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PROPERTIES OF HYBRID STRUCTURES IN GROUPOIDS

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ABSTRACT. Classical mathematical methods are insufficient for resolving certain issues in real-life human problems due to the uncertainty of the data. Researchers from around the world have created innovative mathematical models, like soft and fuzzy set theories, to model the uncertainties that arise in different areas. Jun recently developed a hybrid structure that combined fuzzy and soft set concepts. The hybrid structure principle is applied to groupoids in this paper, and the properties of hybrid ideals and hybrid subgroupoids in groupoids are also described. Furthermore, the notions of hybrid subgroups, hybrid normal subgroups, and hybrid cosets in a group, as well as their key properties, are discussed. In addition, we show that any member of the collection of hybrid cut sets of a hybrid normal subgroup of a group $\mathscr{G}$ is a normal subgroup of $\mathscr{G}$ in the traditional sense. Finally, we obtain a finite-group hybrid version of Lagrange's theorem.

1. INTRODUCTION

1.1. **Motivations and objectives:** Zadeh presented the idea of fuzzy sets in [22]. Since its inception, fuzzy set theory has undergone various transformations, and it now has applications in a multitude of areas. Rosenfeld[20] used this notion to create fuzzy group theory, which contains many of the basic properties of group theory. He also showed how some fundamental concepts in group theory can be used to generate the theory of fuzzy groups in a simple way, and by using the membership function, he defined all fuzzy groups of a prime cycle group.

In [11], Mukherjee and Bhattacharya went on to study fuzzy group theory and found analogs for numerous group theory findings, as well as fuzzy cosets and fuzzy normal subgroup concepts. They proved that each member of a fuzzy normal subgroup's family of level subgroups is a normal subgroup in the traditional sense, as well as a finite group fuzzy version of Lagrange's theorem. In [1], Akgül defined fuzzy normal subgroups, fuzzy level normal subgroups, and their homomorphisms and provided some of their properties. Malik et al. defined and provided various properties for fuzzy cosets in groups and quotient semigroups of fuzzy subgroups in [8]. Das investigated various properties of level subgroups of a fuzzy subgroup and obtained a similar characterization of all fuzzy subgroups of finite cyclic groups in [21]. Molodtsov [10] pioneered the idea of soft sets as

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a fresh mathematical strategy for handling ambiguous issues. He describes a soft set as a parameterized collection of universe set subsets, each member of which is considered a set of approximate soft set elements. He also effectively implemented the soft set principle in multiple ways ([12, 13]).

In an initial universe set, Jun et al. [7] recently developed the concept of ”hybrid structure” by integrating soft sets and fuzzy sets in a system of parameters. With this notion, the ideas of a hybrid linear space, a hybrid field, and a hybrid subalgebra were established.

In [2], Anis et al. introduced the concepts of hybrid subsemigroups and hybrid ideals in semigroups and examined some of their properties. Elavarasan and Jun [3] established equivalent conditions for a semigroup to be regular and intra-regular in terms of hybrid ideals and hybrid bi-ideals and also used these tools to characterize the left and right simple and completely regular semigroups. Elavarasan et al. characterized the regularity of semigroups in terms of a hybrid generalized bi-ideal, which is an extension of a hybrid bi-ideal, and discussed the properties of a hybrid generalized bi-ideal in [6]. In [17], Porselvi et al. discussed hybrid interior ideals and hybrid simple in an ordered semigroup, as well as characteristic hybrid structures based on ideals and interior ideals.

In [4], Elavarasan et al. came up with the idea of hybrid ideals in zero-symmetric near-rings and looked at how they related to hybrid intersections and hybrid products of hybrid left ideals. Meenakshi et al. presented the idea of hybrid ideals in near-subtraction semigroups in [9] and provided several properties pertaining to hybrid ideals.

Muhiuddin et al. [14] introduced hybrid ideals and $k-$hybrid ideals in a ternary semiring, proved various properties of $k-$hybrid ideals, and characterized hybrid intersections with respect to these $k-$hybrid ideals. Muhiuddin et al. discussed the idea of hybrid subsemimodules in [15], where they also looked at hybrid ideals over semirings. A variety of outcomes have been obtained when hybrid structures are applied to various algebraic systems ([5, 16, 18, 19]).

This work describes the notions of hybrid subgroupoid and hybrid ideals in groupoid, as well as their properties. We show every homomorphic hybrid preimage of a hybrid subgroupoid (resp., left ideal, right ideal) is also a hybrid subgroupoid (resp., left ideal, right ideal). We also describe hybrid subgroups, hybrid normal subgroups, and hybrid cosets in a group, as well as their important properties. Finally, a hybrid version of Lagrange's theorem is established.

1.2. **Applications of Hybrid Structures:**

- Decision-Making Systems: Utilizing hybrid structures in fuzzy logic can enhance decision-making in uncertain environments, such as in finance and healthcare.
- Data Analysis: Hybrid models can improve data classification and clustering in machine learning, effectively handling ambiguous data.
- Control Systems: Implementing hybrid structures in control theory can optimize systems where both precise and imprecise information is present.
- Game Theory: Hybrid structures can enrich strategies in games involving uncertainty, allowing for better predictions and outcomes.
- Artificial Intelligence: Incorporating hybrid soft and fuzzy sets in AI can enhance reasoning capabilities in environments with incomplete information.

2. Preliminaries

In this section, we collect several key concepts that we will need later in our main findings. Throughout this paper, unless stated otherwise, $\mathbb{A}$ is a groupoid with respect to the multiplication "." and the notation for the power set of a non-empty set $\mathscr{Z}$ is $\mathscr{P}(\mathscr{Z})$.

Definition 2.1. [7] Let $\mathscr{T}$ be the universal set and $\mathbb{I} = [0,1]$. Define a hybrid structure in $\mathbb{A}$ over $\mathscr{T}$ as a mapping

$$\tilde{l}_\theta := (\tilde{l}, \theta) : \mathbb{A} \to \mathscr{P}(\mathscr{T}) \times [0,1],\ c_1 \mapsto (\tilde{l}(c_1), \theta(c_1)),$$

where $\tilde{l} : \mathbb{A} \to \mathscr{P}(\mathscr{T})$ and $\theta : \mathbb{A} \to [0,1]$ are mappings.

Consider the family $\mathbb{H}(\mathbb{A})$ of all hybrid structures in $\mathbb{A}$ over $\mathscr{T}$. As follows, define a relationship $\ll$ on $\mathbb{H}(\mathbb{A})$:

$$\left(\forall\, \tilde{k}_\theta, \tilde{z}_\varepsilon \in \mathbb{H}(\mathbb{A})\right)\left(\tilde{k}_\theta \ll \tilde{z}_\varepsilon \Leftrightarrow \tilde{k} \,\tilde{\subseteq}\, \tilde{z}, \theta \succeq \varepsilon\right),$$

where $\tilde{k} \,\tilde{\subseteq}\, \tilde{z}$ implying $\tilde{k}(q) \subseteq \tilde{z}(q)$ and $\theta \succeq \varepsilon$ implying $\theta(q) \geq \varepsilon(q)$ $\forall q \in \mathbb{A}$. The set $(\mathbb{H}(\mathbb{A}), \ll)$ is then partially ordered.

Definition 2.2. [7] Let $\tilde{j}_\varepsilon, \tilde{k}_\vartheta \in \mathbb{H}(\mathbb{A})$.

(i) The hybrid union and hybrid intersection are described as below:

(a) $(\tilde{k}_\vartheta \Cup \tilde{j}_\varepsilon) := (\tilde{k}\tilde{\cup}\tilde{j}, \vartheta \wedge \varepsilon)$, where

$$(\forall r_0 \in \mathbb{A})\begin{pmatrix} (\tilde{k}\tilde{\cup}\tilde{j})(r_0) = \tilde{k}(r_0) \cup \tilde{j}(r_0) \\ (\vartheta \wedge \varepsilon)(r_0) = \vartheta(r_0) \wedge \varepsilon(r_0) \end{pmatrix}.$$

(b) $(\tilde{k}_\vartheta \Cap \tilde{j}_\varepsilon) := (\tilde{k}\tilde{\cap}\tilde{j}, \vartheta \vee \varepsilon)$, where

$$(\forall r_0 \in \mathbb{A})\begin{pmatrix} (\tilde{k}\tilde{\cap}\tilde{j})(r_0) = \tilde{k}(r_0) \cap \tilde{j}(r_0) \\ (\vartheta \vee \varepsilon)(r_0) = \vartheta(r_0) \vee \varepsilon(r_0) \end{pmatrix}.$$

(ii) The hybrid product $\tilde{k}_\vartheta \odot \tilde{j}_\varepsilon := (\tilde{k}\tilde{\circ}\tilde{j}; \vartheta\tilde{\circ}\varepsilon)$ is described as follows: $\forall\ s_0 \in \mathbb{A}$, if $\exists\, w_0, c_0 \in \mathbb{A} : s_0 = w_0 c_0$, then

$$(\tilde{k}\tilde{\circ}\tilde{j})(s_0) = \bigcup_{s_0 = w_0 c_0} \{\tilde{k}(w_0) \cap \tilde{j}(c_0)\};$$

$$(\vartheta\tilde{\circ}\varepsilon)(s_0) = \bigwedge_{s_0 = w_0 c_0} \{\vartheta(w_0) \vee \varepsilon(c_0)\}.$$

Otherwise $(\tilde{k}\tilde{\circ}\tilde{j})(s_0) = \varnothing; (\vartheta\tilde{\circ}\varepsilon)(s_0) = 1$.

Definition 2.3. Let $\tilde{k}_\eta \in \mathbb{H}(\mathbb{A})$. If $\tilde{k}$ and η are constant functions, then $\tilde{k}_\eta$ is described as a constant hybrid structure in $\mathbb{A}$.

Definition 2.4. [7] For $\varnothing \neq Q \subseteq \mathbb{A}$ and $\tilde{m}_\theta \in H(\mathbb{A})$, the characteristic hybrid structure $\chi_Q(\tilde{m}_\theta)$ in $\mathbb{A}$ over $\mathscr{T}$ is known as follows:

$$\chi_Q(\tilde{m}_\theta) = (\chi_Q(\tilde{m}), \chi_Q(\theta)) : \mathbb{A} \longrightarrow \mathscr{P}(\mathscr{T}) \times \mathbb{I},$$
$$q \mapsto (\chi_Q(\tilde{m})(q), \chi_Q(\theta)(q)),$$

where

$$\chi_Q(\tilde{m}) : \mathbb{A} \to \mathscr{P}(\mathscr{T}), q \mapsto \begin{cases} \mathscr{T} & if\ q \in Q \\ \varnothing & otherwise, \end{cases}$$

$$\chi_Q(\theta) : \mathbb{A} \to \mathbb{I}, q \mapsto, \begin{cases} 0 & if\ q \in Q \\ 1 & otherwise \end{cases}$$

for any $q \in \mathbb{A}$.

3. Hybrid subgroupoids and hybrid ideals

In this section, we define hybrid sub-groupoids and hybrid ideals in groupoids and discuss their properties. Additionally, results based on the homomorphic hybrid preimage of a hybrid subgroupoid's (resp., left, right) ideals are proved.

Definition 3.1. Let $\tilde{l}_\theta \in \mathbb{H}(\mathbb{A})$. $\tilde{l}_\theta$ is described as a *hybrid subgroupoid* of $\mathbb{A}$ over $\mathscr{T}$ if

$$(\forall a, s \in \mathbb{A}) \left(\begin{array}{c} \tilde{l}(as) \supseteq \tilde{l}(a) \cap \tilde{l}(s) \\ \theta(as) \leq \theta(a) \vee \theta(s) \end{array} \right).$$

Definition 3.2. Let $\tilde{l}_\theta \in \mathbb{H}(\mathbb{A})$. $\tilde{l}_\theta$ is described as a hybrid ideal in $\mathbb{A}$ over $\mathscr{T}$ if it satisfies the below conditions:

(H1) $(\forall b, s \in \mathbb{A}) \left(\begin{array}{c} \tilde{l}(sb) \supseteq \tilde{l}(b) \\ \theta(sb) \leq \theta(b) \end{array} \right)$.

(H2) $(\forall a, q \in \mathbb{A}) \left(\begin{array}{c} \tilde{l}(aq) \supseteq \tilde{l}(a) \\ \theta(aq) \leq \theta(a) \end{array} \right)$.

Note that if $\tilde{l}_\theta$ satisfies $(H1)$, it is a hybrid left ideal in $\mathbb{A}$, and if $\tilde{l}_\theta$ satisfies $(H2)$, it is a hybrid right ideal in $\mathbb{A}$.

A hybrid right (resp., left) ideal is obviously a hybrid subgroupoid. However, the converse is not always valid, as the succeeding example illustrates.

Example 3.3. Let $\mathbb{A}$ be a family of positive integers. In terms of usual addition, $\mathbb{A}$ is a groupoid. For $Q \in \mathscr{P}(\mathscr{T})\backslash\{\varnothing\}$, define $\tilde{l}_\beta \in \mathbb{H}(\mathbb{A})$ as follows: For $t \in \mathbb{A}$, $\tilde{l}(t) = \begin{cases} Q & if\ t = 2q\ for\ q \in \mathbb{A} \\ \varnothing & otherwise \end{cases}$ and a constant mapping β, $\tilde{l}_\beta$ of $\mathbb{A}$ is a hybrid subgroupoid, but it is not a hybrid left ideal of $\mathbb{A}$ as $\varnothing = \tilde{l}(1+2) \not\supseteq \tilde{l}(2) = Q$.

Definition 3.4. For a group $\mathbb{A}$ and $\tilde{l}_\theta \in \mathbb{H}(\mathbb{A})$, $\tilde{l}_\theta$ is described as a hybrid subgroup of $\mathbb{A}$ over $\mathscr{T}$ if it satisfies the below conditions:

(i) $\tilde{l}_\theta$ is a hybrid subgroupoid in $\mathbb{A}$,

(ii) $(\forall s \in \mathbb{A}) \left(\begin{array}{c} \tilde{l}(s^{-1}) \supseteq \tilde{l}(s) \\ \theta(s^{-1}) \leq \theta(s) \end{array} \right)$.

Theorem 3.1. *Let $\mathbb{A}$ be a group and $\tilde{y}_\varpi$ a hybrid subgroup in $\mathbb{A}$. The conditions mentioned below are valid.*

(i) $(\forall s_0 \in \mathbb{A}) \left(\begin{array}{c} \tilde{y}(s_0) = \tilde{y}(s_0^{-1}) \\ \varpi(s_0) = \varpi(s_0^{-1}) \end{array} \right)$.

(ii) $(\forall s_0 \in \mathbb{A}) \left(\begin{array}{c} \tilde{y}(s_0) \subseteq \tilde{y}(e) \\ \varpi(s_0) \geq \varpi(e) \end{array} \right)$.

(iii) $(\forall s_0, s_1 \in \mathbb{A}) \left(\begin{array}{c} \tilde{y}(s_0 s_1^{-1}) = \tilde{y}(e) \\ \varpi(s_0 s_1^{-1}) = \varpi(e) \end{array} \right) \Longrightarrow \left(\begin{array}{c} \tilde{y}(s_0) = \tilde{y}(s_1) \\ \varpi(s_0) = \varpi(s_1) \end{array} \right)$.

Proof: (i) Let $s_0 \in \mathbb{A}$. Then $\tilde{y}(s_0) = \tilde{y}((s_0^{-1})^{-1}) \supseteq \tilde{y}(s_0^{-1}) \supseteq \tilde{y}(s_0)$; $\varpi(s_0) = \varpi((s_0^{-1})^{-1}) \leq \varpi(s_0^{-1}) \leq \varpi(s_0)$. Hence $\tilde{y}(s_0) = \tilde{y}(s_0^{-1})$ and $\varpi(s_0) = \varpi(s_0^{-1})$.

(ii) Let $s_0 \in \mathbb{A}$. Then $\tilde{y}(e) = \tilde{y}(s_0 s_0^{-1}) \supseteq \tilde{y}(s_0) \cap \tilde{y}(s_0^{-1}) = \tilde{y}(s_0)$; $\varpi(e) = \varpi(s_0 s_0^{-1}) \leq \varpi(s_0) \vee \varpi(s_0^{-1}) = \varpi(s_0)$. Hence $\tilde{y}(s_0) \subseteq \tilde{y}(e)$ and $\varpi(s_0) \geq \varpi(e)$.

(iii) Let $s_0, s_1 \in \mathbb{A}$. Then $\tilde{y}(s_0) = \tilde{y}((s_0 s_1^{-1})s_1) \supseteq \tilde{y}(s_0 s_1^{-1}) \cap \tilde{y}(s_1) \supseteq \tilde{y}(e) \cap \tilde{y}(s_1) = \tilde{y}(s_1) = \tilde{y}((s_1 s_0^{-1})s_0) \supseteq \tilde{y}(s_1 s_0^{-1}) \cap \tilde{y}(s_0) \supseteq \tilde{y}(e) \cap \tilde{y}(s_0) = \tilde{y}(s_0)$; $\varpi(s_0) = \varpi((s_0 s_1^{-1})s_1) \leq \varpi((s_0 s_1^{-1})) \vee \varpi(s_1) \leq \varpi(e) \vee \varpi(s_1) = \varpi(s_1) = \varpi(s_1 s_0^{-1}) \vee \varpi(s_0) \leq \varpi(e) \vee \varpi(s_0) = \varpi(s_0)$. Hence $\tilde{y}(s_0) = \tilde{y}(s_1)$ and $\varpi(s_0) = \varpi(s_1)$.

Theorem 3.2. *For a group $\mathbb{A}$ and $\tilde{z}_\varphi \in \mathbb{H}(\mathbb{A})$, the following assertions are equivalent:*
(i) $\tilde{z}_\varphi$ is a hybrid subgroup of $\mathbb{A}$,
(ii) $(\forall v_1, v_2 \in \mathbb{A}) \left(\begin{array}{c} \tilde{z}(v_1 v_2^{-1}) \supseteq \tilde{z}(v_1) \cap \tilde{z}(v_2) \\ \varphi(v_1 v_2^{-1}) \leq \varphi(v_1) \vee \varphi(v_2) \end{array} \right)$.

Proof: Assume that $\tilde{z}_\varphi$ is a hybrid subgroup in $\mathbb{A}$ and let $v_1, v_2 \in \mathbb{A}$. Then $\tilde{z}(v_1 v_2^{-1}) \supseteq \tilde{z}(v_1) \cap \tilde{z}(v_2^{-1}) = \tilde{z}(v_1) \cap \tilde{z}(v_2)$, $\varphi(v_1 v_2^{-1}) \leq \varphi(v_1) \vee \varphi(v_2^{-1}) = \varphi(v_1) \vee \varphi(v_2)$.

Conversely, let $v_2 = v_1$ in (ii). Then we get $\tilde{z}(e) \supseteq \tilde{z}(v_1)$ and $\varphi(e) \leq \varphi(v_1)$ for all $v_1 \in \mathbb{A}$. So $\tilde{z}(v_2^{-1}) = \tilde{z}(e v_2^{-1}) \supseteq \tilde{z}(e) \cap \tilde{z}(v_2) = \tilde{z}(v_2)$; $\varphi(v_2^{-1}) = \varphi(e v_2^{-1}) \leq \varphi(e) \vee \varphi(v_2) = \varphi(v_2)$. Now, $\tilde{z}(v_1 v_2) = \tilde{z}(v_1 (v_2^{-1})^{-1}) \supseteq \tilde{z}(v_1) \cap \tilde{z}(v_2^{-1}) \supseteq \tilde{z}(v_1) \cap \tilde{z}(v_2)$; $\varphi(v_1 v_2) = \varphi(v_1 (v_2^{-1})^{-1}) \leq \varphi(v_1) \vee \varphi(v_2^{-1}) \leq \varphi(v_1) \vee \varphi(v_2)$. So $\tilde{z}_\varphi$ is a hybrid subgroup in $\mathbb{A}$.

Lemma 3.3. *In a finite group $\mathbb{A}$, if $\tilde{k}_\delta$ is a hybrid subgroupoid, then $\tilde{k}_\delta$ in $\mathbb{A}$ is a hybrid subgroup.*

Proof: Let $s_1 \in \mathbb{A}$. Then there exists an integer n such that $s_1^n = e$ which implies $s_1^{-1} = s_1^{n-1}$. By repeatedly applying the definition of a hybrid subgroupoid, we obtain $\tilde{k}(s_1^{-1}) = \tilde{k}(s_1^{n-1}) = \tilde{k}(s_1^{n-2} s_1) \supseteq \tilde{k}(s_1)$ and $\delta(s_1^{-1}) = \delta(s_1^{n-1}) = \delta(s_1^{n-2} s_1) \leq \delta(s_1)$. Hence $\tilde{k}_\delta$ is a hybrid subgroup in $\mathbb{A}$.

Lemma 3.4. *For $\varnothing \neq Q \subseteq \mathbb{A}$ and $\tilde{k}_\delta \in \mathbb{H}(\mathbb{A})$, we have*

(i) Q is a subgroupoid(resp., right ideal, left ideal) of $\mathbb{A}$ if and only if $\chi_Q(\tilde{k}_\delta)$ is a hybrid subgroupoid(resp., right ideal, left ideal) in $\mathbb{A}$.

(ii) If $\mathbb{A}$ is a group, then Q is a subgroup of $\mathbb{A}$ if and only if $\chi_Q(\tilde{k}_\delta)$ is a hybrid subgroup of $\mathbb{A}$.

Proof: The proof of (i) is identical to those for Theorem 3.5. of [2].

(ii) Assume that Q is a subgroup of $\mathbb{A}$. Then by (i), $\chi_Q(\tilde{k}_\delta)$ is a hybrid subgroupoid in $\mathbb{A}$. Let $w \in \mathbb{A}$.

If $w \in Q$, then $w^{-1} \in Q$. So $\chi_Q(\tilde{k})(w) = \mathscr{T} = \chi_Q(\tilde{k})(w^{-1})$ and $\chi_Q(\delta)(w) = 0 = \chi_Q(\delta)(w^{-1})$. If $w \notin Q$, then $w^{-1} \notin Q$. So $\chi_Q(\tilde{k})(w) = \varnothing = \chi_Q(\tilde{k})(w^{-1})$ and $\chi_Q(\delta)(w) = 1 = \chi_Q(\delta)(w^{-1})$. Therefore $\chi_Q(\tilde{k}_\delta)$ is a hybrid subgroup in $\mathbb{A}$.

Conversely, let $w \in Q$. Then $\mathscr{T} = \chi_Q(\tilde{k})(w) = \chi_Q(\tilde{k})(w^{-1})$ and $0 = \chi_Q(\delta)(w) = \chi_Q(\delta)(w^{-1})$ which imply $w^{-1} \in Q$.

Theorem 3.5. *For $\tilde{l}_\delta \in \mathbb{H}(\mathbb{A})$, the below criteria are equivalent:*
(i) $\tilde{l}_\delta$ in $\mathbb{A}$ is a hybrid subgroupoid (resp., right ideal, left ideal, ideal),
(ii) for any $Q \in \mathscr{P}(\mathscr{T})$ and $\partial \in \mathbb{I}$, the sets

$$\varnothing \neq L_{\tilde{l}}^Q := \{s \in \mathbb{A} : \tilde{l}(x) \supseteq Q\} \text{ and } \varnothing \neq L_\delta^\partial := \{s \in \mathbb{A} : \ \delta(s) \leq \partial\}$$

are subgroupoids (resp., right ideal, left ideal, ideal) of $\mathbb{A}$.

Proof: (i) Consider $L_{\tilde{l}}^Q \neq \varnothing$ and $L_\delta^\partial \neq \varnothing$ for any $(Q, \partial) \in \mathscr{P}(\mathscr{T}) \times \mathbb{I}$.

If $\tilde{l}_\delta$ is a hybrid subgroupoid of $\mathbb{A}$, then for $s, t \in L_{\tilde{l}}^Q \cap L_\delta^\partial$, we get $\tilde{l}(st) \supseteq \tilde{l}(t) \cap \tilde{l}(s) \supseteq Q$ and $\delta(st) \leq \delta(s) \vee \delta(t) \leq \partial$ imply that $st \in L_{\tilde{l}}^Q \cap L_\delta^\partial$. So $L_{\tilde{l}}^Q$ and L_δ^∂ are subgroupoids of $\mathbb{A}$.

If $\tilde{l}_\delta$ is hybrid left ideal and $r \in \mathbb{A}$, then $\tilde{l}(rt) \supseteq \tilde{l}(t) \supseteq Q$ and $\delta(rt) \leq \delta(t) \leq \partial$ which imply $rt \in L_{\tilde{l}}^Q \cap L_\delta^\partial$. So $L_{\tilde{l}}^Q$ and L_δ^∂ are left ideals of $\mathbb{A}$.

Assume, on the other hand, that $\forall(Q,\partial)\in\mathscr{P}(\mathscr{T})\times\mathbb{I}$, $L^{Q}_{\tilde{l}}\neq\varnothing$ and $L^{\partial}_{\delta}\neq\varnothing$ are subgroupoids of $\mathbb{A}$. For $s,r\in\mathbb{A}$, let $\tilde{l}(r)=Q_r;\tilde{l}(s)=Q_s$ and $\delta(r)=\partial_r;\delta(s)=\partial_s$ for some $Q_r,Q_s\in\mathscr{P}(\mathscr{T});\partial_r,\partial_s\in\mathbb{I}$.

Set $Q=Q_r\cap Q_s$ and $\partial=\partial_r\vee\partial_s$. Then $r,s\in L^{Q}_{\tilde{l}}\cap L^{\partial}_{\delta}$. By assumption, $L^{Q}_{\tilde{l}}$ and L^{∂}_{δ} are subgroupoids, we have $rs\in L^{Q}_{\tilde{l}}\cap L^{\partial}_{\delta}$ which implies $\tilde{l}(rs)\supseteq Q=Q_r\cap Q_s=\tilde{l}(r)\cap\tilde{l}(s)$ and $\delta(rs)\leq\partial=\partial_r\vee\partial_s=\delta(r)\vee\delta(s)$. So $\tilde{l}_\delta$ is a hybrid subgroupoid of $\mathbb{A}$.

Assume that $L^{Q}_{\tilde{l}}$and L^{∂}_{δ} are left ideals of $\mathbb{A}$ $\forall(Q,\partial)\in\mathscr{P}(\mathscr{T})\times\mathbb{I}$. Let $c\in\mathbb{A}$. Then $\tilde{l}(c)=Q_c;\delta(c)=\partial_c$ for some $(Q_c,\partial_c)\in\mathscr{P}(\mathscr{T})\times\mathbb{I}$ with $sc\in L^{Q_c}_{\tilde{l}}\cap L^{\partial_c}_{\delta}$ which implies $\tilde{l}(sc)\supseteq Q_c=\tilde{l}(c)$ and $\delta(sc)\leq\partial_c=\delta(c)$. So $\tilde{l}_\delta$ of $\mathbb{A}$ is a hybrid left ideal.

Definition 3.5. For $\tilde{d}_\varrho\in\mathbb{H}(\mathbb{A})$ and $(\nabla,\xi)\in\mathscr{P}(\mathscr{T})\times\mathbb{I}$, the set

$$\tilde{d}_\varrho[\nabla,\xi]:=\{q_1\in\mathbb{A}\mid\tilde{d}(q_1)\supseteq\nabla,\varrho(q_1)\leq\xi\}$$

is termed as the $[\nabla,\xi]$ -hybrid cut of $\tilde{d}_\varrho$.

For a hybrid groupoid (resp., ideal, right ideal, left ideal) $\tilde{d}_\varrho\in\mathbb{H}(\mathbb{A})$, $\tilde{d}_\varrho[\nabla,\xi]$ of $\mathbb{A}$ is a groupoid (resp., ideal, right ideal, left ideal).

Theorem 3.6. *Let $\tilde{j}_\varkappa\in\mathbb{H}(\mathbb{A})$. For any $(\Upsilon,\mu)\in\mathscr{P}(\mathscr{T})\times\mathbb{I}$, the assertions mentioned below are equivalent:*

(i) $\tilde{j}_\varkappa[\Upsilon,\mu]$ of $\mathbb{A}$ is a subgroupoid (resp., left ideal, right ideal, ideal),

(ii) $\tilde{j}_\varkappa$ of $\mathbb{A}$ is a hybrid subgroupoid (resp., left ideal, right ideal, ideal).

Proof: The proof is similar to Theorem 3.5.

Theorem 3.7. *For $\tilde{q}_\delta$, $\tilde{m}_\zeta\in\mathbb{H}(\mathbb{A})$, the assertions listed below are valid:*

(i) If $\tilde{q}_\delta$ and $\tilde{m}_\zeta$ are hybrid subgroupoids in $\mathbb{A}$, then $\tilde{q}_\delta\Cap\tilde{m}_\zeta$ in $\mathbb{A}$ is a hybrid subgroupoid.

(ii) If $\tilde{q}_\delta$ and $\tilde{m}_\zeta$ are hybrid left (resp., right) ideals in $\mathbb{A}$, then $\tilde{q}_\delta\Cap\tilde{m}_\zeta$ and $\tilde{q}_\delta\Cup\tilde{m}_\zeta$ in $\mathbb{A}$ are hybrid left (resp., right) ideals.

(iii) If $\tilde{q}_\delta$ and $\tilde{m}_\zeta$ are hybrid subgroups of a group $\mathbb{A}$, then $\tilde{q}_\delta\Cap\tilde{m}_\zeta$ in $\mathbb{A}$ is a hybrid subgroup.

Proof: (i) Assume that $\tilde{q}_\delta$ and $\tilde{m}_\zeta$ are hybrid subgroupoids of $\mathbb{A}$ and let $a,s\in\mathbb{A}$. Then $(\tilde{q}\tilde{\cap}\tilde{m})(as)=\tilde{q}(as)\cap\tilde{m}(as)\supseteq\{\tilde{q}(a)\cap\tilde{q}(s)\}\cap\{\tilde{m}(a)\cap\tilde{m}(s)\}=\{\tilde{q}(a)\cap\tilde{m}(a)\}\cap\{\tilde{q}(s)\cap\tilde{m}(s)\}=(\tilde{q}\cap\tilde{m})(a)\cap(\tilde{q}\cap\tilde{m})(s),(\delta\vee\zeta)(as)=\delta(as)\vee\zeta(as)\leq\{\delta(a)\vee\delta(s)\}\vee\{\zeta(a)\vee\zeta(s)\}\leq\{\delta(a)\vee\zeta(a)\}\vee\{\delta(s)\vee\zeta(s)\}=(\delta\vee\zeta)(a)\vee(\delta\vee\zeta)(s)$. Hence $\tilde{q}_\delta\Cap\tilde{m}_\zeta$ is a hybrid subgroupoid in $\mathbb{A}$.

(ii) Consider that $\tilde{q}_\delta$ and $\tilde{m}_\zeta$ are hybrid left ideals in $\mathbb{A}$. For $a,s\in\mathbb{A}$, we get $(\tilde{q}\tilde{\cap}\tilde{m})(as)=\tilde{q}(as)\cap\tilde{m}(as)\supseteq\tilde{q}(s)\cap\tilde{m}(s)=(\tilde{q}\cap\tilde{m})(s),(\delta\vee\zeta)(as)=\delta(as)\vee\zeta(as)\leq\delta(s)\vee\zeta(s)=(\delta\vee\zeta)(s)$. So $\tilde{q}_\delta\Cap\tilde{m}_\zeta$ is a hybrid left ideal in $\mathbb{A}$.

Also, $(\tilde{q}\tilde{\cup}\tilde{m})(as)=\tilde{q}(as)\cup\tilde{m}(as)\supseteq\tilde{q}(s)\cup\tilde{m}(s)=(\tilde{q}\cup\tilde{m})(s),(\delta\wedge\zeta)(as)=\delta(as)\wedge\zeta(as)\leq\delta(s)\wedge\zeta(s)=(\delta\wedge\zeta)(s)$. So $\tilde{q}_\delta\Cup\tilde{m}_\zeta$ is a hybrid left ideal in $\mathbb{A}$.

(iii) If $\tilde{q}_\delta$ and $\tilde{m}_\zeta$ are hybrid subgroups and $b_0\in\mathbb{A}$. Then $(\tilde{q}\tilde{\cap}\tilde{m})(b_0)=\tilde{q}(b_0)\cap\tilde{m}(b_0)=\tilde{q}(b_0^{-1})\cap\tilde{m}(b_0^{-1})=(\tilde{q}\cap\tilde{m})(b_0^{-1}),(\delta\vee\zeta)(b_0)=\delta(b_0)\vee\zeta(b_0)=\delta(b_0^{-1})\vee\zeta(b_0^{-1})=(\delta\vee\zeta)(b_0^{-1})$. So $\tilde{q}_\delta\Cap\tilde{m}_\zeta$ is a hybrid subgroup in $\mathbb{A}$.

It is important to note that the hybrid union of two hybrid subgroupoids does not have to be a hybrid subgroupoid.

Example 3.6. Let $\mathbb{A}$ be the set of positive integers. Then, with respect to usual addition, $\mathbb{A}$ is a groupoid. Define the hybrid structures $\tilde{l}_\eta$ and $\tilde{g}_\mu$ of $\mathbb{A}$ over $P(\neq\varnothing)$ as follows: For

$s \in \mathbb{A}$,

$$\tilde{l}(s) = \begin{cases} P & if\ s = 2x\ for\ x \in \mathbb{A} \\ \varnothing & otherwise, \end{cases}$$

$$\tilde{g}(s) = \begin{cases} P & if\ s = 3x\ for\ x \in \mathbb{A} \\ \varnothing & otherwise \end{cases}$$

and any constant mappings η and μ, $\tilde{l}_\eta$ and $\tilde{g}_\mu$ are hybrid subgroupoids of $\mathbb{A}$, but $\tilde{l}_\eta \Cup \tilde{g}_\mu$ of $\mathbb{A}$ is not a hybrid subgropoid as $\varnothing = (\tilde{l} \cup \tilde{g})(2+3) \not\supseteq (\tilde{l} \cup \tilde{g})(2) \cup (\tilde{l} \cup \tilde{g})(3)$.

Theorem 3.8. *Let $\mathbb{A}$ be a group and $\tilde{p}_\eta \in \mathbb{H}(\mathbb{A})$. Then $\tilde{p}_\eta$ is a constant hybrid structure in $\mathbb{A}$ if and only if $\tilde{p}_\eta$ is a hybrid left (resp., right) ideal in $\mathbb{A}$.*

Proof: If $\tilde{p}_\eta$ is a constant hybrid structure in $\mathbb{A}$, then clearly $\tilde{p}_\eta$ is a hybrid left ideal in $\mathbb{A}$.

Conversely, assume that $\tilde{p}_\eta$ is a hybrid left ideal in $\mathbb{A}$ and let $a_0 \in \mathbb{A}$. Then $\tilde{p}(a_0) = \tilde{p}(a_0 e) \supseteq \tilde{p}(e)$ and $\tilde{p}(e) = \tilde{p}(a_0^{-1} a_0) \supseteq \tilde{p}(a_0)$ which imply $\tilde{p}(e) = \tilde{p}(a_0)$. Also, $\eta(a_0) = \eta(a_0 e) \leq \eta(e)$ and $\eta(e) = \eta(a_0^{-1} a_0) \leq \eta(a_0)$ which imply $\eta(e) = \eta(a_0)$.

Thus $\tilde{p}$ and η are constant functions and hence $\tilde{p}_\eta$ is a constant hybrid structure in $\mathbb{A}$.

Definition 3.7. [7] Consider the non-empty sets $\mathbb{A}_1$ and $\mathbb{A}_2$. Let $\Omega : \mathbb{A}_1 \to \mathbb{A}_2$ be a mapping from $\mathbb{A}_1$ to $\mathbb{A}_2$. For $\tilde{k}_\eta \in \mathbb{H}(\mathbb{A}_2)$, define a hybrid structure $\Omega^{-1}(\tilde{k}_\eta) := (\Omega^{-1}(\tilde{k}), \Omega^{-1}(\eta))$ in $\mathbb{A}_1$ over $\mathscr{T}$, where $\Omega^{-1}(\tilde{k}(q)) = \tilde{k}(\Omega(q))$ and $\Omega^{-1}(\eta)(q) = \eta(\Omega(q))$ for all $q \in \mathbb{A}_1$. We describe that $\Omega^{-1}(\tilde{k}_\eta)$ is the hybrid preimage of $\tilde{k}_\eta$ under Ω.

For $\tilde{k}_\eta \in \mathbb{H}(\mathbb{A}_1)$, the hybrid image of $\tilde{k}_\eta$ under Ω is the hybrid structure $\Omega(\tilde{k}_\eta) := (\Omega(\tilde{k}), \Omega(\eta))$ in $\mathbb{A}_2$ over $\mathscr{T}$, where

$$(\Omega(\tilde{k}))(d) = \begin{cases} \bigcup_{q \in \Omega^{-1}(d)} \tilde{k}(q) & if\ \Omega^{-1}(d) \neq \varnothing \\ \varnothing & otherwise, \end{cases}$$

$$(\Omega(\eta))(d) = \begin{cases} \bigwedge_{q \in \Omega^{-1}(d)} \eta(q) & if\ \Omega^{-1}(d) \neq \varnothing \\ 1 & otherwise \end{cases}$$

for every $d \in \mathbb{A}_2$.

Theorem 3.9. *Every homomorphic hybrid preimage of a hybrid subgroupoid (resp., right ideal, left ideal, ideal) is also a hybrid subgroupoid (resp., right ideal, left ideal, ideal).*

Proof: Consider a groupoid homomorphism $\Omega : \mathbb{A}_1 \to \mathbb{A}_2$ and let $\tilde{l}_\mu$ be a hybrid subgroupoid of $\mathbb{A}_2$ over $\mathscr{T}$.

Let $w, s \in \mathbb{A}_1$. Then $\Omega^{-1}(\tilde{l})(ws) = \tilde{l}(\Omega(ws)) = \tilde{l}(\Omega(w)\Omega(s)) \supseteq \tilde{l}(\Omega(w)) \cap \tilde{l}(\Omega(s)) = \Omega^{-1}(\tilde{l})(w) \cap \Omega^{-1}(\tilde{l})(s), \Omega^{-1}(\mu)(ws) = \mu(\Omega(ws)) = \mu(\Omega(w)\Omega(s)) \leq \mu(\Omega(w)) \vee \mu(\Omega(s)) = \Omega^{-1}(\mu)(w) \vee \Omega^{-1}(\mu)(s)$. So $\Omega^{-1}(\tilde{l}_\mu)$ is a hybrid subgroupoid of $\mathbb{A}_1$.

For a hybrid left ideal $\tilde{l}_\mu$ in $\mathbb{A}_2$, we get $\Omega^{-1}(\tilde{l})(sw) = \tilde{l}(\Omega(sw)) = \tilde{l}(\Omega(s)\Omega(w)) \supseteq \tilde{l}(\Omega(w)) = \Omega^{-1}(\tilde{l})(w)$, $\Omega^{-1}(\mu)(sw) = \mu(\Omega(sw)) = \mu(\Omega(s)\Omega(w)) \leq \mu(\Omega(w)) = \Omega^{-1}(\mu)(w)$ for any $w, s \in \mathbb{A}_1$. So $\Omega^{-1}(\tilde{l}_\mu)$ of $\mathbb{A}_1$ is a hybrid left ideal.

Theorem 3.10. *Let $\Omega : \mathbb{A}_1 \to \mathbb{A}_2$ be a onto homomorphism of groupoids. For $\tilde{l}_\tau \in \mathbb{H}(\mathbb{A}_2)$, if the preimage $\Omega^{-1}(\tilde{l}_\tau)$ of $\tilde{l}_\tau$ under Ω is a hybrid subgroupoid (resp., right ideal, left ideal,*

ideal) of $\mathbb{A}_1$ over $\mathscr{T}$, then $\tilde{l}_\tau$ is a hybrid subgroupoid (resp., right ideal, left ideal, ideal) of $\mathbb{A}_2$ over $\mathscr{T}$.

Proof: Let $\Omega : \mathbb{A}_1 \to \mathbb{A}_2$ be a onto homomorphism and let $\tilde{l}_\tau \in \mathbb{H}(\mathbb{A}_2)$.

Assume that the preimage $\Omega^{-1}(\tilde{l}_\tau)$ of $\tilde{l}_\tau$ under Ω is a hybrid subgroupoid of $\mathbb{A}_1$.

Let $t, s, r \in \mathbb{A}_2$. Then $\Omega(r_1) = r, \Omega(s_1) = s$ and $\Omega(t_1) = t$ for some $r_1, s_1, t_1 \in \mathbb{A}_1$. Now, $\tilde{l}(rs) = \tilde{l}(\Omega(r_1)\Omega(s_1)) = \tilde{l}(\Omega(r_1 s_1)) = \Omega^{-1}(\tilde{l})(r_1 s_1) \supseteq \Omega^{-1}(\tilde{l})(r_1) \cap \Omega^{-1}(\tilde{l})(s_1) = \tilde{l}(\Omega(r_1)) \cap \tilde{l}(\Omega(s_1)) = \tilde{l}(r) \cap \tilde{l}(s), \tau(rs) = \tau(\Omega(r_1)\Omega(s_1)) = \tau(\Omega(r_1 s_1)) = \Omega^{-1}(\tau)(r_1 s_1) \leq \Omega^{-1}(\tau)(r_1) \vee \Omega^{-1}(\tau)(s_1) = \tau(\Omega(r_1)) \vee \tau(\Omega(s_1)) = \tau(r) \vee \tau(s)$. So $\tilde{l}_\tau$ of $\mathbb{A}_2$ is a hybrid subgroupoid.

If $\Omega^{-1}(\tilde{l}_\tau)$ is a hybrid left ideal of $\mathbb{A}_1$, then $\tilde{l}(rs) = \tilde{l}(\Omega(r_1)\Omega(s_1)) = \tilde{l}(\Omega(r_1 s_1)) = \Omega^{-1}(\tilde{l})(r_1 s_1) \supseteq \Omega^{-1}(\tilde{l})(s_1) = \tilde{l}(\Omega(s_1)) = \tilde{l}(s), \tau(rs) = \tau(\Omega(r_1)\Omega(s_1)) = \tau(\Omega(r_1 s_1)) = \Omega^{-1}(\tau)(r_1 s_1) \leq \Omega^{-1}(\tau)(s_1) = \tau(\Omega(s_1)) = \tau(s)$. So $\tilde{l}_\tau$ of $\mathbb{A}_2$ is a hybrid left ideal.

We describe that $\tilde{k}_\delta \in \mathbb{H}(\mathbb{A})$ has the *sup property* if for $Q \in \mathscr{P}(\mathbb{A})$, $\exists b_0 \in Q$: $\tilde{k}(b_0) = \bigcup_{b \in Q} \tilde{k}(b)$ and $\delta(b_0) = \bigwedge_{b \in Q} \delta(b)$.

Theorem 3.11. *A homomorphic image of a hybrid subgroupoid (resp., right ideal, left ideal) which has the sup property is a hybrid subgroupoid (resp., right ideal, left ideal).*

Proof: Let $\Omega : \mathbb{A}_1 \to \mathbb{A}_2$ be a groupoid homomorphism and $\tilde{a}_\eta$ be a hybrid structure of $\mathbb{A}_1$ with the sup property and $\tilde{l}_\mu$ be the image of $\tilde{a}_\eta$ under Ω.

For $\Omega(v), \Omega(s) \in \Omega(\mathbb{A}_1)$, let $v_0 \in \Omega^{-1}(\Omega(v))$, $s_0 \in \Omega^{-1}(\Omega(s))$ be such that

$$\tilde{a}(v_0) = \bigcup_{s \in \Omega^{-1}(\Omega(v))} \tilde{a}(s); \eta(v_0) = \bigwedge_{s \in \Omega^{-1}(\Omega(v))} \eta(s) \ \tilde{a}(s_0) = \bigcup_{s \in \Omega^{-1}(\Omega(s))} \tilde{a}(s);$$
$$\eta(s_0) = \bigwedge_{s \in \Omega^{-1}(\Omega(s))} \eta(s).$$

If $\tilde{a}_\eta$ is a hybrid subgroupoid of $\mathbb{A}_1$, then

$$\begin{aligned}
\tilde{l}(\Omega(v)\Omega(s)) &= \bigcup_{s \in \Omega^{-1}(\Omega(v)\Omega(s))} \tilde{a}(s) \\
&\supseteq \tilde{a}(v_0 s_0) \supseteq \tilde{a}(v_0) \cap \tilde{a}(s_0) \\
&= \{\bigcup_{s \in \Omega^{-1}(\Omega(v))} \tilde{a}(s)\} \cap \{\bigcup_{s \in \Omega^{-1}(\Omega(s))} \tilde{a}(s)\} \\
&= \tilde{l}(\Omega(v)) \cap \tilde{l}(\Omega(s)), \\
\mu(\Omega(v)\Omega(s)) &= \bigwedge_{s \in \Omega^{-1}(\Omega(v)\Omega(s))} \eta(s) \\
&\leq \eta(v_0 s_0) \leq \eta(v_0) \vee \eta(s_0) \\
&= \{\bigwedge_{s \in \Omega^{-1}(\Omega(v))} \eta(s)\} \vee \{\bigwedge_{s \in \Omega^{-1}(\Omega(s))} \eta(s)\} \\
&= \eta(\Omega(v)) \vee \eta(\Omega(s)).
\end{aligned}$$

Hence $\Omega(\tilde{a}_\eta)$ is a hybrid subgroupoid in $\mathbb{A}_2$.

If $\tilde{a}_\eta$ is a hybrid left ideal of $\mathbb{A}_1$, then $\tilde{l}(\Omega(v)\Omega(s)) = \bigcup_{s\in\Omega^{-1}(\Omega(v)\Omega(s))} \tilde{a}(s) \supseteq \tilde{a}(v_0 s_0) \supseteq \tilde{a}(s_0) = \bigcup_{s\in\Omega^{-1}(\Omega(v))} \tilde{a}(s) = \tilde{l}(\Omega(s)), \mu(\Omega(v)\Omega(s)) = \bigwedge_{s\in\Omega^{-1}(\Omega(v)\Omega(s))} \eta(s) \leq \eta(v_0 s_0) \leq \eta(s_0) = \bigwedge_{s\in\Omega^{-1}(\Omega(v))} (\mu(s)) = \mu(\Omega(s))$. Hence $\Omega(\tilde{a}_\eta)$ is a hybrid left ideal of $\mathbb{A}_2$. In a similar way, we can prove for hybrid right ideal.

4. Hybrid normal subgroups

In this section, we define hybrid normal subgroups and hybrid cosets within a group and discuss some of their key properties. Finally, we show the finite-group hybrid version of Lagrange's theorem.

Definition 4.1. A hybrid structure $\tilde{k}_\theta \in \mathbb{H}(\mathbb{A})$ is described as a hybrid normal subgroup in $\mathbb{A}$ if it satisfies the below conditions:

(i) $\tilde{k}_\theta$ is a hybrid subgroup in $\mathbb{A}$.

(ii) $(\forall r, t \in \mathbb{A}) \begin{pmatrix} \tilde{k}(rt) = \tilde{k}(tr) \\ \theta(rt) = \theta(tr) \end{pmatrix}$.

Theorem 4.1. *If $\tilde{k}_\theta \in \mathbb{H}(\mathbb{A})$ is a hybrid subgroup of a group $\mathbb{A}$, then the below assertions are equivalent:*

(i) $(\forall t, s \in \mathbb{A}) \begin{pmatrix} \tilde{k}(sts^{-1}) = \tilde{k}(t) \\ \theta(sts^{-1}) = \theta(t) \end{pmatrix}$,

(ii) $\tilde{k}_\theta$ of $\mathbb{A}$ is a hybrid normal subgroup.

Proof: For any $s, t \in \mathbb{A}$, we get $\tilde{k}(st) = \tilde{k}(s(ts)s^{-1}) = \tilde{k}(ts)$ and $\theta(st) = \theta(s(ts)s^{-1}) = \theta(ts)$.

Conversely, let $s, t \in \mathbb{A}$. Then

$$\tilde{k}(sts^{-1}) = \tilde{k}((st)s^{-1}) = \tilde{k}(s^{-1}(st)) = \tilde{k}(s^{-1}st) = \tilde{k}(t),$$
$$\theta(sts^{-1}) = \theta((st)s^{-1}) = \theta(s^{-1}(st)) = \theta(s^{-1}st) = \theta(t).$$

Theorem 4.2. *If $\tilde{d}_\varrho \in \mathbb{H}(\mathbb{A})$ is a hybrid subgroup of a group $\mathbb{A}$, then the below criteria are equivalent:*

(i) $\tilde{d}_\varrho$ in $\mathbb{A}$ is a hybrid normal subgroup,

(ii) $(\forall t_0, s_0 \in \mathbb{A}) \begin{pmatrix} \tilde{d}(s_0^{-1}t_0^{-1}s_0 t_0) \supseteq \tilde{d}(s_0) \\ \varrho(s_0^{-1}t_0^{-1}s_0 t_0) \leq \varrho(s_0) \end{pmatrix}$.

Proof: If $\tilde{d}_\varrho$ of $\mathbb{A}$ is a hybrid normal subgroup, then, by Theorem 4.1, for $s_0, t_0 \in \mathbb{A}$,

$$\begin{aligned} \tilde{d}(s_0^{-1}t_0^{-1}s_0 t_0) &\supseteq \tilde{d}(s_0^{-1}) \cap \tilde{d}(t_0^{-1}s_0 t_0) \\ &= \tilde{d}(s_0) \cap \tilde{d}(s_0) = \tilde{d}(s_0), \\ \varrho(s_0^{-1}t_0^{-1}s_0 t_0) &\leq \varrho(s_0^{-1}) \vee \varrho(t_0^{-1}s_0 t_0) \\ &= \varrho(s_0) \vee \varrho(s_0) = \varrho(s_0). \end{aligned}$$

Assume that (ii) is valid and let $s_0, t_0, a_0 \in \mathbb{A}$. Then $\tilde{d}(s_0^{-1}a_0s_0) = \tilde{d}(a_0a_0^{-1}s_0^{-1}a_0s_0) \supseteq \tilde{d}(a_0) \cap \tilde{d}(a_0^{-1}s_0^{-1}a_0s_0) \supseteq \tilde{d}(a_0) \cap \tilde{d}(a_0) = \tilde{d}(a_0)$, $\varrho(s_0^{-1}a_0s_0) = \varrho(a_0a_0^{-1}s_0^{-1}a_0s_0) \leq \varrho(a_0) \vee \varrho(a_0^{-1}s_0^{-1}a_0s_0) \leq \varrho(a_0) \vee \varrho(a_0) = \varrho(a_0)$.

If $a_0 = s_0t_0$, then we get $\tilde{d}(t_0s_0) \supseteq \tilde{d}(s_0t_0)$ and $\varrho(t_0s_0) \leq \varrho(s_0t_0)$.

Replace s_0 by s_0^{-1} in the above steps, we can get $\tilde{d}(s_0a_0s_0^{-1}) \supseteq \tilde{d}(a_0)$ and $\varrho(s_0a_0s_0^{-1}) \leq \varrho(a_0)$. If $a_0 = t_0s_0$ in this case, we get $\tilde{d}(s_0t_0) \supseteq \tilde{d}(t_0s_0)$ and $\varrho(s_0t_0) \leq \varrho(t_0s_0)$.

Therefore $\tilde{d}_\varrho$ of $\mathbb{A}$ is a hybrid normal subgroup.

Theorem 4.3. *For a group $\mathbb{A}$ and $\tilde{l}_\theta \in \mathbb{H}(\mathbb{A})$, the below assertions are equivalent:*

(i) $\tilde{l}_\theta$ of $\mathbb{A}$ is a hybrid normal subgroup,

(ii) for any $X \in \mathscr{P}(\mathscr{T}); \Delta \in \mathbb{I}$, the sets

$$\varnothing \neq \mathbb{A}_{\tilde{l}}^X := \{s \in \mathbb{A} : \tilde{l}(x) \supseteq X\} \text{ and } \varnothing \neq \mathbb{A}_\theta^\Delta := \{s \in \mathbb{A} : \theta(s) \leq \Delta\}$$

are normal subgroups of $\mathbb{A}$.

Proof: For $(X, \Delta) \in \mathscr{P}(\mathscr{T}) \times \mathbb{I}$, by Theorem 3.5, $\mathbb{A}_{\tilde{l}}^X$ and $\mathbb{A}_\theta^\Delta$ are subgroups of $\mathbb{A}$. Let $s \in \mathbb{A}$ and $t \in \mathbb{A}_{\tilde{l}}^X \cap \mathbb{A}_\theta^\Delta$. Then $\tilde{l}(t) \supseteq X$ and $\theta(t) \leq \Delta$. By Theorem 3.5, $\tilde{l}(s^{-1}ts) = \tilde{l}(t) \supseteq X$ and $\theta(s^{-1}ts) = \theta(t) \leq \Delta$ which imply $s^{-1}ts \in \mathbb{A}_{\tilde{l}}^X \cap \mathbb{A}_\theta^\Delta$. Hence $\mathbb{A}_{\tilde{l}}^X$ and $\mathbb{A}_\theta^\Delta$ are normal subgroups of $\mathbb{A}$.

Conversely, assume $\mathbb{A}_{\tilde{l}}^X \neq \varnothing$ and $\mathbb{A}_\theta^\Delta \neq \varnothing$ for all $(X, \Delta) \in \mathscr{P}(\mathscr{T}) \times \mathbb{I}$. Then, by Theorem 3.5, $\tilde{l}_\theta$ is a hybrid subgroup of $\mathbb{A}$. Let $t \in \mathbb{A}$. Then $\tilde{l}(t) = X_t; \theta(t) = \beta_t$ for some $(X_t, \beta_t) \in \mathscr{P}(\mathscr{T}) \times \mathbb{I}$ and $s^{-1}ts \in \mathbb{A}_{\tilde{l}}^{X_t} \cap \mathbb{A}_\theta^{\beta_t}$ which imply $\tilde{l}(s^{-1}ts) \supseteq X_t = \tilde{l}(t)$ and $\theta(s^{-1}ts) \leq \beta_t = \theta(t)$. Hence $\tilde{l}_\theta$ of $\mathbb{A}$ is a hybrid normal subgroup.

Theorem 4.4. *Let $\tilde{l}_\eta \in \mathbb{H}(\mathbb{A})$ and $\mathbb{A}$ be a group. For any $(\Psi, t_1) \in \mathscr{P}(\mathscr{T}) \times \mathbb{I}$, the below criteria are equivalent:*

(i) $\tilde{l}_\eta[\Psi, t_1]$ of $\mathbb{A}$ is a normal subgroup,

(ii) $\tilde{l}_\eta$ of $\mathbb{A}$ is a hybrid normal subgroup.

Proof: The proof is similar to Theorem 4.3.

Theorem 4.5. *Every hybrid normal subgroup's homomorphic hybrid preimage is also a hybrid normal subgroup.*

Proof: Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two groups. Let $\Omega : \mathbb{A}_1 \to \mathbb{A}_2$ be a homomorphism and $\tilde{l}_\beta$ a hybrid normal subgroup of $\mathbb{A}_2$. Let $k, s \in \mathbb{A}_1$. Then $\Omega^{-1}(\tilde{l})(ks) = \tilde{l}(\Omega(ks)) = \tilde{l}(\Omega(k)\Omega(s)) = \tilde{l}(\Omega(s)\Omega(k)) = \tilde{l}(\Omega(sk)) = \Omega^{-1}(\tilde{l})(sk), \Omega^{-1}(\beta)(ks) = \beta(\Omega(ks)) = \beta(\Omega(k)\Omega(s)) = \beta(\Omega(s)\Omega(k)) = \beta(\Omega(sk)) = \Omega^{-1}(\beta)(sk)$. Hence $\Omega^{-1}(\tilde{l}_\beta)$ of $\mathbb{A}_1$ is a hybrid normal subgroup.

Theorem 4.6. *Let $\mathbb{A}$ be a group with identity element e and $\tilde{k}_\eta$ be a hybrid normal subgroup in $\mathbb{A}$. Let $\mathbb{A}_{\tilde{k}}^\eta := \{b \in \mathbb{A} : \tilde{k}(b) = \tilde{k}(e) \text{ and } \eta(b) = \eta(e)\}$. Then $\mathbb{A}_{\tilde{k}}^\eta$ is a normal subgroup of $\mathbb{A}$.*

Define a map $\widehat{\tilde{k}_\eta} := (\widehat{\tilde{k}}, \widehat{\eta}) : \mathbb{A}/\mathbb{A}_{\tilde{k}}^\eta \to \mathscr{P}(\mathscr{T}) \times \mathbb{I}$ as follows: For $l \in \mathbb{A}$,

$$\widehat{\tilde{k}}(l\mathbb{A}_{\tilde{k}}^\eta) = \tilde{k}(l) \text{ and } \widehat{\eta}(l\mathbb{A}_{\tilde{k}}^\eta) = \eta(l),$$

$\widehat{\tilde{k}_\eta}$ is well defined and is a hybrid normal subgroup in $\mathbb{A}/\mathbb{A}_{\tilde{k}}^\eta$.

Conversely, if Q of $\mathbb{A}$ is normal subgroup and $\widehat{\tilde{m}_\partial}$ of $\mathbb{A}/Q$ is a hybrid normal subgroup such that $\widehat{\tilde{m}_\partial}(tQ) = \widehat{\tilde{m}_\partial}(Q)$ only when $t \in Q$, then $\exists$ hybrid normal subgroup $\tilde{k}_\eta$ in $\mathbb{A} : \mathbb{A}^\eta_{\tilde{k}} = Q$ and $\widehat{\tilde{k}_\eta} = \widehat{\tilde{m}_\partial}$.

Proof: Consider $\tilde{k}_\eta$ is a hybrid normal subgroup in $\mathbb{A}$. Then by using Theorem 3.1(ii), $\mathbb{A}^\eta_{\tilde{k}}$ is a normal subgroup of $\mathbb{A}$.

If $q\mathbb{A}^\eta_{\tilde{k}} = y\mathbb{A}^\eta_{\tilde{k}}$ for some $q, y \in \mathbb{A}$, then $qy^{-1} \in \mathbb{A}^\eta_{\tilde{k}}$ and so $\tilde{k}(qy^{-1}) = \tilde{k}(e)$ and $\eta(qy^{-1}) = \eta(e)$. By Theorem 3.1(iii), we get $\tilde{k}(q) = \tilde{k}(y)$ and $\eta(q) = \eta(y)$. *i.e.*, $\widehat{\tilde{k}}(q\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\tilde{k}}(y\mathbb{A}^\eta_{\tilde{k}})$ and $\widehat{\eta}(q\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\eta}(y\mathbb{A}^\eta_{\tilde{k}})$. Therefore $\widehat{\tilde{k}_\eta}$ is well defined structure.

Now, we claim that $\widehat{\tilde{k}_\eta}$ is a hybrid normal subgroup in $\mathbb{A}/\mathbb{A}^\eta_{\tilde{k}}$.

Let $q\mathbb{A}^\eta_{\tilde{k}}, y\mathbb{A}^\eta_{\tilde{k}} \in \mathbb{A}/\mathbb{A}^\eta_{\tilde{k}}$. Then $\widehat{\tilde{k}}(q\mathbb{A}^\eta_{\tilde{k}}y\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\tilde{k}}(qy\mathbb{A}^\eta_{\tilde{k}}) = \tilde{k}(qy) \supseteq \tilde{k}(q) \cap \tilde{k}(y) = \widehat{\tilde{k}}(q\mathbb{A}^\eta_{\tilde{k}}) \cap \widehat{\tilde{k}}(y\mathbb{A}^\eta_{\tilde{k}}), \widehat{\eta}(q\mathbb{A}^\eta_{\tilde{k}}y\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\eta}(qy\mathbb{A}^\eta_{\tilde{k}}) = \eta(qy) \leq \eta(q) \vee \eta(y) = \widehat{\eta}(q\mathbb{A}^\eta_{\tilde{k}}) \vee \widehat{\eta}(y\mathbb{A}^\eta_{\tilde{k}})$. So $\widehat{\tilde{k}_\eta}$ is a hybrid subgroup in $\mathbb{A}/\mathbb{A}^\eta_{\tilde{k}}$.

Also, $\widehat{\tilde{k}}(q\mathbb{A}^\eta_{\tilde{k}}y\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\tilde{k}}(qy\mathbb{A}^\eta_{\tilde{k}}) = \tilde{k}(qy) = \tilde{k}(yq) = \widehat{\tilde{k}}(yq\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\tilde{k}}(y\mathbb{A}^\eta_{\tilde{k}}q\mathbb{A}^\eta_{\tilde{k}}), \widehat{\eta}(q\mathbb{A}^\eta_{\tilde{k}}y\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\eta}(qy\mathbb{A}^\eta_{\tilde{k}}) = \eta(qy) = \eta(yq) = \widehat{\eta}(yq\mathbb{A}^\eta_{\tilde{k}}) = \widehat{\eta}(y\mathbb{A}^\eta_{\tilde{k}}q\mathbb{A}^\eta_{\tilde{k}})$.

Therefore $\widehat{\tilde{k}_\eta}$ is a hybrid normal subgroup in $\mathbb{A}/\mathbb{A}^\eta_{\tilde{k}}$.

Conversely, for the given hybrid normal subgroup $\widehat{\tilde{m}_\partial}$ in $\mathbb{A}/Q$, define a hybrid structure $\tilde{k}_\eta := (\tilde{k}, \eta) : \mathbb{A} \to \mathscr{P}(\mathscr{T}) \times \mathbb{I}$ as follows:

$$\tilde{k}(q) = \widehat{\tilde{m}}(qQ) \text{ and } \eta(q) = \widehat{\partial}(qQ).$$

It is easy to confirm that $\tilde{k}_\eta$ is well defined and a hybrid subgroup in $\mathbb{A}$.

Let $q, y \in \mathbb{A}$. Then, by Theorem 4.1, $\tilde{k}(y^{-1}qy) = \widehat{\tilde{m}}(y^{-1}qyQ) = \widehat{\tilde{m}}(y^{-1}QqQyQ) = \widehat{\tilde{m}}(qQ) = \tilde{k}(q), \eta(y^{-1}qy) = \widehat{\partial}(y^{-1}qyQ) = \widehat{\partial}(y^{-1}QqQyQ) = \widehat{\partial}(qQ) = \eta(q)$. By Theorem 4.1, $\tilde{k}_\eta$ is a hybrid normal subgroup in $\mathbb{A}$.

Further, for $n \in Q$, $\tilde{k}(n) = \widehat{\tilde{m}}(nQ) = \widehat{\tilde{m}}(Q) = \tilde{k}(e)$ and $\eta(n) = \widehat{\partial}(nQ) = \widehat{\partial}(Q) = \eta(e)$. Thus $Q \subseteq \mathbb{A}^\eta_{\tilde{k}}$. If $q \in \mathbb{A}^\eta_{\tilde{k}}$, then $\tilde{k}(q) = \tilde{k}(e)$ and $\eta(q) = \eta(e)$ which imply $\widehat{\tilde{m}}(qQ) = \widehat{\tilde{m}}(Q)$ and $\eta(qQ) = \eta(Q)$. Consequently, $qQ = Q$, so $q \in Q$. Thus $\mathbb{A}^\eta_{\tilde{k}} \subseteq Q$ and hence $\mathbb{A}^\eta_{\tilde{k}} = Q$, it follows that $\widehat{\tilde{k}_\eta} = \widehat{\tilde{m}_\partial}$.

Definition 4.2. Let $\tilde{l}_\eta$ be a hybrid subgroup in a group $\mathbb{A}$. For $s \in \mathbb{A}$, define a hybrid structure $\widehat{\tilde{l}^s_\eta} := (\widehat{\tilde{l}^s}, \widehat{\eta^s}) : \mathbb{A} \to \mathscr{P}(\mathscr{T}) \times \mathbb{I}$ as follows:

$$\widehat{\tilde{l}^s}(t) = \tilde{l}(ts^{-1}) \text{ and } \widehat{\eta^s}(t) = \eta(ts^{-1}) \; \forall t \in \mathbb{A}.$$

$\widehat{\tilde{l}^s_\eta}$ is described as the hybrid coset in $\mathbb{A}$ determined by s and $\tilde{l}_\eta$.

Theorem 4.7. *Let $\mathbb{A}$ be a group. If $\tilde{b}_\eta$ of $\mathbb{A}$ is a hybrid normal subgroup, then for $x \in \mathbb{A}$, we get* $\begin{pmatrix} \widehat{\tilde{b}^x}(xt) = \widehat{\tilde{b}^x}(tx) = \tilde{b}(t) \\ \widehat{\eta^x}(xt) = \widehat{\eta^x}(tx) = \eta(t) \end{pmatrix}$ $(\forall t \in \mathbb{A})$.

Proof: Consider $\tilde{b}_\eta$ of $\mathbb{A}$ is a hybrid normal subgroup. Then for any $t \in \mathbb{A}$, $\widehat{\tilde{b}^x}(xt) = \tilde{b}(xtx^{-1}) = \tilde{b}(t), \widehat{\tilde{b}^x}(tx) = \tilde{b}(txx^{-1}) = \tilde{b}(t)$. Also $\widehat{\eta^x}(xt) = \eta(xtx^{-1}) = \eta(t), \widehat{\eta^x}(tx) = \eta(txx^{-1}) = \eta(t)$.

Theorem 4.8. *Let $\tilde{l}_\eta$ be a hybrid normal subgroup in a group $\mathbb{A}$ and $\mathscr{F}$ be the gathering of every hybrid cosets of $\tilde{l}_\eta$. Then $\mathscr{F}$ forms a group under the hybrid composition*

$$\widehat{\tilde{l}_\eta^x} \odot \widehat{\tilde{l}_\eta^y} = \widehat{\tilde{l}_\eta^{xy}} \; \forall x, y \in \mathbb{A},$$

Define a hybrid structure $\overline{\tilde{l}_\eta} : \mathscr{F} \to \mathscr{P}(\mathscr{T}) \times \mathbb{I}$ as follows:

$$\bar{\tilde{l}}\left(\widehat{\tilde{l}^x}\right) = \tilde{l}(x) \text{ and } \bar{\widehat{\eta}}\left(\widehat{\eta^x}\right) = \eta(x) \; \forall x \in \mathbb{A}. \tag{4.1}$$

Then $\overline{\tilde{l}_\eta}$ on $\mathscr{F}$ is a hybrid subgroup.

Proof: We first claim that the hybrid composition defined on $\mathscr{F}$ given by Definition 4.2 is well defined.

Let $x, y, x_0, y_0 \in \mathbb{A} : \widehat{\tilde{l}_\eta^x} = \widehat{\tilde{l}_\eta^{x_0}}$ and $\widehat{\tilde{l}_\eta^y} = \widehat{\tilde{l}_\eta^{y_0}}$. Then we must prove that $\widehat{\tilde{l}_\eta^x} \odot \widehat{\tilde{l}_\eta^y} = \widehat{\tilde{l}_\eta^{x_0}} \odot \widehat{\tilde{l}_\eta^{y_0}}$, i.e., $\widehat{\tilde{l}_\eta^{xy}} = \widehat{\tilde{l}_\eta^{x_0 y_0}}$.

For $t \in \mathbb{A}$, by assumption, we have

$$\begin{pmatrix} \tilde{l}(tx^{-1}) = \tilde{l}(tx_0^{-1}) \\ \eta(tx^{-1}) = \eta(tx_0^{-1}) \end{pmatrix} \tag{4.2}$$

$$\begin{pmatrix} \tilde{l}(ty^{-1}) = \tilde{l}(ty_0^{-1}) \\ \eta(ty^{-1}) = \eta(ty_0^{-1}) \end{pmatrix} \tag{4.3}$$

Take t as $x_0 y_0 y^{-1}$ in (4.2), we get

$$\begin{aligned} \tilde{l}(x_0 y_0 y^{-1} x^{-1}) &= \tilde{l}(x_0 y_0 y^{-1} x_0^{-1}) \\ &= \tilde{l}(y_0 y^{-1}) \text{ (as } \tilde{l}_\eta \text{ is hybrid normal)} \\ &= \tilde{l}(e), \text{ (by 4.3)} \\ \eta(x_0 y_0 y^{-1} x^{-1}) &= \eta(x_0 y_0 y^{-1} x_0^{-1}) \\ &= \eta(y_0 y^{-1}) \text{ (as } \tilde{l}_\eta \text{ is hybrid normal)} \\ &= \eta(e) \text{ (by 4.3)}. \end{aligned}$$

Now

$$\begin{aligned} \widehat{\tilde{l}^{xy}}(t) &= \tilde{l}(ty^{-1}x^{-1}) = \tilde{l}(ty_0^{-1}x_0^{-1}x_0 y_0 y^{-1} x^{-1}) \\ &\supseteq \tilde{l}(ty_0^{-1}x_0^{-1}) \cap \tilde{l}(x_0 y_0 y^{-1} x^{-1}) \\ &= \tilde{l}(ty_0^{-1}x_0^{-1}) \cap \tilde{l}(e) \\ &= \tilde{l}(ty_0^{-1}x_0^{-1}) = \widehat{\tilde{l}^{x_0 y_0}}(t), \\ \widehat{\eta^{xy}}(t) &= \eta(ty^{-1}x^{-1}) = \eta(ty_0^{-1}x_0^{-1}x_0 y_0 y^{-1} x^{-1}) \\ &\leq \eta(ty_0^{-1}x_0^{-1}) \vee \eta(x_0 y_0 y^{-1} x^{-1}) \\ &= \eta(ty_0^{-1}x_0^{-1}) \vee \eta(e) \\ &= \eta(ty_0^{-1}x_0^{-1}) = \widehat{\eta^{x_0 y_0}}(t). \end{aligned}$$

So, $\widehat{\tilde{l}_\eta^{xy}} \gg \widehat{\tilde{l}_\eta^{x_0 y_0}}$. Similarly, we can get $\widehat{\tilde{l}_\eta^{xy}} \ll \widehat{\tilde{l}_\eta^{x_0 y_0}}$. Thus $\widehat{\tilde{l}_\eta^{xy}} = \widehat{\tilde{l}_\eta^{x_0 y_0}}$ and hence the hybrid product defined on $\mathscr{F}$ given by Definition 4.2 is well defined.

Let $\widehat{\tilde{l}_\eta^x}, \widehat{\tilde{l}_\eta^y}, \widehat{\tilde{l}_\eta^z} \in \mathscr{F}$. Then $\widehat{\tilde{l}_\eta^x} \odot \left(\widehat{\tilde{l}_\eta^y} \odot \widehat{\tilde{l}_\eta^z}\right) = \widehat{\tilde{l}_\eta^{x(yz)}} = \widehat{\tilde{l}_\eta^{(xy)z}} = \left(\widehat{\tilde{l}_\eta^x} \odot \widehat{\tilde{l}_\eta^y}\right) \odot \widehat{\tilde{l}_\eta^z}$.

So, the hybrid product defined in the Definition 4.2 is associative, Also $\widehat{\tilde{l}_\eta^e}$ is the identity element of $\mathscr{F}$, and $\widehat{\tilde{l}_\eta^{x^{-1}}}$ is the inverse of $\widehat{\tilde{l}_\eta^x}$ in $\mathscr{F}$. Hence $(\mathscr{F}, \odot)$ forms a group.

Let $w, b \in \mathbb{A}$. Then $\bar{\tilde{l}}(\widehat{\tilde{l}^w} \circ \widehat{\tilde{l}^b}) = \bar{\tilde{l}}(\widehat{\tilde{l}^{wb}}) = \tilde{l}(wb) \supseteq \tilde{l}(w) \cap \tilde{l}(b) = \bar{\tilde{l}}(\widehat{\tilde{l}^w}) \cap \bar{\tilde{l}}(\widehat{\tilde{l}^b}), \overline{\eta}(\widehat{\eta^w} \circ \widehat{\eta^b}) = \overline{\eta}(\widehat{\eta^{wb}}) = \eta(wb) \leq \eta(w) \vee \eta(b) = \overline{\eta}(\widehat{\eta^w}) \vee \overline{\eta}(\widehat{\eta^b})$.

Also $\bar{\tilde{l}}(\widehat{\tilde{l}^w}) = \tilde{l}(w) = \tilde{l}(w^{-1}) = \bar{\tilde{l}}(\widehat{\tilde{l}^{w^{-1}}}); \overline{\eta}(\widehat{\eta^w}) = \eta(w) = \eta(w^{-1}) = \overline{\eta}(\widehat{\eta^{w^{-1}}})$.

Hence $\overline{\tilde{l}_\eta}$ is a hybrid subgroup.

Definition 4.3. Let $\mathbb{A}$ be a group. For a hybrid normal subgroup $\tilde{l}_\eta$ in $\mathbb{A}$, the hybrid structure defined in (4.1) is described as the hybrid quotient group determined by $\tilde{l}_\eta$.

Lemma 4.9. *Let $\mathbb{A}$ be a group with identity element e. With the same notation as in Theorem 4.8, consider a map $\theta : \mathbb{A} \to \mathscr{F}$ which is defined as follows:*

$$\theta(x) = \widehat{\tilde{l}_\eta^x}. \tag{4.4}$$

Then θ is a homomorphism with kernal given by

$$\mathbb{A}_{\tilde{l}_\eta} = \{q \in \mathbb{A} \,:\, \tilde{l}(q) = \tilde{l}(e) \text{ and } \eta(q) = \eta(e)\}.$$

Proof: Let $s, y \in \mathbb{A}$. Then $\theta(sy) = \widehat{\tilde{l}_\eta^{sy}} = \widehat{\tilde{l}_\eta^s} \odot \widehat{\tilde{l}_\eta^y} = \theta(s) \odot \theta(y)$. Further, the kernal of θ consists of all $s \in \mathbb{A} : \widehat{\tilde{l}_\eta^s} = \widehat{\tilde{l}_\eta^e}$ which implies that $\tilde{l}(s) = \tilde{l}(e)$ and $\eta(s) = \eta(e)$.

We now have an analogue of a result of the "fundamental theorem of homomorphism of groups" for hybrid subgroups.

Theorem 4.10. *Let $\tilde{l}_\delta$ be a hybrid normal subgroup of $\mathbb{A}$ and $\mathscr{F}$ be the family of every hybrid cosets of $\tilde{l}_\eta$. Then each hybrid (resp., normal) subgroup of $\mathscr{F}$ corresponds in a natural way to a hybrid (resp., normal) subgroup of $\mathbb{A}$.*

Proof: Let $\tilde{g^*}_{\eta^*}$ be a hybrid subgroup in $\mathscr{F}$ over the universal set $\mathscr{T}$. Define a hybrid structure $\tilde{m}_\tau \in \mathbb{H}(\mathbb{A})$ as $\tilde{m}_\tau := (\tilde{m}, \tau) : \mathbb{A} \to \mathscr{P}(\mathscr{T}) \times \mathbb{I}$, where $\tilde{m}(b) = \tilde{g^*}(\widehat{\tilde{l}^b})$ and $\tau(b) = \eta^*(\widehat{\delta^b})\ \forall b \in \mathbb{A}$.

We now claim that $\tilde{m}_\tau$ is a hybrid subgroup in $\mathbb{A}$. Let $b, y \in \mathbb{A}$. Then $\tilde{m}(by) = \tilde{g^*}(\widehat{\tilde{l}^{by}}) = \tilde{g^*}(\widehat{\tilde{l}^b} \circ \widehat{\tilde{l}^y}) \supseteq \tilde{g^*}(\widehat{\tilde{l}^b}) \cap \tilde{g^*}(\widehat{\tilde{l}^y}) = \tilde{m}(b) \cap \tilde{m}(y), \tau(by) = \eta^*(\widehat{\delta^{by}}) = \eta^*(\widehat{\delta^b} \circ \widehat{\delta^y}) \leq \eta^*(\widehat{\delta^b}) \vee \eta^*(\widehat{\delta^y}) = \tau(b) \vee \tau(y)$. Also $\tilde{m}(b) = \tilde{g^*}(\widehat{\tilde{l}^b}) = \tilde{g^*}(\widehat{\tilde{l}^{b^{-1}}}) = \tilde{m}(b^{-1}); \tau(b) = \eta^*(\widehat{\delta^b}) = \eta^*(\widehat{\delta^{b^{-1}}}) = \tau(b^{-1})$. Hence $\tilde{m}_\tau$ is a hybrid subgroup in $\mathbb{A}$.

If $\tilde{g^*}_{\eta^*}$ is a hybrid normal subgroup of $\mathscr{F}$, then for any $y, x \in \mathbb{A}$, $\tilde{m}(xy) = \tilde{g^*}(\widehat{\tilde{l}^{xy}}) = \tilde{g^*}(\widehat{\tilde{l}^x} \circ \widehat{\tilde{l}^y}) = \tilde{g^*}(\widehat{\tilde{l}^y} \circ \widehat{\tilde{l}^x}) = \tilde{g^*}(\widehat{\tilde{l}^{yx}}) = \tilde{m}(yx), \tau(xy) = \eta^*(\widehat{\delta^{xy}}) = \eta^*(\widehat{\delta^x} \circ \widehat{\delta^y}) = \eta^*(\widehat{\delta^y} \circ \widehat{\delta^x}) = \eta^*(\widehat{\delta^{yx}}) = \tau(yx)$. Therefore $\tilde{m}_\tau$ is a hybrid normal subgroup in $\mathbb{A}$.

We now construct a hybrid analogue of Lagrange's theorem for finite groups, which is a fundamental result in group theory.

Let $\tilde{l}_\eta$ be a hybrid subgroup of a finite group $\mathbb{A}$. The collection of hybrid cosets of $\tilde{l}_\eta$ is given by

$$\mathscr{F} = \{\, \widehat{\tilde{l}_\eta^x} \,:\, x \in \mathbb{A} \,\},$$

where $\widehat{\tilde{l}_\eta^x}$ is defined by Definition 4.2. The order of $\mathscr{F}$ is described as the index of the hybrid subgroup $\tilde{l}_\eta$. If $\mathbb{A}$ is a finite group, then $\mathscr{F}$ is a finite set.

Theorem 4.11. *(Hybrid Lagrange's Theorem) Let $\tilde{l}_\eta$ be a hybrid subgroup of a finite group $\mathbb{A}$. Then the index of $\tilde{l}_\eta$ divides the order of $\mathbb{A}$.*

Proof: From Lemma 4.9, there exists a homomorphism $\theta : \mathbb{A} \to \mathscr{F}$ such that $\theta(x) = \widehat{\tilde{l}^{x}_{\eta}}$.

Define a subgroup Q of $\mathbb{A}$ by $Q := \{h \in \mathbb{A} : \widehat{\tilde{l}^{h}_{\eta}} = \widehat{\tilde{l}^{e}_{\eta}}\}$. Let $h \in Q$. Then for any $t \in \mathbb{A}$, $\widehat{\tilde{l}^{h}_{\eta}}(t) = \widehat{\tilde{l}^{e}_{\eta}}(t)$ implies $\tilde{l}(th^{-1}) = \tilde{l}(t)$ and $\eta(th^{-1}) = \eta(t)$. In particular, we get $\tilde{l}(h^{-1}) = \tilde{l}(e)$ and $\eta(h^{-1}) = \eta(e)$. i.e., $\tilde{l}(h) = \tilde{l}(e)$ and $\eta(h) = \eta(e)$.

Next, if $\tilde{l}(h) = \tilde{l}(e)$ and $\eta(h) = \eta(e)$, then clearly $\widehat{\tilde{l}^{h}_{\eta}} = \widehat{\tilde{l}^{e}_{\eta}}$.

Thus $Q = \{h \in \mathbb{A} : \tilde{l}(h) = \tilde{l}(e) \text{ and } \eta(h) = \eta(e)\}$.

Now, decompose $\mathbb{A}$ as a disjoint union of the cosets of $\mathbb{A}$ with respect to Q :

$$\mathbb{A} = Qx_1 \cup Qx_2 \cup ... \cup Qx_k, \tag{4.5}$$

where $Qx_1 = Q$. We now claim that corresponding to each coset Qx_i given in (4.5), there is a hybrid cosets belonging to $\mathscr{F}$ and this correspondence is one-to-one.

Consider the coset Qx_i and let $h \in Q$. Then $\theta(hx_i) = \widehat{\tilde{l}^{hx_i}_{\eta}} = \widehat{\tilde{l}^{h}_{\eta}} \odot \widehat{\tilde{l}^{x_i}_{\eta}} = \widehat{\tilde{l}^{h}_{\eta}} \odot \widehat{\tilde{l}^{x_i}_{\eta}} = \widehat{\tilde{l}^{x_i}_{\eta}}$. Thus θ maps each element of Qx_i into the hybrid coset $\widehat{\tilde{l}^{x_i}_{\eta}}$.

We now set up a natural correspondence $\overline{\theta}$ between the set $\{Qx_i : 1 \leq i \leq k\}$ and the set $\mathscr{F}$ by $\overline{\theta}(Qx_i) = \widehat{\tilde{l}^{x_i}_{\eta}}, 1 \leq i \leq k$.

If $\widehat{\tilde{l}^{x_i}_{\eta}} = \widehat{\tilde{l}^{x_j}_{\eta}}$, then $\widehat{\tilde{l}^{x_i x_j^{-1}}_{\eta}} = \widehat{\tilde{l}^{e}_{\eta}}$ implies $x_i x_j^{-1} \in Q$ and $Qx_i = Qx_j$. So $\overline{\theta}$ is one-to-one.

As shown in the previous discussion, the number of distinct Q cosets in $\mathbb{A}$ is equal to the number of hybrid $\tilde{l}_{\eta}$ cosets.

As the number of distinct cosets of Q in $\mathbb{A}$ is a divisor of the order of $\mathbb{A}$, we have the index of $\tilde{l}_{\eta}$ divides the order of $\mathbb{A}$.

5. Conclusion

In this study, we examined and discussed the notions of hybrid ideals and hybrid subgroupoids in groupoid. We obtained some equivalent assertions for a hybrid structure to be a hybrid subgroupoid and a hybird ideal. We proved that every homomorphic hybrid preimage of a hybrid subgroupoid (left ideal, right ideal) is also a hybrid subgroupoid (left ideal, right ideal). The properties of a group's hybrid subgroup, hybrid normal subgroup, and hybrid coset, as well as their important properties, were defined and established. Finally, we used a hybrid structure to prove Lagrange's theorem. It is intended to define the concept of hybrid prime (resp., semi) ideals and derive their various properties and equivalent conditions for a hybrid ideal to be a hybrid prime (resp., semi) ideal in groupoids using the ideas and findings of this paper.

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References

[1] M. Akgül, Some properties of fuzzy groups, *J. Math. Anal. Appl.* **133** (1988), 93-100.
[2] S. Anis, M. Khan and Y. B. Jun, Hybrid ideals in semigroups, *Cogent Mathematics* **4** (2017), 1352117
[3] B. Elavarasan and Y. B. Jun, Regularity of semigroups in terms of hybrid ideals and hybrid bi-ideals, *Kragujev. J. Math.* **46(6)** (2022), 857-864.
[4] B. Elavarasan, G. Muhiuddin, K. Porselvi and Y. B. Jun, Hybrid structures applied to ideals in near-rings, *Complex Intell. Syst.* **7(3)** (2021), 1489–1498.

[5] B. Elavarasan and Y. B. Jun, Hybrid ideals in semirings, *Advances in Mathematics: Scientific Journal* **9(3)** (2020), 1349–1357.
[6] B. Elavarasan, K. Porselvi and Y. B. Jun, Hybrid generalized bi-ideals in semigroups, *International Journal of Mathematics and Computer Science* **14(3)** (2019), 601 – 612.
[7] Y. B. Jun, S. Z. Song and G. Muhiuddin, Hybrid structures and applications, *Annals of Communications in Mathematics* **1(1)** (2018), 11–25.
[8] D.S. Malik, J. N. Mordeson and P. S. Nair, Fuzzy normal subgroups in fuzzy subgroups, *J. Korean Math. Sco.* **29(1)** (1992), 1 - 8.
[9] S. Meenakshi, G. Muhiuddin, B. Elavarasan, D. Al-Kadi, Hybrid ideals in near-subtraction semigroups, *AIMS Mathematics* **7**(7) (2022), 13493–13507.
[10] D. Molodtsov, Soft set theory - First results, *Comput. Math. Appl.* **37**(1999), 19-31.
[11] N. P. Muherjee and P. Bhattacharya, Fuzzy normal subgroups and fuzzy cosets, *Information Sciences* **34** (1984) 225 - 239.
[12] D. A. Molodtsov, The description of a dependence with the help of soft sets, *Journal of Computer and Systems Sciences International* **40(6)** (2001), 977-984.
[13] D. A. Molodtsov, The Theory of Soft Sets, URSS Publishers, Moscow (2004).
[14] G. Muhiuddin, D. Al-Kadi, A. Mahboob, Hybrid Structures Applied to Ideals in BCI-Algebras, *Journal of Mathematics* **2020** (2020), Article ID 2365078.
[15] G. Muhiuddin, J. Catherine Grace John, B. Elavarasan, Y. B. Jun, K. Porselvi, Hybrid structures applied to modules over semirings, *J. Intell. Fuzzy Syst.* **42(3)** (2022), 2521–2531.
[16] G. Muhiuddin, J. Catherine Grace John, B. Elavarasan, K. Porselvi, D. Al-Kadi, Properties of k-hybrid ideals in ternary semiring, *J. Intell. Fuzzy Syst.* **4**2(6) (2022), 5799 - 5807.
[17] K. Porselvi and B. Elavarasan, On hybrid interior ideals in semigroups, *Probl. Anal. Issues Anal.* **8(26)(3)** (2019), 137 – 146.
[18] K. Porselvi, G. Muhiuddin, B. Elavarasan, A. Assiry, Hybrid Nil Radical of a Ring, *Symmetry* **14** (2022), 1367.
[19] K. Porselvi, G. Muhiuddin, B. Elavarasan, Y. B. Jun, J. Catherine Grace John, Hybrid ideals in an AG-groupoid, *New Math. Nat. Comput.* **19(01)** (2023) 1–17.
[20] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35** (1971) 512-517.
[21] P. Sivaramakrishna Das, Fuzzy groups and level subgroups, *J. Math. Anal. Appl.* **84** (1981), 264 - 269.
[22] L. A. Zadeh, Fuzzy sets, *Inform. Control* **8** (1965), 338–353.

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