higher percentage of poor populations. These communities may have difficulty accessing clean water and adequate sanitation, and may have poor infrastructure, which can contribute to favorable living conditions for vectors and pathogens. The consequences of infection can, in turn, have economic repercussions, as illness, disability, and death affect labor and productivity. The situation is thus characterized by a circular relationship between disease burden and economic prosperity. As with many acute and chronic diseases, vector-borne disease prevention methods have proven to be an effective method of disease control. Prevention methods include vector control, vaccines and medications, early disease detection, protection from bites, safe hygiene practices, and increased community cooperation. Mathematical models are useful tools for understanding, predicting and controlling the spread of infections. Recently, a number of authors have proposed mathematical models to explore the transmission dynamics of these vector-borne diseases. Rosaire et al. [5] formulated a mathematical model to assess the theoretical effects of hospital ressources on the propagation of a host vector disease. They derived a basic reproduction number and show the existence of a bifurcation that indicates qualitative changes in the epidemic dynamics. In [7], Wang et al. proposed a malaria model that incorporates spatial heterogeneity and treatment plan with ivemectin. They derived that treatment plans with ivermectin have effects on the epidemic level. Zhao et al. [10] proposed a vector-borne diseases model with two control strategies on bipartite network. The authors derived the basic reproduction number, established stability result of the steady states and provided an optimal solution to reduce control cost. In [1], Dumont et al. studied the effectiveness of the sterile insect technique in the disease control. Treatment have a great impact in disease control. Classical epidemic models typically assume that the treatment rate is linearly proportional to the number of infected individuals. This linear assumption is appropriate when the number of infections is not significant, it becomes inadequate as the number of infective individuals increases due to the low medical ressources such as the medical staff, hospital beds, and medicine. For that reason, Wang [8] formulated the following piecewise-smooth treatment function: $\mathcal{T}(I) = \min(rI; rI_0)$, where r is a positive constant, I_0 is the infective level at which the health care system reaches its capacity. This functions indicates that the treatment rate increases linearly with the number of infective individuals, up to the point at which the treatment capacity is saturated. When the number of infective individuals is higher than the treatment capacity, the maximum capacity represented by the constant value rI_0 is taken [4]. This reflects a potential scenario in which the number of patients requiring hospitalization exceeds the available hospital bed capacity. In this work, we propose and analyse a mathematical model that incorporates a treatment function T defined above. According to the individual's disease status, the populations are categorized into compartments. The human population is divided into three compartments, namely susceptibles (S_h), infected (I_h), and recovered (R_h). The vectors population is divided into two compartments, namely susceptible (S_v) and infected (I_v). We define I_{h_0} as the infective human population level at which the health care system reaches its capacity, so $T(I_h) = \min\left(rI_h, rI_{h_0}\right)$.

To investigate the combined impact of social behavior change and limited medical ressources on the transmission dynamics of the host-vectors diseases, we consider the following model

$$\begin{aligned} \frac{dS_h}{dt} &= b_h - \mu_h S_h - \beta I_v S_h, \\ \frac{dI_h}{dt} &= \beta I_v S_h - (\mu_h + d + \gamma) I_h - \min(rI_{h_0}; rI_h), \\ \frac{dR_h}{dt} &= \gamma I_h + \min(rI_{h0}; rI_h) - \mu_h R_h, \\ \frac{dS_v}{dt} &= b_v - \mu_v S_v - \xi I_h S_v, \\ \frac{dI_v}{dt} &= \xi I_h S_v - \mu_v I_v, \end{aligned} \tag{1.1}$$

where, μ_h denotes the natural mortality rate of humans; b_h is the recruitment rate of susceptible human population; β represents the disease transmission rate from vectors to humans; d is the disease-induced mortality rate of infected individuals; γ is the recovery rate; b_v represents the recruitment rate of the vector population; μ_v denotes the natural mortality rate of vectors; and ξ is the disease transmission rate from humans to vectors.

The rest of the paper is structured as follows: in section 2, we derived some basic properties of the model (1.1). We show that model (1.1) can be divided into two subsystems according to the piecewise smooth nature of the processing function, called respectively the left subsystem and the right subsystem. Section 3 is devoted to the analysis of the global dynamics of the left subsystem. In section 4, we study the global dynamics of the right subsystem. Finally, in section 5 the paper end with a conclusion.

2 Basic properties

In this section, we first show that the model (1.1) is biologically well-posed, that is all solutions remain positive and bounded. We first state the following result.

Proposition 1: *Let* $(S_h, I_h, R_h, S_v, I_v)$ *be any solution of system* (1.1) *with positive initial conditions. Then, the solution remains positive for all* $t \geq 0$ *and satisfies the following bounds :*

$$\lim_{t\to\infty}(S_h(t) + I_h(t) + R_h(t)) \leq \frac{b_h}{\mu_h} \quad \textit{and} \quad \lim_{t\to\infty}(S_v(t) + I_v(t)) = \frac{b_v}{\mu_v}.$$

Proof: We easily find that within $\mathbf{R}_+^5$,

$$\begin{aligned}
&\frac{dS_h}{dt}\Big|_{S_h=0} = b_h \geq 0,\\
&\frac{dI_h}{dt}\Big|_{I_h=0} = \beta I_v S_v \geq 0,\\
&\frac{dR_h}{dt}\Big|_{R_h=0} = \gamma I_h + \min(rI_{h_0}, rI_h) \geq 0,\\
&\frac{dS_v}{dt}\Big|_{S_v=0} = b_v \geq 0,\\
&\frac{dI_v}{dt}\Big|_{I_v=0} = \xi I_h S_v \geq 0.
\end{aligned} \tag{2.1}$$

Assuming that $S_h(0), I_h(0), R_h(0), S_h(0), S_v(0)$ and $I_v(0)$ are positive, it follows from equation (2.1) that the vector field defined by (1.1) is directed toward the interior of $\mathbf{R}_+^5$. That is, the solution $(S_h(t), I_h(t), R_h(t), S_v(t), I_v(t))$ remains in $\mathbf{R}_+^5$. Further, let $N_h = S_h + I_h + R_h$ and $N_v = S_v + I_v$. Thus, from equation (1.1) we deduce that

$$\frac{dN_h}{dt} \leq b_h - \mu_h N_h \quad \text{and} \quad \frac{dN_v}{dt} = b_v - \mu_v N_v.$$

That is,

$$N_h \leq \frac{b_h}{\mu_h} + \left(N_h(0) - \frac{b_h}{\mu_h}\right)e^{-\mu_h t} \quad \text{and} \quad N_v = \frac{b_v}{\mu_v} + \left(N_v(0) - \frac{b_v}{\mu_v}\right)e^{-\mu_v t}.$$

Therefore, $\lim\limits_{t\to\infty} N_h \leq \dfrac{b_h}{\mu_h}$ and $\lim\limits_{t\to\infty} N_v = \dfrac{b_v}{\mu_v}$. □

Since the removed human population does not appear in the remaining equation of (1.1), it is sufficient to consider the following system:

$$\begin{cases}
\dfrac{dS_h}{dt} = b_h - \mu_h S_h - \beta I_v S_h,\\[2ex]
\dfrac{dI_h}{dt} = \beta I_v S_h - (\mu_h + d + \gamma) I_h - \min(rI_{h_0}; rI_h),\\[2ex]
\dfrac{dS_v}{dt} = b_v - \mu_v S_v - \xi I_h S_v,\\[2ex]
\dfrac{dI_v}{dt} = \xi I_h S_v - \mu_v I_v.
\end{cases} \tag{2.2}$$

Moreover, Proposition 1 shows that it is sufficient to work in the following set

$$\Omega = \Big\{ (S_h(t), I_h(t), S_v(t), I_v(t)) \in \mathbf{R}^4_+, S_h(t) + I_h(t) \leq \frac{b_h}{\mu_h},\ S_v(t) + I_v(t) \leq \frac{b_v}{\mu_h} \Big\}.$$

Now, we set

$$x = \frac{\beta S_h}{\mu_h + d + \gamma},\ y = \frac{\beta I_h}{\mu_h + d + \gamma},\ z = \frac{\xi S_v}{\mu_h + d + \gamma},\ w = \frac{\xi I_v}{\mu_h + d + \gamma} \tag{2.3}$$

and $\tau = (\mu_h + d + \gamma)t$. Thus, we obtain

$$\begin{aligned} \frac{dx}{dt} &= \frac{\beta}{\mu_h + d + \gamma} \frac{dS_h}{dt} \\ &= \frac{\beta}{\mu_h + d + \gamma} (b_h - \mu_h S_h - \beta I_v S_h). \end{aligned}$$

By using (2.3), we get

$$\frac{dx}{dt} = \frac{\beta b_h}{\mu_h + d + \gamma} - \mu_h x - \frac{\beta(\mu_h + d + \gamma)}{\xi} x w,$$

that is

$$\frac{1}{(\mu_h + d + \gamma)} \frac{dx}{dt} = \frac{\beta b_h}{(\mu_h + d + \gamma)^2} - \frac{\mu_h}{\mu_h + d + \gamma} x - \frac{\beta}{\xi} x w.$$

Further, setting $\mathcal{A}_1 = \dfrac{\beta b_h}{(\mu_h + d + \gamma)^2}$ and $m_1 = \dfrac{\mu_h}{\mu_h + d + \gamma}$, we get

$$\frac{dx}{d\tau} = \mathcal{A}_1 - m_1 x - \frac{\beta}{\xi} x w. \tag{2.4}$$

Further,

$$\begin{aligned} \frac{dy}{dt} &= \frac{\beta}{\mu_h + d + \gamma} \frac{dI_h}{dt} \\ &= \frac{\beta}{\mu_h + d + \gamma} \big(\beta I_v S_h - (\mu_h + d + \gamma) I_h - \min(r I_{h_0}; r I_h) \big) \\ &= \frac{\beta^2}{\mu_h + d + \gamma} I_v S_h - \beta I_h - \frac{\beta}{\mu_h + d + \gamma} \min(r I_{h_0}; r I_h), \end{aligned}$$

which gives us

$$\frac{1}{\mu_h + d + \gamma} \frac{dy}{dt} = \frac{\beta I_v x}{\mu_h + d + \gamma} - \frac{\beta I_h}{\mu_h + d + \gamma} \frac{\beta}{(\mu_h + d + \gamma)^2} \min(r I_{h_0}; r I_h).$$

Then,

$$\frac{dy}{dt} = \frac{\beta}{\xi}xw - y - \frac{\beta}{(\mu_h + d + \gamma)^2} \min(rI_{h_0}; rI_h).$$

Let define the function $\mathcal{T}$ by

$$\mathcal{T}(y) = \frac{\beta}{(\mu_h + d + \gamma)^2} \min(rI_{h_0}; rI_h).$$

Thus,

$$\mathcal{T}(y) = \frac{2ny}{\mid y_0 - y \mid + y_0 + y} \text{ where } n = \frac{\beta r I_{h_0}}{(\mu_h + d + \gamma)^2} \text{ and } y_0 = \frac{\beta I_{h_0}}{\mu_h + d + \gamma}.$$

Hence

$$\frac{dy}{d\tau} = \frac{\beta}{\xi}xw - y - \mathcal{T}(y). \tag{2.5}$$

By using equation (2.3), we obtain

$$\begin{aligned}\frac{dz}{dt} &= \frac{\xi}{\mu_h + d + \gamma}\frac{dS_v}{dt} \\ &= \frac{\xi b_v}{\mu_h + d + \gamma} - \frac{\xi \mu_v S_v}{\mu_h + d + \gamma} - \frac{\xi^2 I_h S_v}{\mu_h + d + \gamma}.\end{aligned}$$

That is,

$$\frac{1}{(\mu_h + d + \gamma)}\frac{dz}{dt} = \frac{\xi b_v}{(\mu_h + d + \gamma)^2} - \frac{\mu_v}{\mu_h + d + \gamma}z - \frac{\xi}{\beta}yz.$$

Then,

$$\frac{dz}{dt} = \frac{\xi b_v}{(\mu_h + d + \gamma)^2} - \frac{\mu_v}{\mu_h + d + \gamma}z - \frac{\xi}{\beta}yz.$$

By setting

$$\mathcal{A}_2 = \frac{\xi b_v}{(\mu_h + d + \gamma)^2} \text{ and } m_2 = \frac{\mu_v}{\mu_h + d + \gamma},$$

we obtain

$$\frac{dz}{d\tau} = \mathcal{A}_2 - m_2 z - \frac{\xi}{\beta}yz. \tag{2.6}$$

The derivative of w with respect to the time t is given by:

$$\begin{aligned}\frac{dw}{dt} &= \frac{\xi}{\mu_h + d + \gamma}\frac{dI_v}{dt}\\ &= \frac{\xi^2}{\mu_h + d + \gamma}I_h S_v - \frac{\xi}{\mu_h + d + \gamma}\mu_v I_v.\end{aligned}$$

That is,

$$\frac{1}{(\mu_h + d + \gamma)}\frac{dw}{dt} = \frac{\xi}{\beta}yz - \frac{\mu_h}{\mu_h + d + \gamma}w.$$

Then,

$$\frac{dw}{d\tau} = \frac{\xi}{\beta}yz - m_2 w. \tag{2.7}$$

Thus, from (2.4)-(2.7), we derive that system (2.2) becomes

$$\begin{cases}\dfrac{dx}{dt} = \mathcal{A}_1 - m_1 x - \dfrac{\beta}{\xi}xw,\\[2ex] \dfrac{dy}{dt} = \dfrac{\beta}{\xi}xw - y - \mathcal{T}(y),\\[2ex] \dfrac{dz}{dt} = \mathcal{A}_2 - m_2 z - \dfrac{\xi}{\beta}yz,\\[2ex] \dfrac{dw}{dt} = \dfrac{\xi}{\beta}yz - m_2 w.\end{cases} \tag{2.8}$$

• If $0 \leq y \leq y_0$ $\left(\text{i.e } 0 \leq I_h \leq I_{h_0}\right)$, then

$$\mathcal{T}(y) = \frac{ny}{y_0}.$$

Thus, the second equation of (2.8) becomes

$$\frac{dy}{dt} = \frac{\beta}{\xi}xw - \left(1 + \frac{n}{y_0}\right)y,$$

that is,

$$\frac{dy}{dt} = \frac{\beta}{\xi}xw - sy, \quad \text{where} \quad s = 1 + \frac{n}{y_0}.$$

Hence, system (2.8) can be rewritten as

$$\begin{cases} \dfrac{dx}{dt} = \mathcal{A}_1 - m_1 x - \dfrac{\beta}{\xi} x w, \\[2ex] \dfrac{dy}{dt} = \dfrac{\beta}{\xi} x w - s y, \\[2ex] \dfrac{dz}{dt} = \mathcal{A}_2 - m_2 z - \dfrac{\xi}{\beta} y z, \\[2ex] \dfrac{dw}{dt} = \dfrac{\xi}{\beta} y z - m_2 w. \end{cases} \tag{2.9}$$

From equation (2.9), one can verify that $0 \leq x \leq \dfrac{\mathcal{A}_1}{m_1}$, $0 \leq y \leq \min\left(\dfrac{\mathcal{A}_1}{m_1}, y_0\right)$, and $0 \leq w, z \leq \dfrac{\mathcal{A}_2}{m_2}$. Thus, the set

$$\Omega_1 = \left\{(x, y, z, w) \mid 0 \leq x \leq \frac{\mathcal{A}_1}{m_1}, 0 \leq y \leq \min\left(\frac{\mathcal{A}_1}{m_1}, y_0\right), 0 \leq z \leq \frac{\mathcal{A}_2}{m_2}, 0 \leq w \leq \frac{\mathcal{A}_2}{m_2}\right\}$$

is a positively invariant for system (2.9).

• If $0 \leq y_0 \leq y$ $\left(\text{i.e } I_h \geq I_{h_0}\right)$, then

$$\mathcal{T}(y) = n.$$

Thus, system (2.8) becomes

$$\begin{cases} \dfrac{dx}{dt} = \mathcal{A}_1 - m_1 x - \dfrac{\beta}{\xi} x w, \\[2ex] \dfrac{dy}{dt} = \dfrac{\beta}{\xi} x w - y - n, \\[2ex] \dfrac{dz}{dt} = \mathcal{A}_2 - m_2 z - \dfrac{\xi}{\beta} y z, \\[2ex] \dfrac{dw}{dt} = \dfrac{\xi}{\beta} y z - m_2 w. \end{cases} \tag{2.10}$$

One can verify that $0 \le x, y \le \dfrac{\mathcal{A}_1}{m_1}$ and $0 \le w, z \le \dfrac{\mathcal{A}_2}{m_2}$. Hence,

$$\Omega_2 = \left\{ (x, y, z, w) \mid 0 \le x, y \le \frac{\mathcal{A}_1}{m_1}, 0 \le w, z \le \frac{\mathcal{A}_2}{m_2} \right\}$$

is a positively invariant set for system (2.10).

3 Dynamics of the left subsystem (2.9)

3.1 Existence of equilibrium points

By direct calculation, we show that system (2.9) admits a disease-free equilibrium

$$\mathcal{E}^0 = \left(\frac{\mathcal{A}_1}{m_1}, 0, \frac{\mathcal{A}_2}{m_2}, 0 \right).$$

Further, following the idea of Driesche and Watmough ([6]), the basic reproduction number is given by

$$\mathcal{R}_0 = \frac{1}{m_2} \sqrt{\frac{\mathcal{A}_1 \mathcal{A}_2}{s m_1}}.$$

We now determine the endemic equilibrium points that will provide information regarding long-term behavior of the disease. We state the following result:

Proposition 2: *If* $\mathcal{R}_0 > 1$ *the model* (2.9) *admits a unique positive endemic equilibrium* $\mathcal{E}^* = (x^*, y^*, z^*, w^*)$ *where*

$$x^* = \frac{m_2 \xi \mathcal{A}_1 + m_2^2 s \beta}{\mathcal{A}_2 \beta + m_1 m_2 \xi}, \quad y^* = \frac{\beta m_1 m_2^2 \left(\mathcal{R}_0^2 - 1 \right)}{\mathcal{A}_2 \beta + m_1 m_2 \xi},$$
$$z^* = \frac{\xi m_1 m_2 s + \mathcal{A}_2 s \beta}{\xi \mathcal{A}_1 + m_2 s \beta}, \quad w^* = \frac{\xi m_1 m_2^2 s \left(\mathcal{R}_0^2 - 1 \right)}{m_2 \xi \mathcal{A}_1 + m_2^2 s \beta}.$$

Proof: Let (x, y, z, w) be an equilibrium equilibrium of model (2.9), thus the following equations holds:

$$\mathcal{A}_1 - m_1 x - \frac{\beta}{\xi} x w = 0, \tag{3.1}$$

$$\frac{\beta}{\xi} x w - s y = 0, \tag{3.2}$$

$$\mathcal{A}_2 - m_2 z - \frac{\xi}{\beta} y z = 0, \tag{3.3}$$

$$\frac{\xi}{\beta} y z - m_2 w = 0. \tag{3.4}$$

Summing (3.1) and (3.2) give us $\mathcal{A}_1 - m_1 x - sy = 0$, which yields

$$y = \frac{\mathcal{A}_1 - m_1 x}{s}. \tag{3.5}$$

From equation (3.1), we derive that

$$w = \frac{\xi}{\beta}\frac{\mathcal{A}_1 - m_1 x}{x}. \tag{3.6}$$

By using equations (3.3) and (3.5) we establish that

$$z = \frac{\mathcal{A}_2 \beta s}{m_2 \beta s + \xi(\mathcal{A}_1 - m_1 x)}. \tag{3.7}$$

Using equations (3.5), (3.6) and (3.7) to replace y, z and w into (3.4) gives us

$$(\mathcal{A}_1 - m_1 x)\frac{\mathcal{A}_2 \beta x - m_2^2 s\beta - m_2 \xi \mathcal{A}_1 + m_2 \xi m_1 x}{m_2 s \beta x + \xi(\mathcal{A}_1 - m_1 x)} = 0. \tag{3.8}$$

Equation (3.8) has two solutions, namely, $x_1 = \dfrac{\mathcal{A}_1}{m_1}$ and $x_2 = \dfrac{m_2 \xi \mathcal{A}_1 + m_2^2 s\beta}{\mathcal{A}_2 \beta + m_1 m_2 \xi}$. The solution $x = \dfrac{\mathcal{A}_1}{m_1}$ gives us the disease-free equilibrium $\mathcal{E}^0$ defined above. For $x_2 = \dfrac{m_2 \xi \mathcal{A}_1 + m_2^2 s\beta}{\mathcal{A}_2 \beta + m_1 m_2 \xi}$, we derive that

$$y = \frac{\beta m_1 m_2^2 \left(\mathcal{R}_0^2 - 1\right)}{\mathcal{A}_2 \beta + m_1 m_2 \xi}, \quad z = \frac{\xi m_1 m_2 s + \mathcal{A}_2 s\beta}{\xi \mathcal{A}_1 + m_2 s\beta}, \quad w = \frac{\xi m_1 m_2^2 s\left(\mathcal{R}_0^2 - 1\right)}{m_2 \xi \mathcal{A}_1 + m_2^2 s\beta}.$$

Clearly y and w are positive whenever $\mathcal{R}_0 > 1$. The proof is completed. □

3.2 Stability analysis of the disease-free equilibrium

In this part, we establish the local and global behavior of the disease-free equilibrium $\mathcal{E}^0$. We state the following result:

Proposition 3: *If $\mathcal{R}_0 < 1$, the disease-free equilibrium $\mathcal{E}^0 = \left(\dfrac{\mathcal{A}_1}{m_1}, 0, \dfrac{\mathcal{A}_2}{m_2}, 0\right)$ is locally asymptotically stable. It is unstable if $\mathcal{R}_0 > 1$.*

Proof: Let $\mathcal{M}_0$ be the Jacobian matrix associated to the system (2.9) at the disease-free equilibrium point $\mathcal{E}^0 = \left(\dfrac{\mathcal{A}_1}{m_1}, 0, \dfrac{\mathcal{A}_2}{m_2}, 0\right)$. Then,

$$\mathcal{M}_0 = \begin{pmatrix} -m_1 & 0 & 0 & -\frac{\beta}{\xi}\frac{\mathcal{A}_1}{m_1} \\ 0 & -s & 0 & \frac{\beta}{\xi}\frac{\mathcal{A}_1}{m_1} \\ 0 & -\frac{\xi}{\beta}\frac{\mathcal{A}_2}{m_2} & -m_2 & 0 \\ 0 & \frac{\xi}{\beta}\frac{\mathcal{A}_2}{m_2} & 0 & -m_2 \end{pmatrix}.$$

Further, the characteristic polynomial of the matrix $\mathcal{M}_0$ is given by

$$P_{\mathcal{M}_0}(\lambda) = (\lambda + m_1)(\lambda + m_2)\big(\lambda^2 + (m_2 + s)\lambda + m_1 m_2^2 s\big(1 - \mathcal{R}_0^2\big)\big).$$

By direct calculation, we easily find that the root of $P_{\mathcal{M}_0}$ are

$$\begin{aligned} \lambda_1 &= -m_1, \quad \lambda_2 = -m_2, \\ \lambda_3 &= \frac{-(m_2 + s) - \sqrt{(m_2 + s)^2 - 4m_1 m_2^2 s\big(1 - \mathcal{R}_0^2\big)}}{2}, \\ \lambda_4 &= \frac{-(m_2 + s) + \sqrt{(m_2 + s)^2 - 4m_1 m_2^2 s\big(1 - \mathcal{R}_0^2\big)}}{2}. \end{aligned}$$

If $\mathcal{R}_0 < 1$, all roots are strictly negative, that is $\mathcal{E}^0$ is locally asymptotically stable. When $\mathcal{R}_0 > 1$, the root λ_4 is strictly positive, so $\mathcal{E}^0$ becomes unstable. □

We now analyze the global behavior of the disease-free equilibrium $\mathcal{E}^0$. The following result holds.

Proposition 4: *If $\mathcal{R}_0 < 1$, the disease-free equilibrium $\mathcal{E}^0$ is globally asymptotically stable,*

Proof: The infected components satisfy

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\beta}{\xi} xw - sy, \\ \dfrac{dw}{dt} = \dfrac{\xi}{\beta} yz - m_2 w. \end{cases} \tag{3.9}$$

Let $\mathbf{Z} = (y, w)^T$ and $\theta = \left(1, \mathcal{R}_0 \dfrac{\beta m_2^2}{\xi \mathcal{A}_2}\right)$.
Thus equation (3.9) gives us

$$\frac{d\mathbf{Z}}{dt} \leq (F + V)\mathbf{Z}, \tag{3.10}$$

where

$$F = \begin{pmatrix} 0 & \dfrac{\beta}{\xi}\dfrac{\mathcal{A}_1}{m_1} \\ \dfrac{\xi}{\beta}\dfrac{\mathcal{A}_2}{m_2} & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -s & 0 \\ 0 & -m_2 \end{pmatrix}.$$

It is easy to verify that

$$\rho(-FV^{-1}) = \rho(-V^{-1}F) = \mathcal{R}_0 \quad \text{and} \quad \theta V^{-1} F = \mathcal{R}_0 \theta.$$

Further, we define the following Lyapunov candidate function:

$$\mathcal{V} = -\theta V^{-1} \mathbf{Z}.$$

The time derivative of $\mathcal{V}$ is given by

$$\frac{d\mathcal{V}}{dt} = -\theta V^{-1} \frac{d\mathbf{Z}}{dt}.$$

Then, by using inequality (3.10) we obtain

$$\frac{d\mathcal{V}}{dt} \leq -\theta V^{-1}(F + V)\mathbf{Z},$$

that is,

$$\frac{d\mathcal{V}}{dt} \leq \theta(\mathcal{R}_0 - 1)\mathbf{Z}.$$

Since $\mathcal{R}_0 < 1$, we obtain

$$\frac{d\mathcal{V}}{dt} \leq 0.$$

Moreover, if $\mathcal{R}_0 < 1$, the condition $\dfrac{d\mathcal{V}}{dt} = 0$ implies that $\theta\mathbf{Z} = 0$. Since all components of θ are positive, it follows that $y = w = 0$. Therefore, when $\mathcal{R}_0 < 1$, setting the right-hand side of system (2.9) to zero and substituting y and w by 0, we obtain $x = \dfrac{\mathcal{A}_1}{m_1}$ and $z = \dfrac{\mathcal{A}_2}{m_2}$. Moreover, the largest invariant subset of $\left\{(x, y, w, z) \in \mathbf{R}_4^+ | \dfrac{d\mathcal{V}}{dt} = 0\right\}$ is reduced to the singleton $\left\{\mathcal{E}^0\right\}$. Hence, by using LaSalle's invariant principle (see [2], [3]), we claim that the disease-free equilibrium $\mathcal{E}^0$ is globally asymptotically stable within the domain Ω_1. This ends the proof. □

3.3 Stability analysis of the endemic equilibrium

Here, we provide the local and global behavior of the endemic equilibrium $\mathcal{E}^*$. The following result holds.

Proposition 5: *If $\mathcal{R}_0 > 1$, then the endemic equilibrium $\mathcal{E}^*$ is locally asymptotically stable.*

Proof: Let $\mathcal{M}^*$ be the Jacobian matrix evaluated at the endemic equilibrium $\mathcal{E}^*$. Thus,

$$\mathcal{M}^* = \begin{pmatrix} -m_1 - \frac{\beta}{\xi} w^* & 0 & 0 & -\frac{\beta}{\xi} x^* \\ \frac{\beta}{\xi} w^* & -s & 0 & \frac{\beta}{\xi} x^* \\ 0 & -\frac{\xi}{\beta} z^* & -m_2 - \frac{\xi}{\beta} y^* & 0 \\ 0 & \frac{\xi}{\beta} z^* & \frac{\xi}{\beta} y^* & -m_2 \end{pmatrix}.$$

So, the characteristic equation is given by

$$(\lambda + m_2)\big(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3\big) = 0, \tag{3.11}$$

where

$$\begin{aligned} b_1 &= m_1 + m_2 + s + \frac{\beta}{\xi} w^* + \frac{\xi}{\beta} y^* > 0, \\ b_2 &= \left(m_1 + \frac{\beta}{\xi} w^*\right)\left(s + m_2 + \frac{\xi}{\beta} y^*\right) + 2sm_2 + s\frac{\xi}{\beta} y^* > 0, \\ b_3 &= \left(m_1 + \frac{\beta}{\xi} w^*\right)\left(sm_2 + s\frac{\xi}{\beta} y^*\right) + sm_1m_2 > 0. \end{aligned}$$

Clearly, $-m_2$ is a solution of equation (3.11) and the remaining solution solve equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0. \tag{3.12}$$

Furthermore, we derived that

$$\begin{aligned} b_1b_2 - b_3 &= \Big(m_1 + m_2 + s + \frac{\beta}{\xi}w^* + \frac{\xi}{\beta}y^*\Big)\Big((m_1 + \frac{\beta}{xi}w^*)(s + m_2 + \frac{\xi}{\beta}y^*) \\ &\quad +2sm_2 + s\frac{\xi}{\beta}y^*\Big) - \Big((m_1 + \frac{\beta}{\xi}w^*)(sm_2 + s\frac{\xi}{\beta}y^*) + sm_1m_2\Big) \\ &= sm_1^2 + m_1^2m_2 + 2sm_1\frac{\beta}{\xi}w^* + 2m_1m_2\frac{\beta}{\xi}w^* + 2m_1y^*w^* + s(\frac{\beta}{\xi}w^*)^2 \\ &\quad + m_2(\frac{\beta}{\xi}w^*)^2 + \frac{\beta}{\xi}y^*(w^*)^2 + m_1s^2 + s^2\frac{\beta}{\xi}w^* + sm_2\frac{\beta}{\xi}w^* + m_1m_2^2 \\ &\quad + 2m_1m_2\frac{\xi}{\beta}y^* + m_2^2\frac{\beta}{\xi}w^* + \frac{\xi}{\beta}w^*(y^*)^2 + m_1(\frac{\xi}{\beta}y^*)^2 + 2sy^*w^* + 2m_2y^*w^* \\ &\quad + 2sm_1m_2 + 2sm_2\frac{\beta}{\xi}w^* + 2s^2m_2 + 2sm_2^2 + 2sm_2\frac{\xi}{\beta}y^* + 2sm_1\frac{\xi}{\beta}y^* \\ &\quad + s^2\frac{\xi}{\beta}y^* + sm_2\frac{\xi}{\beta}y^* + s\left(\frac{\xi}{\beta}y^*\right)^2 > 0. \end{aligned}$$

Thus, by using Routh-Hurwitz criterion, we claim that all solution of (3.12) have a negative real part, so the endemic equilibrium is locally asymptotically stable. □

Proposition 6: *For $\mathcal{R}_0 > 1$ the endemic equilibrium $\mathcal{E}^*$ is globally asymptotically stable.*

Proof: Let us define the function $h(x) = x - 1 - \ln x$ which is positive for all $x > 0$ with a global minimum at $x = 1$. At the endemic equilibrium $\mathcal{E}^* = (x^*, y^*, z^*, w^*)$, the following relationships hold:

$$\mathcal{A}_1 = m_1x^* + \frac{\beta}{\xi}x^*w^*, \tag{3.13}$$

$$s = \frac{\beta}{\xi}\frac{x^*w^*}{y^*}, \tag{3.14}$$

$$\mathcal{A}_2 = m_2z^* + \frac{\xi}{\beta}y^*z^*, \tag{3.15}$$

$$m_2 = \frac{\xi}{\beta}\frac{y^*z^*}{w^*}. \tag{3.16}$$

We define the Lyapunov function

$$\mathcal{L}(x, y, z, w) = \sum_{i=1}^{4} k_i\mathcal{L}_i(x, y, z, w),$$

where

$$\mathcal{L}_1(x,y,z,w) = h\left(\frac{x}{x^*}\right), \quad \mathcal{L}_2(x,y,z,w) = h\left(\frac{y}{y^*}\right)$$
$$\mathcal{L}_3(x,y,z,w) = h\left(\frac{z}{z^*}\right), \quad \mathcal{L}_4(x,y,z,w) = h\left(\frac{w}{w^*}\right),$$

and

$$k_1 = \frac{\xi}{\beta} y^* z^*, \quad k_2 = \frac{\xi}{\beta}\frac{(y^*)^2 z^*}{x^*}, \quad k_3 = \frac{\beta}{\xi} z^* w^* \quad k_4 = \frac{\beta}{\xi}(w^*)^2.$$

$\mathcal{L}$ is non-negative and strictly minimized at the equilibrium $\mathcal{E}^*$. To avoid long expression, the derivative of $\mathcal{L}_i, i = 1, \cdots, 4$ will be calculate separately and combine to get that of $\mathcal{L}$. The derivative of $\mathcal{L}_1$ with respect to the time is given by:

$$\begin{aligned}
\frac{d\mathcal{L}_1}{dt} &= \frac{1}{x^*}\left(1 - \frac{x}{x^*}\right)\frac{dx}{dt} \\
&= \frac{1}{x^*}\left(1 - \frac{x}{x^*}\right)\left(\mathcal{A}_1 - m_1 x - \frac{\beta}{\xi} x w\right).
\end{aligned}$$

Using equation (3.13), we obtain

$$\begin{aligned}
\frac{d\mathcal{L}_1}{dt} &= \frac{1}{x^*}\left(1 - \frac{x}{x^*}\right)\left(m_1 x^* + \frac{\beta}{\xi} x^* w^* - m_1 x - \frac{\beta}{\xi} x w\right) \\
&= \frac{1}{x^*}\left(1 - \frac{x}{x^*}\right)\left(m_1 x^*\left(1 - \frac{x}{x^*}\right) + \frac{\beta}{\xi} x^* w^*\left(1 - \frac{xw}{x^* w^*}\right)\right) \\
&= \frac{1}{x^*} m_1 x^*\left(2 - \frac{x}{x^*} - \frac{x^*}{x}\right) + \frac{\beta}{\xi} w^*\left(1 - \frac{xw}{x^* w^*} - \frac{x^*}{x} + \frac{w}{w^*}\right) \\
&= m_1\left(2 - \frac{x}{x^*} - \frac{x^*}{x}\right) + \frac{\beta}{\xi} w^*\left(-h\left(\frac{xw}{x^* w^*}\right) - h\left(\frac{x^*}{x}\right) + h\left(\frac{w}{w^*}\right)\right).
\end{aligned} \tag{3.17}$$

The derivative of $\mathcal{L}_2$ with respect to the time t is given by:

$$\begin{aligned}
\frac{d\mathcal{L}_2}{dt} &= \frac{1}{y^*}\left(1 - \frac{y}{y^*}\right)\frac{dy}{dt} \\
&= \frac{1}{y^*}\left(1 - \frac{y}{y^*}\right)\left(\frac{\beta}{\xi} x w - s y\right).
\end{aligned}$$

By using (3.14) to substitute s, we get

$$\begin{aligned}
\frac{d\mathcal{L}_2}{dt} &= \frac{1}{y^*}\left(1-\frac{y}{y^*}\right)\left(\frac{\beta}{\xi}xw - \frac{\beta}{\xi}\frac{x^*w^*}{y^*}y\right)\\
&= \frac{\beta}{\xi}\frac{x^*w^*}{y^*}\left(\frac{xw}{x^*w^*} - \frac{y}{y^*} - \frac{xwy^*}{x^*w^*y} + 1\right)\\
&= \frac{\beta}{\xi}\frac{x^*w^*}{y^*}\left(h\left(\frac{xw}{x^*w^*}\right) - h\left(\frac{y}{y^*}\right) - h\left(\frac{xwy^*}{x^*w^*y}\right)\right). \qquad (3.18)
\end{aligned}$$

Differentiating $\mathcal{L}_3$ with respect to the time t gives us

$$\begin{aligned}
\frac{d\mathcal{L}_3}{dt} &= \frac{1}{z^*}\left(1-\frac{z}{z^*}\right)\frac{dz}{dt}\\
&= \frac{1}{z^*}\left(1-\frac{z}{z^*}\right)\left(\mathcal{A}_2 - m_2 z - \frac{\xi}{\beta}yz\right).
\end{aligned}$$

Using equation (3.15), we obtain

$$\begin{aligned}
\frac{d\mathcal{L}_3}{dt} &= \frac{1}{z^*}\left(1-\frac{z}{z^*}\right)\left(m_2 z^* + \frac{\xi}{\beta}y^*z^* - m_2 z - \frac{\xi}{\beta}yz\right)\\
&= \frac{1}{z^*}\left(1-\frac{z}{z^*}\right)\left(m_2 z^*\left(1-\frac{z}{z^*}\right) + \frac{\xi}{\beta}y^*z^*\left(1-\frac{yz}{y^*z^*}\right)\right)\\
&= m_2\left(2-\frac{z}{z^*}-\frac{z^*}{z}\right) + \frac{\xi}{\beta}y^*\left(1-\frac{yz}{y^*z^*}-\frac{z^*}{z}+\frac{y}{y^*}\right)\\
&= m_2\left(2-\frac{z}{z^*}-\frac{z^*}{z}\right) + \frac{\xi}{\beta}y^*\left(-h\left(\frac{yz}{y^*z^*}\right) - h\left(\frac{z^*}{z}\right) + h\left(\frac{y}{y^*}\right)\right). \qquad (3.19)
\end{aligned}$$

Differentiating $\mathcal{L}_4$ with respect to the time t yields

$$\begin{aligned}
\frac{d\mathcal{L}_4}{dt} &= \frac{1}{w^*}\left(1-\frac{w}{w^*}\right)\frac{dw}{dt}\\
&= \frac{1}{w^*}\left(1-\frac{w}{w^*}\right)\left(\frac{\xi}{\beta}yz - m_2 w\right).
\end{aligned}$$

Using equation (3.16), we get

$$\begin{aligned}
\frac{d\mathcal{L}_4}{dt} &= \frac{1}{w^*}\left(1-\frac{w}{w^*}\right)\left(\frac{\xi}{\beta}yz - \frac{\xi}{\beta}\frac{y^*z^*}{w^*}w\right)\\
&= \frac{\xi}{\beta}\frac{y^*z^*}{w^*}\left(1-\frac{yzw^*}{y^*z^*w}+\frac{yz}{y^*z^*}-\frac{w}{w^*}\right)\\
&= \frac{\xi}{\beta}\frac{y^*z^*}{w^*}\left(-h\left(\frac{yzw^*}{y^*z^*w}\right) + h\left(\frac{yz}{y^*z^*}\right) - h\left(\frac{w}{w^*}\right)\right). \qquad (3.20)
\end{aligned}$$

Combining equations (3.17)-(3.20) and multiplying by the appropriate coefficient, we derive that

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= k_1 m_1\left(2-\frac{x}{x^*}-\frac{x^*}{x}\right)+k_1\frac{\beta}{\xi}w^*\left(-h\left(\frac{xw}{x^*w^*}\right)-h\left(\frac{x^*}{x}\right)+h\left(\frac{w}{w^*}\right)\right)\\ &+k_2\frac{\beta}{\xi}\frac{x^*w^*}{y^*}\left(h\left(\frac{xw}{x^*w^*}\right)-h\left(\frac{y}{y^*}\right)-h\left(\frac{xwy^*}{x^*w^*y}\right)\right)\\ &+k_3 m_2\left(2-\frac{z}{z^*}-\frac{z^*}{z}\right)+k_3\frac{\xi}{\beta}y^*\left(-h\left(\frac{yz}{y^*z^*}\right)-h\left(\frac{z^*}{z}\right)+h\left(\frac{y}{y^*}\right)\right)\\ &+k_4\frac{\beta}{\xi}\frac{x^*w^*}{y^*}\left(h\left(\frac{xw}{x^*w^*}\right)-h\left(\frac{y}{y^*}\right)-h\left(\frac{xwy^*}{x^*w^*y}\right)\right).\end{aligned}$$

By replacing k_1, k_2, k_3 and k_4 and with their respective expressions and after a bit of algebra, we obtain

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \frac{\xi}{\beta}y^*z^*m_1\left(2-\frac{x}{x^*}-\frac{x^*}{x}\right)-y^*z^*w^*h\left(\frac{xw}{x^*w^*}\right)\\ &-y^*z^*w^*h\left(\frac{xwy^*}{x^*w^*y}\right)+\frac{\beta}{\xi}z^*w^*m_2\left(2-\frac{z}{z^*}-\frac{z^*}{z}\right)\\ &-y^*z^*w^*h\left(\frac{z^*}{z}\right)-y^*z^*w^*h\left(\frac{xwy^*}{x^*w^*y}\right).\end{aligned}$$

That is,

$$\frac{d\mathcal{L}}{dt} \leq 0.$$

Moreover, we observe that $\frac{d\mathcal{L}}{dt} = 0$ if and only if $(x, y, z, w) = (x^*, y^*, z^*, w^*)$. Hence, by using LaSalle's invariant principle (see [2], [3]), we deduce that $\mathcal{E}^*$ is globally asymptotically stable within the domain Ω_1. This concludes the proof. □

4 Dynamics of the right subsystem (2.10)

Let $\mathcal{E}^{**} = (x^{**}, y^{**}, z^{**}, w^{**})$ be an equilibrium point of (2.10). Thus,

$$\mathcal{A}_1 = m_1 x^{**} + \frac{\beta}{\xi}x^{**}w^{**}, \tag{4.1}$$

$$\frac{\beta}{\xi}x^{**}w^{**} = y^{**} + n, \tag{4.2}$$

$$\mathcal{A}_2 = m_2 z^{**} + \frac{\xi}{\beta}y^{**}z^{**}, \tag{4.3}$$

$$\frac{\xi}{\beta}y^{**}z^{**} = m_2 w^{**}. \tag{4.4}$$

That is,

$$y^{**} = \mathcal{A}_1 - m_1 x^{**} - n,$$
$$w^{**} = \frac{\xi(\mathcal{A}_1 - m_1 x^{**})}{\beta x^{**}},$$
$$z^{**} = \frac{\mathcal{A}_2 \beta x^{**} - m_2 \xi(\mathcal{A}_1 - m_1 x^{**})}{m_2 \beta x^{**}},$$

where x^{**} is a solution of the following quadratic equation:

$$\left(-m_1\mathcal{A}_2\beta - m_1^2 m_2 \xi\right)x^2 + \Big((\mathcal{A}_1 - n)(\mathcal{A}_2\beta + m_1 m_2 \xi) + m_1 m_2 \xi \mathcal{A}_1 + m_1 m_2^2 \xi\Big)x + \mathcal{A}_1 m_2[\xi(n - \mathcal{A}_1) - m_2\beta] = 0. \tag{4.5}$$

Let define the function g by:

$$g(x) = px^2 + qx + r,$$

where

$$p = -m_1\beta\mathcal{A}_2 - m_1^2 m_2 \xi < 0,$$
$$q = (A_1 - n)(A_2\beta + m_1 m_2 \xi) + m_1 m_2 \xi A_1 + m_1 m_2^2 \xi > 0,$$
$$r = m_2\mathcal{A}_1\Big(\xi(n - \mathcal{A}_1) - m_2\beta\Big) < 0.$$

Therefore, any positive root x^{**} of g gives us an endemic equilibrium
$\mathcal{E}^{**} = (x^{**}, y^{**}, z^{**}, w^{**})$. Further, we note that the function g admits a global maximum at $x_0 = -\dfrac{q}{2p}$ equal to $-\dfrac{\Delta_0}{4p}$ where $\Delta_0 = q^2 - 4pr$. Thus, the following result holds.

Proposition 7: *Consider system* (2.10)*, then*
i) *if* $\Delta_0 < 0$, *there is no endemic equilibrium;*
ii) *if* $\Delta_0 = 0$, *there is one endemic equilibrium;*
iii) *if* $\Delta_0 > 0$, *there is two endemic equilibria.*

We now derive the result about the stability of the endemic equilibrium $\mathcal{E}^{**}$. Assume that $\Delta_0 \geq 0$, and that $\mathcal{E}^{**} = (x^{**}, y^{**}, z^{**}, w^{**})$ is an endemic equilibrium of (2.10). The linearization of system (2.10) at the equilibrium $\mathcal{E}^{**}$ give us the following Jacobian matrix:

$$\mathcal{M}^{**} = \begin{pmatrix} -m_1 - \frac{\beta}{\xi} w^{**} & 0 & 0 & -\frac{\beta}{\xi} x^{**} \\ \frac{\beta}{\xi} w^{**} & -1 & 0 & \frac{\beta}{\xi} x^{**} \\ 0 & -\frac{\xi}{\beta} z^{**} & -m_2 - \frac{\xi}{\beta} y^{**} & 0 \\ 0 & \frac{\xi}{\beta} z^{**} & \frac{\xi}{\beta} y^{**} & -m_2 \end{pmatrix}.$$

The characteristic equation of the matrix $\mathcal{M}^{**}$ is given by:

$$(\lambda + 1)\left(\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3\right) = 0,$$

where

$$c_1 = 1 + m_1 + m_2 + \frac{\beta}{\xi} w^{**} + \frac{\xi}{\beta} y^{**},$$
$$c_2 = m_1 + m_2 + \frac{\beta}{\xi} w^{**} + \frac{\xi}{\beta} y^{**} + \left(m_1 + \frac{\beta}{\xi}\right)\left(m_2 + \frac{\xi}{\beta} y^{**}\right) - x^{**} z^{**},$$
$$c_3 = \left(m_1 + \frac{\beta}{\xi}\right)\left(m_2 + \frac{\xi}{\beta} y^{**}\right) - m_1 x^{**} z^{**}.$$

Clearly, $\lambda = -1 < 0$ is a solution of above characteristic equation. Using equations (4.1)- (4.4), we derive that

$$c_1 = 1 + \frac{\mathcal{A}_1}{x^{**}} + \frac{\mathcal{A}_2}{z^{**}}, \quad c_2 = \frac{\mathcal{A}_1}{x^{**}} + \frac{\mathcal{A}_2}{z^{**}} + \frac{\mathcal{A}_1\mathcal{A}_2}{x^{**} z^{**}} - x^{**} z^{**}, \quad c_3 = \frac{\mathcal{A}_1\mathcal{A}_2}{x^{**} z^{**}} - m_1 x^{**} z^{**}.$$

Let

$$\mathcal{R}_1 = \frac{\mathcal{A}_1\mathcal{A}_2}{(x^{**})^2 (z^{**})^2} \quad \text{and} \quad \mathcal{R}_2 = \frac{\mathcal{A}_1\mathcal{A}_2}{m_1 (x^{**})^2 (z^{**})^2}.$$

If $\mathcal{R}_1 > 1$, thus $\mathcal{R}_2 > 1$, $c_2 > 0$ and $c_3 > 0$, so

$$\begin{aligned} c_1 c_2 - c_3 = {} & \frac{\mathcal{A}_1}{x^{**}} + \frac{\mathcal{A}_2}{z^{**}} + \left(\frac{\mathcal{A}_1}{x^{**}}\right)^2 + \left(\frac{\mathcal{A}_2}{z^{**}}\right)^2 + \frac{\mathcal{A}_1\mathcal{A}_2}{x^{**} z^{**}} + m_1 x^{**} z^{**} + \mathcal{A}_1 z^{**}(\mathcal{R}_1 - 1) \\ & + \mathcal{A}_2 x^{*}(\mathcal{R}_1 - 1) > 0. \end{aligned}$$

Hence, using Routh-Hurwitz criteria, we conclude that all eigenvalues of the of the matrix $\mathcal{M}^{**}$ have negative real part. Therefore, we state the following result:

Proposition 8: *If $\mathcal{R}_1 > 1$, the endemic equilibrium $\mathcal{E}^{**}$ is locally asymptotically stable.*

We now present the global stability result of system (2.10). We define the following Lyapunov function:

$$\mathcal{G}(t) = a_1 h\Big(\frac{x}{x^{**}}\Big) + a_2 h\Big(\frac{y}{y^{**}}\Big) + a_3 h\Big(\frac{z}{z^{**}}\Big) + a_4 h\Big(\frac{w}{w^{**}}\Big)$$

where a_1, a_2, a_3 and a_4 are strictly positive constants and h is defined in Section 3.3. To avoid long expression, the derivatives $\frac{dh}{dt}\Big(\frac{x}{x^{**}}\Big)$, $\frac{dh}{dt}\Big(\frac{y}{y^{**}}\Big)$, $\frac{dh}{dt}\Big(\frac{z}{z^{**}}\Big)$, $\frac{dh}{dt}\Big(\frac{w}{w^{**}}\Big)$ will be calculate separately and combine to get $\frac{d\mathcal{G}}{dt}$.

$$\begin{aligned}\frac{dh}{dt}\Big(\frac{x}{x^{**}}\Big) &= \frac{1}{x^{**}}\Big(1-\frac{x}{x^{**}}\Big)\Big(m_1 x^{**} + \frac{\beta}{\xi}x^{**}w^{**} - m_1 x - \frac{\beta}{\xi}xw\Big)\\ &= m_1\Big(2 - \frac{x}{x^{**}} - \frac{x^{**}}{x}\Big) + \frac{\beta}{\xi}w^{**}\Big(-h\Big(\frac{xw}{x^{**}w^{**}}\Big) - h\Big(\frac{x^{**}}{x}\Big) + h\Big(\frac{w}{w^{**}}\Big)\Big). \qquad (4.6)\end{aligned}$$

Further,

$$\begin{aligned}\frac{dh}{dt}\Big(\frac{y}{y^{**}}\Big) &= \frac{1}{y^{**}}\Big(1-\frac{y}{y^{**}}\Big)\Big(\frac{\beta}{\xi}xw - y - n\Big)\\ &= \frac{1}{y^{**}}\Big(1-\frac{y}{y^{**}}\Big)\Big(\frac{\beta}{\xi}xw - y - \frac{\beta}{\xi}x^{**}w^{**} + y^{**}\Big)\\ &= \frac{1}{y^{**}}\Big(1-\frac{y}{y^{**}}\Big)\Big(\frac{\beta}{\xi}x^{**}w^{**}\Big(\frac{xw}{x^{**}w^{**}} - 1\Big) + y^{**}\Big(1-\frac{y}{y^{**}}\Big)\Big)\\ &= \frac{\beta}{\xi}\frac{x^{**}w^{**}}{y^{**}}\Big(\frac{xw}{x^{**}w^{**}} - 1 - \frac{xwy^{**}}{x^{**}w^{**}y} + \frac{y^{**}}{y}\Big) + \Big(2 - \frac{y^{**}}{y} - \frac{y}{y^{**}}\Big)\\ &= \frac{\beta}{\xi}\frac{x^{**}w^{**}}{y^{**}}\Big(h\Big(\frac{xw}{x^{**}w^{**}}\Big) - h\Big(\frac{xwy^{**}}{x^{**}w^{**}y}\Big) + h\Big(\frac{y^{**}}{y}\Big)\Big)\\ &+ \Big(2 - \frac{y^{**}}{y} - \frac{y}{y^{**}}\Big). \qquad (4.7)\end{aligned}$$

The derivative $\frac{dh}{dt}\Big(\frac{z}{z^{**}}\Big)$ is given by:

$$\begin{aligned}\frac{dh}{dt}\Big(\frac{z}{z^{**}}\Big) &= \frac{1}{z^{**}}\Big(1-\frac{z}{z^{**}}\Big)\Big(m_2 z^{**} + \frac{\xi}{\beta}y^{**}z^{**} - m_2 z - \frac{\xi}{\beta}yz\Big)\\ &= m_2\Big(2 - \frac{z}{z^{**}} - \frac{z^{**}}{z}\Big) + \frac{\xi}{\beta}y^{**}\Big(-h\Big(\frac{yz}{y^{**}z^{**}}\Big) - h\Big(\frac{z^{**}}{z}\Big) + h\Big(\frac{y}{y^{**}}\Big)\Big). \qquad (4.8)\end{aligned}$$

Moreover,

$$\frac{dh}{dt}\left(\frac{w}{w^{**}}\right) = \frac{1}{w^{**}}\left(1-\frac{w}{w^{**}}\right)\left(\frac{\xi}{\beta}yz - \frac{\xi}{\beta}\frac{y^{**}z^{**}}{w^{**}}w\right)$$
$$= \frac{\xi}{\beta}\frac{y^{**}z^{**}}{w^{**}}\left(-h\left(\frac{yzw^{**}}{y^{**}z^{**}w}\right) + h\left(\frac{yz}{y^{**}z^{**}}\right) - h\left(\frac{w}{w^{**}}\right)\right). \tag{4.9}$$

Using equations (4.6)-(4.10), we obtain

$$\begin{aligned}\frac{d\mathcal{G}}{dt} &= a_1m_1\left(2-\frac{x}{x^{**}}-\frac{x^{**}}{x}\right) + a_1\frac{\beta}{\xi}w^{**}\left(-h\left(\frac{xw}{x^{**}w^{**}}\right) - h\left(\frac{x^{**}}{x}\right) + h\left(\frac{w}{w^{**}}\right)\right)\\ &+ a_2\frac{\beta}{\xi}\frac{x^{**}w^{**}}{y^{**}}\left(h\left(\frac{xw}{x^{**}w^{**}}\right) - h\left(\frac{xwy^{**}}{x^{**}w^{**}y}\right) + h\left(\frac{y^{**}}{y}\right)\right) + a_2\left(2-\frac{y^{**}}{y}-\frac{y}{y^{**}}\right)\\ &+ a_3m_2\left(2-\frac{z}{z^{**}}-\frac{z^{**}}{z}\right) + a_3\frac{\xi}{\beta}y^{**}\left(-h\left(\frac{yz}{y^{**}z^{**}}\right) - h\left(\frac{z^{**}}{z}\right) + h\left(\frac{y}{y^{**}}\right)\right)\\ &+ a_4\frac{\beta}{\xi}\frac{x^{**}w^{**}}{y^{**}}\left(h\left(\frac{xw}{x^{**}w^{**}}\right) - h\left(\frac{y}{y^{**}}\right) - h\left(\frac{xwy^{**}}{x^{**}w^{**}y}\right)\right).\end{aligned}$$

Setting

$$a_1 = \frac{x^{**}}{y^{**}},\ a_2 = 1,\ a_3 = \frac{\beta}{2\xi}\frac{w^{**}}{y^{**}z^{**}} \text{ and } a_4 = \frac{\beta}{2\xi}\frac{1}{y^{**}},$$

we get after a simplification,

$$\begin{aligned}\frac{d\mathcal{G}}{dt} &= m_1\frac{x^{**}}{y^{**}}\left(2-\frac{x}{x^{**}}-\frac{x^{**}}{x}\right) - \frac{\beta}{\xi}w^{**}h\left(\frac{x^{**}}{x}\right) - \frac{\beta}{\xi}\frac{x^{**}w^{**}}{y^{*}}h\left(\frac{xwy^{**}}{x^{**}w^{**}y}\right)\\ &- \frac{1}{2}h\left(\frac{yzw^{**}}{y^{**}z^{**}w}\right) + \frac{\beta}{2\xi}\frac{1}{y^{**}}m_2\left(2-\frac{z}{z^{**}}-\frac{z^{**}}{z}\right) - \frac{1}{2}h\left(\frac{z^{**}}{z}\right)\\ &+ \left(2-\frac{y}{y^{**}}-\frac{y^{**}}{y}\right)\left(\frac{1}{2}-\frac{\beta}{\xi}\frac{x^{**}w^{**}}{y^{**}}\right).\end{aligned} \tag{4.10}$$

Clearly,

$$2-\frac{x}{x^{**}}-\frac{x^{**}}{x} < 0, \quad 2-\frac{y}{y^{**}}-\frac{y^{**}}{y} < 0, \quad 2-\frac{z}{z^{**}}-\frac{z^{**}}{z} < 0.$$

If

$$\frac{\beta}{\xi} = \frac{y^{**}}{2x^{**}w^{**}} \tag{4.11}$$

then $\frac{d\mathcal{G}}{dt} < 0$. In addition, we easily verify that $\frac{d\mathcal{G}}{dt} = 0$ if and only if $(x, y, z, w) = (x^{**}, y^{**}, z^{**}, w^{**})$. Hence, by LaSalle's invariance principle [2], [3], the endemic equilibrium is globally asymptotically stable.

Thus, the following result holds:

Proposition 9: *If condition* (4.11) *hold, thus any endemic equilibrium of the system* (2.10) *is globally asymptotically stable.*

5 Conclusion

In this paper, we formulated and analyzed a deterministic model with a monotonic incidence rate and a piecewise-defined treatment function, in order to investigate its role in the control and elimination of a vector-host disease. The piecewise defined treatment function allows the simplified system to be divided into two subsystems. We first examined the global dynamics of the left subsystem, which corresponds to the case where the number of infected individuals remains below the treatment capacity. The analysis of the existence and stability of equilibria shows that the dynamics of this subsystem is completely determined by the basic reproduction number $\mathcal{R}_0$. We established that when $\mathcal{R}_0 < 1$, the disease free state $\mathcal{E}^0$ is globally asymptotically stable. Conversely, when $\mathcal{R}_0 > 1$, an endemic equilibrium $\mathcal{E}^*$ emerges which is globally asymptotically stable. The analysis of the dynamics of the right subsystem, which corresponds to the case of maximum treatment when the number of infected individuals exceeds the treatment capacity. Using the Lyapunov method, we derived the global stability of the different equilibrium points. The results show that by increasing hospital resources and then reducing the contact rate, we can better control the epidemic.

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