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ON A GENERALIZED HARDY INTEGRAL INEQUALITY

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ABSTRACT. In this article, we present a new, generalized version of the Hardy integral inequality. This version depends on an auxiliary function. Thanks to this function, numerous variants can be derived. The theory is complemented by several secondary results, including one demonstrating that the main inequality can be improved upon under certain additional assumptions and another providing a valuable lower bound for the main integral term. Several examples are given for illustration.

1. INTRODUCTION

Integral inequalities are important tools in all areas of mathematics. They have specific applications in differential equations, optimization, statistics, engineering and mathematical physics. Their main contribution to solving practical problems is providing rigorous bounds that simplify analysis. A wide panel of integral inequalities can be found in [11, 2, 19, 1, 21]. The Hardy integral inequality, originally developed in [10], is one of the most famous. To present it, we need $p > 1$, $f : (0, +\infty) \to (0, +\infty)$ an integrable function, and, for any $t \in (0, +\infty)$, the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then the Hardy integral inequality ensures that

$$\int_0^{+\infty} \frac{1}{t^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} [f(t)]^p dt, \tag{1.1}$$

provided that the integrals considered converge. This inequality is useful in several contexts, including the study of integral operators and transforms. The constant factor $[p/(p-1)]^p$ is the optimal one. It guarantees the tightness of the upper bound.

The Hardy integral inequality has been generalized in various ways, starting with the work in [10]. Natural extensions include weighted Hardy-type integral inequalities, leading

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to further refinements and multidimensional versions. See [3, 17, 18, 14, 12, 5, 13, 9, 6, 7, 8].

A notable contemporary result on this topic is [16, Theorem 2.2]. It introduces an auxiliary function $g : (0, +\infty) \to (0, +\infty)$, which can adapt to different mathematical scenarios. More precisely, assuming that $t/g(t)$ is non-increasing, the following inequality is established:

$$\int_0^{+\infty} \frac{1}{[g(t)]^p} [F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{t^p}{[g(t)]^p} [f(t)]^p dt. \tag{1.2}$$

The proof mainly relies on the Hölder integral inequality, an exchange of the order of integration, and the non-increasing assumption on $t/g(t)$. Obviously, by choosing $g(t) = t$, Equation (1.2) reduces to the Hardy integral inequality. Other variants are suggested in [16], including several based on different convexity assumptions. See also [15], which has extended Equation (1.2) considering two parameters, p and q. Important advances in the topic are also discussed in [20, 4].

In this article, we present another variant of the Hardy integral inequality, which is still based on an auxiliary function g. It has the following form:

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p} [F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}} [f(t)]^p dt. \tag{1.3}$$

There are significant differences compared to the inequality in Equation (1.2), mainly with the determinant role of g' appearing in the two main integral terms. Obviously, by taking $g(t) = t$, the Hardy integral inequality is restored. The proof of Equation (1.3) follows the spirit of the original proof in [10]; it relies on integration by parts, several limit results, and a well-calibrated Hölder integral inequality. To the best of our knowledge, although the proof is intuitive, it has not been presented in this form before. We then illustrate this generalized Hardy integral inequality with several examples based on different functions g. A final part is devoted to secondary results, including a refinement of our main inequality under some additional assumptions on f. A tight lower bound on the left term in Equation (1.3) is also given, which may be viewed as of independent interest.

The rest of the article is composed of the following sections: Section 2 contains the details on our main inequality. Examples are described in Section 3. Section 4 is devoted to the secondary results. A conclusion is formulated in Section 5.

2. Main result

Our generalized Hardy integral inequality is formalized below, with all assumptions on the functions involved.

Proposition 2.1. *Let $p > 1$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that f is continuous and integrable, and g is differentiable, non-decreasing and satisfies the following limit properties:*

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = 0, \quad \lim_{t\to+\infty} g(t) \neq 0.$$

For any $t \in (0, +\infty)$, let us consider the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then we have

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

provided that the integrals considered converge.

Proof. We start by setting

$$J = \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt.$$

An integration by parts taking account that the primitive of $g'(t)/[g(t)]^p$ is $-1/\{(p-1)[g(t)]^{p-1}\}$ and the derivative of $[F(t)]^p$ is $p[F(t)]^{p-1}f(t)$ gives

$$J = \left\{-\frac{1}{(p-1)[g(t)]^{p-1}}[F(t)]^p\right\}_{t\to 0}^{t\to+\infty} + \frac{p}{p-1}K,$$

where

$$K = \int_0^{+\infty} \frac{1}{[g(t)]^{p-1}}[F(t)]^{p-1} f(t) dt.$$

Since $F(0) = 0$, $f(0)$ is finite, and $\lim_{t\to 0} t^p/[g(t)]^{p-1} = 0$, we have

$$\lim_{t\to 0} \frac{1}{[g(t)]^{p-1}}[F(t)]^p = \lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} \times \left[\lim_{t\to 0} \frac{F(t)-F(0)}{t-0}\right]^p = 0 \times [f(0)]^p = 0$$

and, using $\lim_{t\to+\infty} g(t) \neq 0$,

$$\lim_{t\to+\infty} \left\{-\frac{1}{[g(t)]^{p-1}}[F(t)]^p\right\} = -\frac{1}{[\lim_{t\to+\infty} g(t)]^{p-1}} \left[\int_0^{+\infty} f(x)dx\right]^p \in (-\infty, 0),$$

that is, negative and finite. Therefore, we have

$$J \leq \frac{p}{p-1}K. \tag{2.1}$$

Since g is differentiable and non-decreasing, we have $g'(t) \geq 0$ for any $t \in (0, +\infty)$, and we can write

$$K = \int_0^{+\infty} \left\{\frac{[g'(t)]^{1-1/p}}{[g(t)]^{p-1}}[F(t)]^{p-1}\right\} \left\{\frac{1}{[g'(t)]^{1-1/p}} f(t)\right\} dt.$$

Applying the Hölder integral inequality with the exponent $p/(p-1) > 1$ and recognizing J, we obtain

$$\begin{aligned} K &\leq \left\{\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt\right\}^{1-1/p} \left\{\int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt\right\}^{1/p} \\ &= J^{1-1/p} \left\{\int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt\right\}^{1/p}. \end{aligned} \tag{2.2}$$

It follows from Equations (2.1) and (2.2) that

$$J \leq \frac{p}{p-1} J^{1-1/p} \left\{\int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt\right\}^{1/p},$$

so

$$J^{1/p} \leq \frac{p}{p-1} \left\{\int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt\right\}^{1/p}$$

and

$$J \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt.$$

Taking into account the definition of J, we get

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt.$$

The desired inequality is established, ending the proof. □

As an immediate example of Proposition 2.1, we can consider $g(t) = t$, satisfying $g'(t) = 1$, g is non-decreasing, $\lim_{t\to 0} t^p/[g(t)]^{p-1} = \lim_{t\to 0} t = 0$ and $\lim_{t\to+\infty} g(t) = \lim_{t\to+\infty} t = +\infty \neq 0$. Then Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{1}{t^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} [f(t)]^p dt.$$

We recognize the original Hardy integral inequality. Proposition 2.1 is a generalization in this sense, and also, an alternative to that in [16, Theorem 2.2], recalled in Equation (1.2). Thus, it is able to offer new variants of the Hardy integral inequalities, depending on the choice of g. To illustrate this claim, some examples are given in the next section.

3. Examples

Seven examples of integral inequalities based on Proposition 2.1 are now presented, using the same notation. Each of them is based on a specific one-parameter function g. The inequalities obtained can be used independently in different mathematical scenarios.

Example 1: We consider $g(t) = t^\alpha$, $t \in (0, +\infty)$, with $\alpha \in (0, p/(p-1))$. We have $g'(t) = \alpha t^{\alpha-1}$, g is non-decreasing, and, since $\alpha \in (0, p/(p-1))$, we have

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} t^{p-\alpha(p-1)} = 0$$

and

$$\lim_{t\to+\infty} g(t) = \lim_{t\to+\infty} t^\alpha = +\infty \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha t^{\alpha-1}}{t^{\alpha p}}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{(\alpha t^{\alpha-1})^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{t^{\alpha(p-1)+1}}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} \frac{1}{t^{(\alpha-1)(p-1)}}[f(t)]^p dt.$$

Setting $\alpha = 1$ gives the original Hardy integral inequality, as expected. The other values of α, i.e., $\alpha \in (0, p/(p-1))/\{1\}$, give manageable integral inequalities.

Example 2: We consider $g(t) = \ln(1+\alpha t)$, $t \in (0,+\infty)$, with $\alpha > 0$. We have $g'(t) = \alpha/(1+\alpha t)$, g is non-decreasing,

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} \frac{t^p}{[\ln(1+\alpha t)]^{p-1}} = \alpha^{1-p} \lim_{t\to 0} t = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} \ln(1+\alpha t) = +\infty \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha}{(1+\alpha t)[\ln(1+\alpha t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[\alpha/(1+\alpha t)]^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(1+\alpha t)[\ln(1+\alpha t)]^p}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} (1+\alpha t)^{p-1}[f(t)]^p dt. \tag{3.1}$$

Example 3: We consider $g(t) = \arctan(\alpha t)$, $t \in (0,+\infty)$, with $\alpha > 0$. We have $g'(t) = \alpha/(1+\alpha^2 t^2)$, g is non-decreasing,

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} \frac{t^p}{[\arctan(\alpha t)]^{p-1}} = \alpha^{1-p} \lim_{t\to 0} t = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} \arctan(\alpha t) = \frac{\pi}{2} \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\begin{aligned}&\int_0^{+\infty} \frac{\alpha}{(1+\alpha^2 t^2)[\arctan(\alpha t)]^p}[F(t)]^p dt \\ &\leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[\alpha/(1+\alpha^2 t^2)]^{p-1}}[f(t)]^p dt,\end{aligned}$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(1+\alpha^2 t^2)[\arctan(\alpha t)]^p}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} (1+\alpha^2 t^2)^{p-1}[f(t)]^p dt.$$

Example 4: We consider $g(t) = e^{\alpha t}$, $t \in (0,+\infty)$, with $\alpha > 0$. We have $g'(t) = \alpha e^{\alpha t}$, g is non-decreasing,

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} \frac{t^p}{e^{\alpha(p-1)t}} = \lim_{t\to 0} t^p = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} e^{\alpha t} = +\infty \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha e^{\alpha t}}{e^{tp}}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{(\alpha e^{\alpha t})^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} e^{-(p-\alpha)t}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} e^{-\alpha(p-1)t}[f(t)]^p dt.$$

In terms of the Laplace transform, i.e., for an integrable function $h : (0,+\infty) \to (0,+\infty)$, $\mathcal{L}(h)(\lambda) = \int_0^{+\infty} e^{-\lambda t} h(t) dt$, with $\lambda > 0$, we have

$$\mathcal{L}(F^p)(p-\alpha) \leq \left[\frac{p}{\alpha(p-1)}\right]^p \mathcal{L}(f^p)[\alpha(p-1)],$$

with $p > \alpha$. This Laplace transform inequality is new to our knowledge.

Example 5: We consider $g(t) = e^{-1/(1+\alpha t)}$, $t \in (0,+\infty)$, with $\alpha > 0$. We have $g'(t) = [\alpha/(1+\alpha t)^2]e^{-1/(1+\alpha t)}$, g is non-decreasing,

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} \frac{t^p}{e^{-(p-1)/(1+\alpha t)}} = e^{p-1} \lim_{t\to 0} t^p = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} e^{-1/(1+\alpha t)} = 1 \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha e^{-1/(1+\alpha t)}}{(1+\alpha t)^2 e^{-p/(1+\alpha t)}}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[\alpha e^{-1/(1+\alpha t)}/(1+\alpha t)^2]^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(1+\alpha t)^2} e^{(p-1)/(1+\alpha t)}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} (1+\alpha t)^{2(p-1)} e^{(p-1)/(1+\alpha t)}[f(t)]^p dt. \tag{3.2}$$

Example 6: We consider $g(t) = 1 - e^{-\alpha t}$, $t \in (0,+\infty)$, with $\alpha > 0$. We have $g'(t) = \alpha e^{-\alpha t}$, g is non-decreasing,

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} \frac{t^p}{(1-e^{-\alpha t})^{p-1}} = \alpha^{1-p} \lim_{t\to 0} t = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} (1 - e^{-\alpha t}) = 1 \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha e^{-\alpha t}}{(1-e^{-\alpha t})^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{(\alpha e^{-\alpha t})^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{e^{-\alpha t}}{(1-e^{-\alpha t})^p}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} e^{\alpha(p-1)t}[f(t)]^p dt. \tag{3.3}$$

Example 7: We consider $g(t) = \sqrt{t+\alpha}$, $t \in (0,+\infty)$, with $\alpha > 0$. We have $g'(t) = 1/\left[2\sqrt{t+\alpha}\right]$, g is non-decreasing,

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = \lim_{t\to 0} \frac{t^p}{(t+\alpha)^{(p-1)/2}} = \alpha^{(1-p)/2} \lim_{t\to 0} t^p = 0$$

and

$$\lim_{t\to+\infty} g(t) = \lim_{t\to+\infty} \sqrt{t+\alpha} = +\infty \neq 0.$$

Proposition 2.1 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{1}{2\sqrt{t+\alpha}[\sqrt{t+\alpha}]^p}[F(t)]^p dt$$
$$\leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{\left\{1/\left[2\sqrt{t+\alpha}\right]\right\}^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(t+\alpha)^{(p+1)/2}}[F(t)]^p dt \leq \left(\frac{2p}{p-1}\right)^p \int_0^{+\infty} (t+\alpha)^{(p-1)/2}[f(t)]^p dt. \tag{3.4}$$

Other examples can be presented in a similar way. These include multi-parameter functions and special functions.

4. Complementary results

This section is devoted to some complementary results of Proposition 2.1.

The result below makes a simple connection between Proposition 2.1 and [16, Theorem 2.2], with an additional assumption on g'.

Proposition 4.1. *Let $p > 1$, $f, g : (0,+\infty) \to (0,+\infty)$ be two functions such that f is continuous and integrable, and g is differentiable, non-decreasing and satisfies the following limit properties:*

$$\lim_{t\to 0} \frac{t^p}{[g(t)]^{p-1}} = 0, \quad \lim_{t\to+\infty} g(t) \neq 0, \quad \inf_{t\in(0,+\infty)} g'(t) > 0.$$

For any $t \in (0,+\infty)$, let us consider the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then we have

$$\int_0^{+\infty} \frac{1}{[g(t)]^p}[F(t)]^p dt \le \frac{1}{\inf_{t\in(0,+\infty)} g'(t)} \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

provided that the integrals considered converge.

Proof. Since $\inf_{t\in(0,+\infty)} g'(t) > 0$, then we have $g'(t)/\inf_{t\in(0,+\infty)} g'(t) \ge 1$ for any $t \in (0,+\infty)$. Proposition 2.1 directly implies that

$$\int_0^{+\infty} \frac{1}{[g(t)]^p}[F(t)]^p dt \le \frac{1}{\inf_{t\in(0,+\infty)} g'(t)} \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt$$

$$\le \frac{1}{\inf_{t\in(0,+\infty)} g'(t)} \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt.$$

This ends the proof. □

Proposition 4.1 provides a simple alternative to [16, Theorem 2.2] under the assumptions considered.

Under new assumptions on g, an alternative result to Proposition 2.1 as a direct consequence of the Hardy integral inequality is proposed below.

Proposition 4.2. *Let $p > 1$, $f, g : (0,+\infty) \to (0,+\infty)$ be two functions such that f is continuous and integrable, and g is differentiable, non-decreasing and satisfies the following limit properties:*

$$\lim_{t\to 0} g(t) = 0, \quad \lim_{t\to+\infty} g(t) = +\infty.$$

For any $t \in (0,+\infty)$, let us consider the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then the Hardy integral inequality implies the result to Proposition 2.1, i.e.,

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

provided that the integrals considered converge.

Proof. Doing the change of variables $u = g(t)$, which incorporates $\lim_{t\to 0} g(t) = 0$ and $\lim_{t\to+\infty} g(t) = +\infty$, we obtain

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt = \int_0^{+\infty} \frac{1}{u^p}\left\{F[g^{-1}(u)\right\}^p du, \tag{4.1}$$

where g^{-1} denotes the inverse function of g and, by the definition of F,

$$F[g^{-1}(u)] = \int_0^{g^{-1}(u)} f(x)dx.$$

Applying the change of variables $x = g^{-1}(v)$ into this integral (using again the limit assumptions on g), we get

$$F[g^{-1}(u)] = \int_0^u \frac{1}{g'[g^{-1}(v)]} f[g^{-1}(v)]dv = \int_0^u k(v)dv, \tag{4.2}$$

where $k(v) = \{1/g'[g^{-1}(v)]\} f[g^{-1}(v)]$.

It follows from Equations (4.1) and (4.2), and the Hardy integral inequality (as recalled in Equation (1.1)) applied to the function k, that

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt = \int_0^{+\infty} \frac{1}{u^p}\left[\int_0^u k(v)dv\right]^p du$$
$$\leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} [k(u)]^p du = \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{\{g'[g^{-1}(u)]\}^p}\left\{f[g^{-1}(u)]\right\}^p du. \tag{4.3}$$

Doing the change of variables $u = g(w)$, we have

$$\int_0^{+\infty} \frac{1}{\{g'[g^{-1}(u)]\}^p}\left\{f[g^{-1}(u)]\right\}^p du = \int_0^{+\infty} \frac{1}{[g'(w)]^p} g'(w)[f(w)]^p dw$$
$$= \int_0^{+\infty} \frac{1}{[g'(w)]^{p-1}}[f(w)]^p dw. \tag{4.4}$$

Combining Equations (4.3) and (4.4), and standardizing the notation, we obtain

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

which is the stated inequality. This concludes the proof. □

Some examples of this proposition are presented below.

Example 1: We consider $g(t) = t^\alpha$, $t \in (0, +\infty)$, with $\alpha > 0$. We have $g'(t) = \alpha t^{\alpha-1}$, g is non-decreasing,

$$\lim_{t\to 0} g(t) = \lim_{t\to 0} t^\alpha = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} t^\alpha = +\infty.$$

Proposition 4.2 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha t^{\alpha-1}}{t^{\alpha p}}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{(\alpha t^{\alpha-1})^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{t^{\alpha(p-1)+1}}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} \frac{1}{t^{(\alpha-1)(p-1)}}[f(t)]^p dt.$$

We see that, in this special case, the restriction of $\alpha \in (0, p/(p-1))$ imposed in Example 1 illustrating Proposition 2.1 is relaxed.

Example 2: We consider $g(t) = \ln(1 + \alpha t)$, $t \in (0, +\infty)$, with $\alpha > 0$. We have $g'(t) = \alpha/(1 + \alpha t)$, g is non-decreasing,

$$\lim_{t\to 0} g(t) = \lim_{t\to 0} \ln(1 + \alpha t) = \ln(1) = 0$$

and

$$\lim_{t\to +\infty} g(t) = \lim_{t\to +\infty} \ln(1 + \alpha t) = +\infty.$$

Proposition 4.2 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha}{(1+\alpha t)[\ln(1+\alpha t)]^p}[F(t)]^p dt \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[\alpha/(1+\alpha t)]^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(1+\alpha t)[\ln(1+\alpha t)]^p}[F(t)]^p dt \leq \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} (1+\alpha t)^{p-1}[f(t)]^p dt.$$

This example is identical to Example 2, which illustrates Proposition 2.1. Note that some of the examples presented in Section 3 cannot be derived from Proposition 4.2 because the limit conditions on g are not satisfied.

The result below shows that, under some additional assumptions on f and g, the inequality of Proposition 2.1 can be refined.

Proposition 4.3. *Let $p > 1$, $f, g : (0,+\infty) \to (0,+\infty)$ be two functions such that f is continuous, non-decreasing and integrable, and g is differentiable, non-decreasing and there exists a $K > 0$ such that, for any $t \in (0,+\infty)$,*

$$tg'(t) \leq Kg(t).$$

For any $t \in (0,+\infty)$, let us consider the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then we have

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq K^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

provided that the integrals considered converge. If $K < p/(p-1)$, this inequality is better than that of Proposition 2.1.

Proof. Since f is non-decreasing, we obviously have

$$F(t) = \int_0^t f(x)dx \leq f(t)\int_0^t dx = tf(t).$$

This combined with the fact that g is differentiable and non-decreasing implies that $g'(t) \geq 0$ for any $t \in (0,+\infty)$, and $tg'(t) \leq Kg(t)$ so that $t^p g'(t)/[g(t)]^p \leq K^p/[g'(t)]^{p-1}$ for any $t \in (0,+\infty)$, gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[tf(t)]^p dt = \int_0^{+\infty} t^p \frac{g'(t)}{[g(t)]^p}[f(t)]^p dt$$
$$\leq K^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt.$$

The desired inequality is established. Clearly, if $K < p/(p-1)$, we get

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq K^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt$$
$$\leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt.$$

The obtained inequality is thus better than that of Proposition 2.1. This ends the proof. □

Example 1: We consider $g(t) = \ln(1+\alpha t)$, $t \in (0,+\infty)$, with $\alpha > 0$. Then g is differentiable and non-decreasing. Furthermore, the following inequality is well-known: $\ln(1+u) \geq u/(1+u)$ for any $u \in (0,+\infty)$, and we obtain

$$tg'(t) = t\frac{\alpha}{1+\alpha t} = \frac{\alpha t}{1+\alpha t} \leq \ln(1+\alpha t) = Kg(t),$$

with $K = 1$. Therefore, in this case, if f is continuous, non-decreasing and integrable, Proposition 4.3 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq K^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha}{(1+\alpha t)[\ln(1+\alpha t)]^p}[F(t)]^p dt \leq \int_0^{+\infty} \frac{1}{[\alpha/(1+\alpha t)]^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(1+\alpha t)[\ln(1+\alpha t)]^p}[F(t)]^p dt \leq \frac{1}{\alpha^p}\int_0^{+\infty} (1+\alpha t)^{p-1}[f(t)]^p dt.$$

Since $K = 1 < p/(p-1)$, this improves the inequality given in Equation (3.1), but under the assumption that f is non-decreasing.

Example 2: We consider $g(t) = e^{-1/(1+\alpha t)}$, $t \in (0,+\infty)$, with $\alpha > 0$. Then g is differentiable and non-decreasing. Furthermore, using $(1+\alpha t)^2 = 1 + 2\alpha t + \alpha^2 t^2 \geq 2\alpha t$, we have

$$tg'(t) = t\frac{\alpha}{(1+\alpha t)^2}e^{-1/(1+\alpha t)} = \frac{\alpha t}{(1+\alpha t)^2}e^{-1/(1+\alpha t)}$$
$$\leq \frac{\alpha t}{2\alpha t}e^{-1/(1+\alpha t)} = \frac{1}{2}e^{-1/(1+\alpha t)} = Kg(t),$$

with $K = 1/2$. Therefore, in this case, if f is continuous, non-decreasing and integrable, Proposition 4.3 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq K^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{\alpha e^{-1/(1+\alpha t)}}{(1+\alpha t)^2 e^{-p/(1+\alpha t)}}[F(t)]^p dt \leq \frac{1}{2^p}\int_0^{+\infty} \frac{1}{[\alpha e^{-1/(1+\alpha t)}/(1+\alpha t)^2]^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(1+\alpha t)^2}e^{(p-1)/(1+\alpha t)}[F(t)]^p dt$$
$$\leq \frac{1}{(2\alpha)^p}\int_0^{+\infty} (1+\alpha t)^{2(p-1)}e^{(p-1)/(1+\alpha t)}[f(t)]^p dt.$$

Since $K = 1/2 < 1 < p/(p-1)$, this improves the inequality obtained in Equation (3.2), but under the assumption that f is non-decreasing.

Example 3: We consider $g(t) = \sqrt{t+\alpha}$, $t \in (0,+\infty)$, with $\alpha > 0$. Then g is differentiable and non-decreasing. Furthermore, since $t + \alpha \geq t$, we have

$\sqrt{t+\alpha} \geq t/\sqrt{t+\alpha}$, and

$$tg'(t) = t\frac{1}{2\sqrt{t+\alpha}} = \frac{t}{2\sqrt{t+\alpha}} \leq \frac{1}{2}\sqrt{t+\alpha} = Kg(t),$$

with $K = 1/2$. Therefore, in this case, if f is continuous, non-decreasing and integrable, Proposition 4.3 gives

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq K^p \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

so that

$$\int_0^{+\infty} \frac{1}{2\sqrt{t+\alpha}[\sqrt{t+\alpha}]^p}[F(t)]^p dt \leq \frac{1}{2^p}\int_0^{+\infty} \frac{1}{\left\{1/\left[2\sqrt{t+\alpha}\right]\right\}^{p-1}}[f(t)]^p dt,$$

which is equivalent to

$$\int_0^{+\infty} \frac{1}{(t+\alpha)^{(p+1)/2}}[F(t)]^p dt \leq \int_0^{+\infty} (t+\alpha)^{(p-1)/2}[f(t)]^p dt.$$

Since $K = 1/2 < 1 < p/(p-1)$, this improves the inequality established in Equation (3.4), but under the assumption that f is non-decreasing.

Other examples could be described in a similar way.

The proposition below supplements Proposition 2.1 by giving a lower bound for the left integral term, under some assumptions on g.

Proposition 4.4. *Let $p > 1$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that f is continuous and integrable, and g is differentiable, non-decreasing and satisfies the following limit properties:*

$$\lim_{t\to 0} g(t) = 0, \quad \lim_{t\to+\infty} g(t) = 1.$$

For any $t \in (0, +\infty)$, let us consider the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then we have

$$\left[\int_0^{+\infty} \{-\ln[g(t)]\} f(t)dt\right]^p \leq \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt,$$

provided that the integrals considered converge.

Proof. Let us set

$$L = \int_0^{+\infty} \frac{g'(t)}{g(t)} F(t)dt.$$

Since g is differentiable, non-decreasing, with $\lim_{t\to 0} g(t) = 0$ and $\lim_{t\to+\infty} g(t) = 1$, it is a valid cumulative distribution function. This implies that g' is a valid probability density function. Since the function $h(t) = t^p$, $t \in (0, +\infty)$ is convex for $p > 1$, the Jensen integral inequality implies that

$$L^p = h\left[\int_0^{+\infty} \frac{g'(t)}{g(t)} F(t)dt\right] \leq \int_0^{+\infty} g'(t)h\left[\frac{1}{g(t)}F(t)\right] dt = \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt. \tag{4.5}$$

Using the definition of F, applying an exchange of order of integration (possible because all the functions involved are positive) and considering $\lim_{t\to+\infty} g(t) = 1$, we get

$$\begin{aligned} L &= \int_0^{+\infty} \frac{g'(t)}{g(t)} \int_0^t f(x)dxdt = \int_0^{+\infty} \int_0^t \frac{g'(t)}{g(t)} f(x)dxdt \\ &= \int_0^{+\infty} \int_x^{+\infty} \frac{g'(t)}{g(t)} f(x)dtdx = \int_0^{+\infty} f(x) \int_x^{+\infty} \frac{g'(t)}{g(t)} dtdx \\ &= \int_0^{+\infty} f(x) \left\{\ln[g(t)]\right\}_x^{+\infty} dx = \int_0^{+\infty} f(x) \left\{\ln[\lim_{t\to+\infty} g(t)] - \ln[g(x)]\right\} dx \\ &= \int_0^{+\infty} f(x) \left\{-\ln[g(x)]\right\} dx. \end{aligned} \tag{4.6}$$

It follows from Equations (4.5) and (4.6), and a standardization of the notations, that

$$\left[\int_0^{+\infty} \left\{-\ln[g(t)]\right\} f(t)dt\right]^p \le \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p} [F(t)]^p dt,$$

which is the desired inequality. This ends the proof. □

Example 1: We consider $g(t) = 1 - e^{-\alpha t}$, $t \in (0, +\infty)$, with $\alpha > 0$. Then g is differentiable, non-decreasing, with $\lim_{t\to 0} g(t) = 0$ and $\lim_{t\to+\infty} g(t) = 1$ (in fact, it is the cumulative distribution function of the exponential distribution with parameter α). Proposition 4.4 gives

$$\left[\int_0^{+\infty} \left\{-\ln[g(t)]\right\} f(t)dt\right]^p \le \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p} [F(t)]^p dt,$$

which is equivalent to

$$\left\{\int_0^{+\infty} \left[-\ln(1 - e^{-\alpha t})\right] f(t)dt\right\}^p \le \alpha \int_0^{+\infty} \frac{e^{-\alpha t}}{(1 - e^{-\alpha t})^p} [F(t)]^p dt.$$

If we combine this result with Equation (3.3), we get the following inequalities:

$$\begin{aligned} &\frac{1}{\alpha} \left\{\int_0^{+\infty} \left[-\ln(1 - e^{-\alpha t})\right] f(t)dt\right\}^p \\ &\le \int_0^{+\infty} \frac{e^{-\alpha t}}{(1 - e^{-\alpha t})^p} [F(t)]^p dt \le \left[\frac{p}{\alpha(p-1)}\right]^p \int_0^{+\infty} e^{\alpha(p-1)t} [f(t)]^p dt. \end{aligned}$$

Example 2: We consider $g(t) = t^\alpha/(1 + t^\alpha)$, $t \in (0, +\infty)$, with $\alpha > 0$. Then g is differentiable, non-decreasing, with $\lim_{t\to 0} g(t) = 0$ and $\lim_{t\to+\infty} g(t) = 1$ (in fact, it is the cumulative distribution function of a simplified version of the Dagum distribution with parameter α). Proposition 4.4 gives

$$\left[\int_0^{+\infty} \left\{-\ln[g(t)]\right\} f(t)dt\right]^p \le \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p} [F(t)]^p dt,$$

which is equivalent to

$$\left\{\int_0^{+\infty} \left[\ln(1 + t^\alpha) - \alpha \ln(t)\right] f(t)dt\right\}^p \le \alpha \int_0^{+\infty} (1 + t^\alpha)^{p-2} t^{-\alpha(p-1)-1} [F(t)]^p dt.$$

Example 3: We consider $g(t) = e^{-t^{-\alpha}}$, $t \in (0, +\infty)$, with $\alpha > 0$. Then g is differentiable, non-decreasing, with $\lim_{t\to 0} g(t) = 0$ and $\lim_{t\to+\infty} g(t) = 1$ (in fact, it is the cumulative distribution function of the inverse Weibull distribution with parameter α). Proposition 4.4 gives

$$\left[\int_0^{+\infty} \{-\ln[g(t)]\} f(t)dt\right]^p \leq \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt,$$

which is equivalent to

$$\left[\int_0^{+\infty} \frac{1}{t^\alpha} f(t)dt\right]^p \leq \alpha \int_0^{+\infty} \frac{1}{t^{\alpha+1}} e^{(p-1)t^{-\alpha}}[F(t)]^p dt.$$

These are just a few notable examples of the use of Proposition 4.4; other examples could be presented in a similar way.

The result below shows that, under some assumptions on f and g, an improvement can be made for the constant factor of Proposition 2.1.

Proposition 4.5. *Let $p > 1$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that f is continuous and integrable, and g is differentiable, non-decreasing, satisfying*

$$\lim_{t\to 0} g(t) = 0$$

and such that f/g' is non-decreasing. For any $t \in (0, +\infty)$, let us consider the primitive of f defined by

$$F(t) = \int_0^t f(x)dx.$$

Then we have

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

provided that the integrals considered converge.

Proof. Since g is differentiable and non-decreasing, we can write

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt = \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}\left[\int_0^t f(x)dx\right]^p dt$$

$$= \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}\left[\int_0^t \frac{f(x)}{g'(x)} g'(x)dx\right]^p dt. \tag{4.7}$$

Using the facts that f/g' is non-decreasing, g non-decreasing and $\lim_{t\to 0} g(t) = 0$, we obtain

$$\int_0^t \frac{f(x)}{g'(x)} g'(x)dx \leq \frac{f(t)}{g'(t)} \int_0^t g'(x)dx = \frac{f(t)}{g'(t)} g(t). \tag{4.8}$$

It follows from Equations (4.7) and (4.8) that

$$\int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}[F(t)]^p dt \leq \int_0^{+\infty} \frac{g'(t)}{[g(t)]^p}\left[\frac{f(t)}{g'(t)} g(t)\right]^p dt = \int_0^{+\infty} \frac{1}{[g'(t)]^{p-1}}[f(t)]^p dt,$$

which is the desired inequality. This ends the proof. □

Therefore, under the additional assumption that f/g' is non-decreasing, the constant factor $[p/(p-1)]^p$ in Proposition 2.1 can be replaced by 1, which is a clear improvement.

5. Conclusion

In this article, we make several contributions to integral inequalities. We establish a new generalization of the Hardy integral inequality and improve this generalization under more stringent assumptions. We also derive a lower bound on the main integral term. We provide numerous examples to demonstrate how the theory can be applied in both simple and general settings. Future work could include adapting our main integral inequality to the multidimensional case and exploring its reversed version.

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