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A REVIEW OF RECENT GENERALIZED PROBABILITY DISTRIBUTION FAMILIES: ADVANCES AND APPLICATION

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ABSTRACT. Probability distributions are essential tools for modeling, prediction, and statistical inference. In recent years, several generalized families of distributions have been proposed to extend classical models and increase their flexibility in capturing complex data behaviors. This paper reviews selected generalized families published between 2023 and 2025, focusing on their construction mechanisms, statistical properties, estimation methods, and real-world applications. The families discussed include trigonometric-based, inverse, Lomax-generated, Topp–Leone, and hybrid forms. To illustrate their performance, five families were combined with the exponential distribution and fitted to a real dataset. The comparison shows that all extended models provide an adequate fit, while the standard exponential model performs poorly. The findings confirm the practical value of generalized families in improving data modeling.

1. INTRODUCTION

Probability distributions are fundamental tools in statistics, forming the basis for inference, uncertainty measurement, and predictive modeling across many fields, including actuarial science, hydrology, bioinformatics, and operations research. As data in modern research become more complex, these tools must be both theoretically rigorous and practically flexible. However, a growing body of evidence ([1, 12, 7, 10, 5, 22, 23, 24]) shows that classical models often fail to capture real-world patterns such as heavy or light tails, skewness, multimodality, and non-monotonic hazard behaviors. For example, financial time series data frequently reveal fat-tailed and skewed distributions, while biomedical survival data may exhibit hazard functions that deviate substantially from the exponential assumption of constant failure rates.

To overcome these limitations, statisticians have developed generalized families of distributions, also called generators. These frameworks extend existing distributions by adding parameters, applying functional transformations, or using compounding

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mechanisms. Such approaches preserve the foundation of classical models while broadening their capacity to handle more complex data.

In recent years, researchers have also explored trigonometric-based transformations, where sine, cosine, or other periodic functions are integrated into a distribution's cumulative or density function. These models are motivated both by the cyclic behavior seen in some datasets and by the additional flexibility trigonometric functions provide. They enable better modeling of patterns such as bimodality and periodic hazard rates.

Building on this context, the aim of this study is twofold: first, to provide a consolidated overview of recent advances in generalized families of distributions, and second, to evaluate their comparative effectiveness in modeling real-world data. This review brings together key contributions published between 2023 and 2025, examining new generator frameworks, their theoretical foundations, estimation techniques, and practical applications. It also includes a comparative analysis of families sharing the same baseline distribution, where competing models are fitted to a real dataset to assess and rank their performance. Relevant studies were gathered using Scopus and Google Scholar, focusing on keywords such as generalized distributions and generated families of distributions.

The structure of this study is as follows: Section 2 discusses some of the the recent generalized family of distributions developed between 2023 and 2025. Application results are given in Section 3. Finally, concluding remarks are presented in Section 4.

2. Prominent Generalized Distribution Families (2023–2025)

2.1. **The type I heavy-tailed odd power generalized Weibull-G family of distributions.** Moakofi and Oluyede [14] proposed the Type I heavy-tailed odd power generalized Weibull-G family of distributions. It was formed by compounding the Type I heavy-tailed-G family with the odd power generalized Weibull-G family. The authors derived the reliability functions of this family. Other properties obtained include the quantile function, moments, and order statistics. A linear representation was also provided. The family includes several subfamilies. Three submodels were developed using different baseline distributions: log-logistic, Weibull, Rayleigh, and standard half-logistic. The authors showed that this family can produce rich density shapes, including left- and right-skewed forms, as well as hazard rate shapes that are monotonic or non-monotonic. To illustrate its practical use, the Weibull-based submodel was applied to four real datasets. It outperformed submodels obtained from other distribution families. The cumulative distribution function (CDF) is given by

$$F\left(x;\theta,\alpha,\beta,\xi\right)=1-\left(\frac{\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\bar{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{\theta}, \qquad (2.1)$$

for $x \geq 0$, $\alpha > 0$, $\beta > 0$, and $\theta > 0$, where ξ denotes the vector of baseline parameters. The corresponding probability density function (PDF) is

$$f(x;\theta,\alpha,\beta,\xi) = \theta^2\alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1}$$
$$\times\exp\left(\theta\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2}$$
$$\times\left(1-\bar{\theta}\left[1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]\right)^{-(\theta+1)}. \tag{2.2}$$

Here, $G(x;\xi)$ and $g(x;\xi)$ denote the CDF and PDF of the baseline distribution, respectively.

2.2. **Sine type II Topp-Leone-G family of distributions.** Isa et al. [11] introduced the sine type II Topp-Leone-G family of distributions by combining the sine generalized family of distributions with the type II Topp-Leone family. Several properties of this family were derived, and its parameters were estimated using the method of maximum likelihood. To demonstrate its practical applicability, the authors developed a submodel based on the Lomax distribution as the baseline. The resulting sine type II Topp-Leone Lomax distribution showed superior performance compared to other submodels when applied to real-life COVID-19 and carbon fiber datasets. The CDF of the family is given by

$$F(x;\kappa,\xi) = \sin\left\{\frac{\pi}{2}\left[1-\left(1-(G(x;\xi))^2\right)^{\kappa}\right]\right\}; x \geq 0,\ \kappa > 0. \tag{2.3}$$

The corresponding PDF is

$$f(x;\kappa,\xi) = \frac{\pi}{2}2\kappa g(x;\xi)\,G(x)\left[1-(G(x;\xi))^2\right]^{\kappa-1}\cos\left\{\frac{\pi}{2}\left[1-\left(1-(G(x;\xi))^2\right)^{\kappa}\right]\right\}. \tag{2.4}$$

2.3. **Ramos-Louzada-G Family of distributions.** Okutu et al. [17] introduced the Ramos-Louzada-G family of distributions by applying the T-X approach to the classical Ramos-Louzada distribution. Several properties were derived, including the moments, quantile function, survival function, and different entropy measures. The parameters of the family were estimated using the method of maximum likelihood. Two submodels were developed: the Ramos-Louzada Weibull distribution and the Ramos-Louzada Kumaraswamy distribution. A simulation study was carried out using the Ramos-Louzada Weibull distribution. The results showed that the model produced reliable estimates as the sample size increased, since both the root mean square error and bias decreased. The Ramos-Louzada Weibull distribution was also applied to carbon fibre, hypertension, and rainfall datasets. It outperformed nine other existing distributions. This highlighted the ability of the Ramos-Louzada family to enhance the flexibility of existing distributions. The CDF of the Ramos-Louzada generator is given by

$$F(x;\theta,\xi) = 1-\left(1-\frac{\log(1-G(x,\xi))}{\theta(\theta-1)}\right)(1-G(x,\xi))^{1/\theta}; x > 0, \theta \geq 2. \tag{2.5}$$

The corresponding PDF is

$$f(x;\theta,\xi) = \frac{g(x,\xi)(1-G(x;\xi))^{(1/\theta)-1}}{\theta^2(\theta-1)}\left(\theta^2-2\theta-\log(1-G(x,\xi)\right). \tag{2.6}$$

2.4. **Odd Lomax-G family of Distributions.** Nooria et al. [16] introduced the odd Lomax-G (OLG) family of distributions using the T-X framework. The study also presented a flexible three-parameter model within this family, called the odd Lomax-G exponential (OLE) distribution. Using binomial, logarithmic, and exponential series expansions, the authors expressed the OLG family and its submodel in expanded forms. Several properties were derived, including moments, moment generating function, quantile function, order statistics, and Rényi entropy. The parameters of the OLE distribution were estimated through the maximum likelihood method. TTo illustrate its practical strengths, two real datasets were analyzed, and the OLE distribution outperformed several existing models based on goodness-of-fit criteria, highlighting the family's flexibility and usefulness in modeling diverse data patterns. The CDF of the family is given by

$$F\left(x;\alpha,\gamma,\xi\right)=1-\left(1-\frac{G\left(x,\xi\right).\log\left(1-G\left(x,\xi\right)\right)}{\gamma}\right)^{-\alpha};x,\alpha,\gamma>0. \tag{2.7}$$

The corresponding PDF is

$$\begin{aligned}f\left(x;\alpha,\gamma,\xi\right)=&\frac{\alpha}{\gamma}g\left(x,\xi\right)\left(1-\frac{G\left(x,\xi\right).\log\left(1-G\left(x,\xi\right)\right)}{\gamma}\right)^{-(\alpha+1)}\\&\times\left[\frac{G\left(x,\xi\right)}{1-G\left(x,\xi\right)}-\log\left(1-G\left(x,\xi\right)\right)\right].\end{aligned} \tag{2.8}$$

2.5. **Odd Muth-G family of distributions.** Ayed et al. [4] proposed the Odd Muth-G family of distributions using the Transformed-Transformer methodology. The family is notable for its ability to generate several special models that have applications in reliability analysis. The authors showed that the density of the model can be expressed as linear combinations of generalized exponentials, which simplifies the derivation of its properties. They derived structural results such as the quantile function, moments, probability-weighted moments, and entropy. Parameter estimation was carried out using the maximum likelihood method. With the Weibull distribution as a baseline, an Odd Muth-Weibull distribution was developed and further investigated. A simulation study demonstrated that the estimators performed well as the mean squared error and bias decreased with larger sample sizes. The study also introduced a plan acceptance sampling design based on the Odd Muth-Weibull distribution, where the median lifetime was used as the quality parameter. To show practical relevance, two real-life datasets were analyzed, and the results indicated that the family has enough flexibility to model a wide range of lifetime data. The CDF of the family is given by

$$F\left(x,a,\xi\right)=1-\exp\left\{\frac{aG\left(x;\xi\right)}{\bar{G}\left(x;\xi\right)}-\frac{1}{a}\left[\exp\left(\frac{aG\left(x;\xi\right)}{\bar{G}\left(x;\xi\right)}\right)-1\right]\right\};a>0,x>0. \tag{2.9}$$

The corresponding PDF is

$$\begin{aligned}f\left(x;a,\xi\right)=&\frac{g\left(x;\xi\right)}{\bar{G}(x;\xi)^2}\left\{\exp\left(\frac{aG\left(x;\xi\right)}{\bar{G}\left(x;\xi\right)}\right)-a\right\}\\&\times\exp\left\{\frac{aG\left(x;\xi\right)}{\bar{G}\left(x;\xi\right)}-\frac{1}{a}\left[\exp\left(\frac{aG\left(x;\xi\right)}{\bar{G}\left(x;\xi\right)}\right)-1\right]\right\}.\end{aligned} \tag{2.10}$$

2.6. **New Lomax G family of distributions.** Sapkota et al. [20] proposed the New Lomax-G family of distributions using the beta-generated transformation technique. The study emphasized the inverse exponential power distribution as the baseline, which can produce reverse-J, inverted bathtub, and monotonically increasing hazard functions. The authors derived the main characteristics of the family and estimated its parameters through the maximum likelihood method. A simulation study was carried out to check the accuracy of the estimation procedure. The results showed that bias and mean square error decreased as the sample size increased, even for small samples. For data fitting purposes, the inverse power exponential submodel of the family was employed in applications to medical datasets. Based on goodness-of-fit measures and model selection criteria, the Lomax inverse power exponential distribution outperformed several competing models. The CDF of the family is given by

$$F\left(x,\alpha,\beta,\xi\right)=\beta^{\alpha}[\beta-\log\left\{G\left(x,\alpha,\beta,\xi\right)\right\}]^{-\alpha};x>0,\alpha,\beta>0. \tag{2.11}$$

The corresponding PDF is

$$f\left(x,\alpha,\beta,\xi\right)=\alpha\beta^{\alpha}\frac{g\left(x,\alpha,\beta,\xi\right)}{G\left(x,\alpha,\beta,\xi\right)}[\beta-\log\left\{G\left(x,\alpha,\beta,\xi\right)\right\}]^{-(\alpha+1)}. \tag{2.12}$$

2.7. **ARCTAN Kavya-Manoharan-G family of distributions.** Alrashidi [2] introduced the ARCTAN Kavya-Manoharan-G family of distributions by combining the arctan-G and Kavya-Manoharan-G families. Several properties of this family were derived, and its parameters were estimated using the maximum likelihood method, expressed in terms of a vector of baseline parameters. The family can produce submodels with different hazard rate functions. Its usefulness was shown with real data, where a submodel provided a better fit than existing models. The CDF is given by

$$F\left(x;\xi\right)=\frac{4}{\pi}\arctan\left(\frac{e}{e-1}\left(1-e^{-G(x;\xi)}\right)\right);x\in\mathbb{R}. \tag{2.13}$$

The corresponding PDF is

$$f\left(x;\xi\right)=\frac{4\frac{e}{e-1}g\left(x;\xi\right)e^{-G(x;\xi)}}{\pi\left[1+\left(\frac{e}{e-1}\left(1-e^{-G(x;\xi)}\right)\right)^{2}\right]}. \tag{2.14}$$

2.8. **DUS Topp–Leone-G Family of Distributions.** Ekemezie et al. [3] introduced the DUS Topp–Leone-G family by extending the Topp–Leone generated family through the Dinesh–Umesh–Sanjay (DUS) transformation. A notable submodel, the DUS Topp–Leone Exponential distribution, was proposed and studied in detail. The authors derived its main statistical properties, considered different estimation methods, and showed that non-Bayesian approaches such as maximum likelihood and least squares performed best. Applications to productivity and mortality data highlighted the flexibility of the family in modeling diverse real-world datasets. The CDF of the DUS Topp–Leone family is given by

$$F\left(x;v,\xi\right)=\frac{e^{\left[1-(1-G(x;,\xi))^{2}\right]^{v}}-1}{e-1};x>0,v>0. \tag{2.15}$$

The corresponding PDF is

$$f\left(x;v,\xi\right)=\frac{2vg\left(x;\xi\right)\left(1-G\left(x;\xi\right)\right)\left[1-\left(1-G\left(x;\xi\right)\right)^{2}\right]^{v-1}e^{\left[1-(1-G(x;\xi))^{2}\right]^{v}}}{e-1}. \tag{2.16}$$

2.9. **Transmuted cosine Topp-Leone G family of distributions.** Osi et al. [18] introduced the transmuted cosine Topp-Leone-G (TrCTLG) family of distributions by combining the transmuted cosine-G family with the Topp-Leone family. Several statistical properties of the family were derived, including the survival and hazard functions, as well as moments and moment-generating functions. Parameter estimation was carried out using the maximum likelihood method. Two special cases were obtained by considering Weibull and exponential baseline distributions, with the Weibull-based submodel studied in detail. A Monte Carlo simulation was carried out to examine the behavior and consistency of the estimators. Applications to lifetime datasets illustrated the practical usefulness of the TrCTLG family and confirmed its flexibility for real-world modeling. The CDF of the family is given by

$$\begin{aligned} F\left(x;\theta,\alpha,\xi\right) &= \left(1+\theta\right)\left[1-\cos\left(\frac{\pi}{2}\left(1-\bar{G}(x;\xi)^2\right)\right)^{\alpha}\right] \\ &\quad -\theta\Big[1-\cos\left(\frac{\pi}{2}\left(1-\bar{G}(x;\xi)^2\right)\right)^{\alpha}\Big]^2; x>0, \alpha>0, |\theta|<1. \end{aligned} \tag{2.17}$$

The corresponding PDF is

$$\begin{aligned} f\left(x;\theta,\alpha,\xi\right) &= \left(1+\theta\right)\pi\alpha g\left(x;\xi\right)\bar{G}\left(x;\xi\right)\left(1-\bar{G}(x;\xi)^2\right)^{\alpha-1}\sin\left[\left(\frac{\pi}{2}\left(1-\bar{G}(x;\xi)^2\right)\right)^{\alpha}\right] \\ &\quad -2\theta\pi\alpha g\left(x;\xi\right)\bar{G}\left(x;\xi\right)\left(1-\bar{G}(x;\xi)^2\right)^{\alpha-1}\sin\left[\left(\frac{\pi}{2}\left(1-\bar{G}(x;\xi)^2\right)\right)^{\alpha}\right] \\ &\quad \times\left[1-\cos\left(\frac{\pi}{2}\left(1-\bar{G}(x;\xi)^2\right)\right)^{\alpha}\right]^2. \end{aligned} \tag{2.18}$$

2.10. **Modified Type II Half Logistic Distribution family of distributions.** Salahuddin et al. [19] proposed the Modified Type II Half-Logistic family of distributions by applying the T-X framework with the Type II variant of the half-logistic distribution. The family was developed to increase the flexibility of baseline distributions. For illustration, the Weibull distribution was studied as a submodel. The authors discussed several properties of the family and estimated parameters using the maximum likelihood method. Applications to real data demonstrated the practical usefulness of the proposed model compared with existing alternatives. The CDF is given by

$$F\left(x;\theta,\xi\right)=\left[\frac{2G(x;\xi)^{\theta}}{1+G(x;\xi)^{\theta}}\right]\left[1+\frac{1-G(x;\xi)^{\theta}}{1+G(x;\xi)^{\theta}}\right]; x>0, \theta>0. \tag{2.19}$$

The corresponding PDF is

$$f\left(x;\theta,\xi\right)=\frac{4\theta G(x;\xi)^{\theta-1}g\left(x;\xi\right)\left[1-G(x;\xi)^{\theta}\right]}{\left(1+G(x;\xi)^{\theta}\right)^3}. \tag{2.20}$$

2.11. **Generalized Alpha-Beta-Power family of distributions.** Semary et al. [21] introduced the Generalized Alpha-Beta-Power (GABP) family of distributions by applying the exponentiated family approach to the Alpha-Beta Power family. The family incorporates three parameters, which increase the flexibility of baseline distributions. A special submodel based on the exponential distribution was studied in detail. The authors showed that the density curves of this submodel can take decreasing and right-skewed heavy-tailed shapes, depending on the values of the shape and scale parameters. The hazard function may be increasing, decreasing, or bathtub-shaped. The study derived several statistical properties, including moments, moment generating function, order statistics, and entropy. Parameters were estimated using the maximum likelihood method, and their performance was examined through simulations. Applications in reliability, engineering, and medicine showed that the submodel provided better fits than some existing models.

The CDF is given by

$$F(x,\alpha,\beta,\eta,\xi)=\left(\frac{\alpha^{G(x;\xi)}-\beta^{G(x;\xi)}}{\alpha-\beta}\right)^{\eta};x\in\mathbb{R},\alpha,\beta,\eta>0,and\alpha\neq\beta. \tag{2.21}$$

The corresponding PDF is

$$f(x,\alpha,\beta,\eta,\xi)=\eta\left(\frac{\alpha^{G(x;\xi)}-\beta^{G(x;\xi)}}{\alpha-\beta}\right)^{\eta-1}\frac{\alpha^{G(x;\xi)}\log\alpha-\beta^{G(x;\xi)}\log\beta}{\alpha-\beta}g(x;\xi). \tag{2.22}$$

2.12. **Cosine Inverse Lomax-G family of distributions.** Bashiru et al. [6] introduced the Cosine Inverse Lomax-G (CIL-G) family of distributions by merging the cosine-G and inverse Lomax-G families. The resulting hybrid family enhances the flexibility of baseline distributions and allows for the development of several submodels suitable for diverse applications. Key statistical properties, including moments, entropy measures, and the quantile function, were derived. Parameters were estimated using maximum likelihood, least squares, and weighted least squares methods, and their performance was evaluated through Monte Carlo simulations. Using the CIL-exponentiated Weibull submodel, the study demonstrated that the estimators are consistent as sample size increases. Using the CIL-exponentiated Weibull submodel in applications to engineering and medical datasets, the study showed that it outperformed the standard exponentiated Weibull model, demonstrating the practical value of the generalized family in improving model fit and predictive performance. The CDF is given by

$$F(x,\alpha,\ \beta,\xi)=1-\cos\left\{\frac{\pi}{2}\left(1+\frac{\beta\bar{G}(x;\xi)}{G(x;\xi)}\right)^{-\alpha}\right\};\quad \alpha,\ \beta,\ x>0. \tag{2.23}$$

The corresponding PDF is

$$f(x,\alpha,\ \beta,\xi)=\frac{\pi}{2}\frac{\alpha\ \beta\ g(x;\xi)}{[G(x;\xi)]^{2}}\left(1+\frac{\beta\bar{G}(x;\xi)}{G(x;\xi)}\right)^{-(\alpha+1)}\sin\left\{\frac{\pi}{2}\left(1+\frac{\beta\bar{G}(x;\xi)}{G(x;\xi)}\right)^{-\alpha}\right\}. \tag{2.24}$$

2.13. **New Inverted Modified Kies-G family of distributions.** Diab et al. [8] introduced the New Inverted Modified Kies-G (NIMK-G) family by applying the inverse transformation to the Modified Kies family. This class of continuous distributions was developed to provide greater flexibility for modeling complex data. The authors derived several properties, including moments, hazard rate function, moment-generating function, Rényi entropy, stress–strength reliability, and quantile function. Special cases of the family include the NIMK-Weibull, NIMK-Exponential, and NIMK-Lomax distributions, which illustrate its adaptability. Parameters were estimated using the maximum likelihood method, and their performance was examined through Monte Carlo simulations. Applications in environmental, biomedical, and engineering fields showed that the NIMK-Weibull distribution offered better fits than existing models. The CDF is given by

$$F(x,\alpha,\gamma,\vartheta,\xi)=e^{-\vartheta\left[\frac{[G(x;\xi)]^{\gamma}}{1-[G(x;\xi)]^{\gamma}}\right]^{-\alpha}};-\infty<x<\infty,\alpha,\gamma,\vartheta>0. \tag{2.25}$$

The corresponding PDF is

$$f(x,\alpha,\gamma,\vartheta,\xi)=\frac{\vartheta\alpha\gamma g(x;\xi)\left[G(x;\xi)\right]^{\gamma-1}\left[\frac{[G(x;\xi)]^{\gamma}}{1-[G(x;\xi)]^{\gamma}}\right]^{-(\alpha+1)}e^{-\vartheta\left[\frac{[G(x;\xi)]^{\gamma}}{1-[G(x;\xi)]^{\gamma}}\right]^{-\alpha}}}{\left[1-[G(x;\xi)]^{\gamma}\right]^{2}}. \tag{2.26}$$

2.14. **Tangent-DUS-G Family of Distributions.** Elgarhy et al. [9] proposed the Tangent-DUS-G (TDUS) family of distributions by combining the Dinesh–Umesh–Sanjay (DUS) transformer with the tangent family of distributions. The family, was used to construct a more flexible Weibull distribution (TDUSW) that retains the same two parameters as the classical Weibull, preserving parsimony. The parameters were estimated using both Bayesian and non-Bayesian approaches. A Monte Carlo simulation study was conducted to examine the behavior of the estimators under different sample sizes. To demonstrate practical value, the TDUSW distribution was applied in fields such as engineering, agriculture, finance, and public health. The results showed that the proposed model performed better than several trigonometric-based models, based on goodness-of-fit tests and model selection criteria. The CDF is given by

$$F(x;\xi)=\tan\left[\frac{\pi}{4}G(x;\xi)\right];x\in\mathbb{R}. \tag{2.27}$$

The corresponding PDF is

$$f(x)=\frac{\pi}{4}g(x;\xi)\sec^{2}\left[\frac{\pi}{4}G(x;\xi)\right]. \tag{2.28}$$

2.15. **Cosine Kumarswamy-G family of distributions.** Ali et al. [13] proposed the cosine Kumaraswamy (CK) family of distributions by combining the cosine and Kumaraswamy families of distributions. The family was used to construct a more flexible Lomax distribution, called the cosine Kumaraswamy Lomax (CKL) distribution, which enhances the baseline Lomax distribution while preserving interpretability. The parameters of the distribution were estimated using the maximum likelihood method. A detailed study of its statistical properties, including the quantile function, moments, moment-generating function, survival and hazard functions, and order statistics, was conducted. To demonstrate practical utility, the CKL distribution was fitted to datasets from medical and engineering fields. The results showed that the CKL distribution provided a better fit than the standard Lomax and other existing models, highlighting the family's strength in extending classical distributions and its usefulness in real-world data modeling. The CDF is given by

$$F(x,\theta,\lambda,\xi)=1-cos\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\theta}\right)^{\lambda}\right]\right\}\ ;x,\theta,\lambda>0. \tag{2.29}$$

The corresponding PDF is

$$f(x)=\frac{\pi}{2}\lambda\theta g(x;\xi)\,G(x;\xi)^{\theta-1}\left[1-G(x;\xi)^{\theta}\right]^{\lambda-1}sin\left\{\frac{\pi}{2}\left[1-\left(1-G(x;\xi)^{\theta}\right)^{\lambda}\right]\right\}. \tag{2.30}$$

3. Real Data Analysis

To demonstrate the practical relevance of the generalized families reviewed in this study, five families were selected and applied using the exponential distribution as the baseline model. Each resulting model represents a specific submodel derived from a distinct generalized family proposed in the literature. Substituting the exponential distribution into these families yields the following models:

- **New Inverted Modified Kies Exponential (NIMKE)** [8]:

$$F\left(x\right)=e^{-\vartheta\left[\frac{\left[G\left(x;\xi\right)\right]^{\gamma}}{1-\left[G\left(x;\xi\right)\right]^{\gamma}}\right]^{-\alpha}}.$$

- **Cosine Inverse Lomax Exponential (CILE)** [6]:

$$F\left(x\right)=1-\cos\left\{\frac{\pi}{2}\left(1+\frac{\beta\left(e^{-\omega x}\right)}{\left(1-e^{-\omega x}\right)}\right)^{-\alpha}\right\}.$$

- **Modified Type II Half Logistic Exponential (MTHLE)** [19]:

$$F\left(x;\xi\right)=\left[\frac{2(1-e^{-\omega x})^{\theta}}{1+\left(1-e^{-\omega x}\right)^{\theta}}\right]\left[1+\frac{1-\left(1-e^{-\omega x}\right)^{\theta}}{1+\left(1-e^{-\omega x}\right)^{\theta}}\right].$$

- **Sine type II Topp-Leone Exponential (STIITE)** [11]:

$$F\left(x\right)=\sin\left\{\frac{\pi}{2}\left[1-\left(1-\left(1-e^{-\omega x}\right)^{2}\right)^{\kappa}\right]\right\}.$$

- **DUS Toppleone Exponential (DTE)** [3]:

$$F\left(x;v,\xi\right)=\frac{e^{\left[1-\left(1-\left(1-e^{-\omega x}\right)\right)^{2}\right]^{v}}-1}{e-1}.$$

For comparison, the standard exponential distribution was also included.

3.1. **Description and analysis of the data.** The models were fitted to a complete dataset previously discussed by Murthy et al. [15], which contains the failure times (in hours) of 24 mechanical components. The observations are:

30.94, 18.51, 16.62, 51.56, 22.85, 22.38, 19.08, 49.56, 17.12, 10.67, 25.43, 10.24, 27.47, 14.70, 14.10, 29.93, 27.98, 36.02, 19.40, 14.97, 22.57, 12.26, 18.14, 18.84.

From the descriptive statistics in Table 1, the data exhibit a clear right-skewed pattern, as indicated by the positive skewness value (1.3454). The mean (22.97) exceeds the median (19.24), suggesting a longer right tail. The kurtosis value (4.3599) indicates a leptokurtic distribution.

TABLE 1. Descriptive Statistics of the Data

Min	Max	Mean	Median	Skewness	Kurtosis
10.24	51.56	22.97	19.24	1.3454	4.3599

Figure 1 displays the histogram, kernel density estimate, and box plot of the data. The histogram and density curve confirm the right-skewed nature of the observations, while the box plot reveals the presence of outliers.

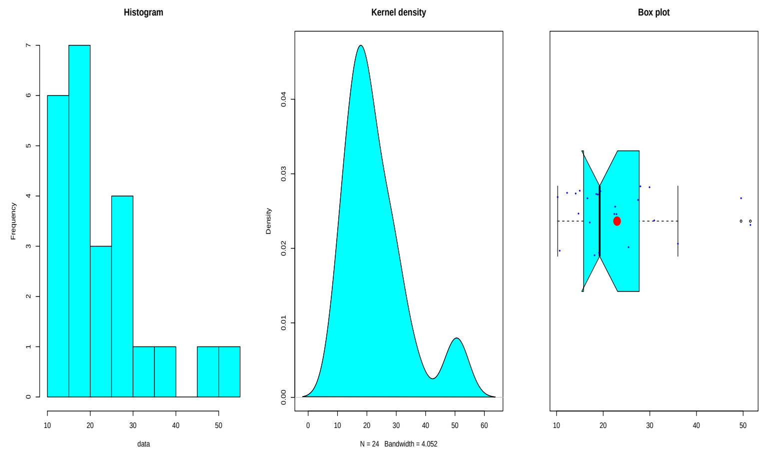

FIGURE 1. Histogram, kernel density and Box plot of the data

The parameters of all competing models were estimated using the maximum likelihood method. Table 2 reports the parameter estimates together with the corresponding goodness-of-fit measures, including the negative log-likelihood (NLL), root mean square error (RMSE), coefficient of determination (COD), and the Kolmogorov–Smirnov (KS) statistic with its p-value.

TABLE 2. Parameter Estimates and Goodness-of-Fit Statistics

Models	Estimates	NLL	RMSE	COD	KS	p-value
NIMKE	$\hat{\vartheta} = 1.0851$ $\hat{\alpha} = 1.1600$ $\hat{\gamma} = 2.3746$ $\hat{\omega} = 0.0805$	85.5058	0.0279	0.9905	0.0811	0.9935
CILE	$\hat{\alpha} = 482.4586$ $\hat{\beta} = 0.0083$ $\hat{\omega} = 0.1149$	85.7745	0.0318	0.9879	0.1075	0.9167
MTHLE	$\hat{\theta} = 6.0070$ $\hat{\omega} = 0.0648$	86.4312	0.0377	0.9830	0.1243	0.8087
STIITE	$\hat{\kappa} = 253.7438$ $\hat{\omega} = 0.0019$	89.2065	0.0587	0.9421	0.1436	0.6535
DTE	$\hat{\nu} = 9.7191$ $\hat{\omega} = 0.0713$	86.4927	0.0419	0.9794	0.1328	0.7421
E	$\hat{\omega} = 0.0435$	99.2231	0.1721	-0.4678	0.3598	0.0027

Among the fitted models, the NIMKE model provides the best overall fit, attaining the smallest NLL, RMSE, and KS values, along with the highest COD and p-value. The CILE and MTHLE models also show satisfactory performance. In contrast, the standard exponential distribution yields the weakest fit. All generalized exponential-based models have p-values exceeding 0.05, indicating adequate agreement with the data, whereas the exponential model fails this criterion.

Figures 2–4 present the fitted probability density functions, cumulative distribution functions, and P–P plots. These graphical assessments further confirm that the generalized families offer a better description of the data than the standard exponential model.

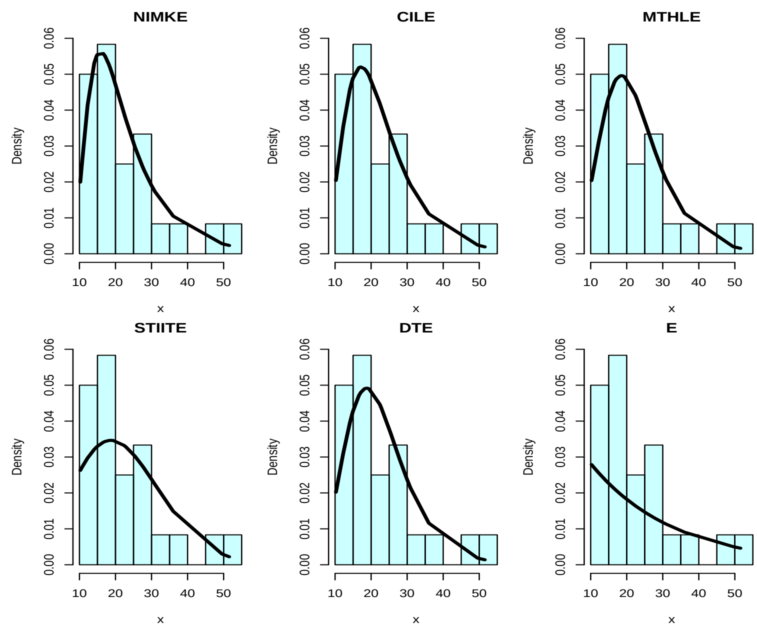

FIGURE 2. Fitted PDF plots of the models for the dataset

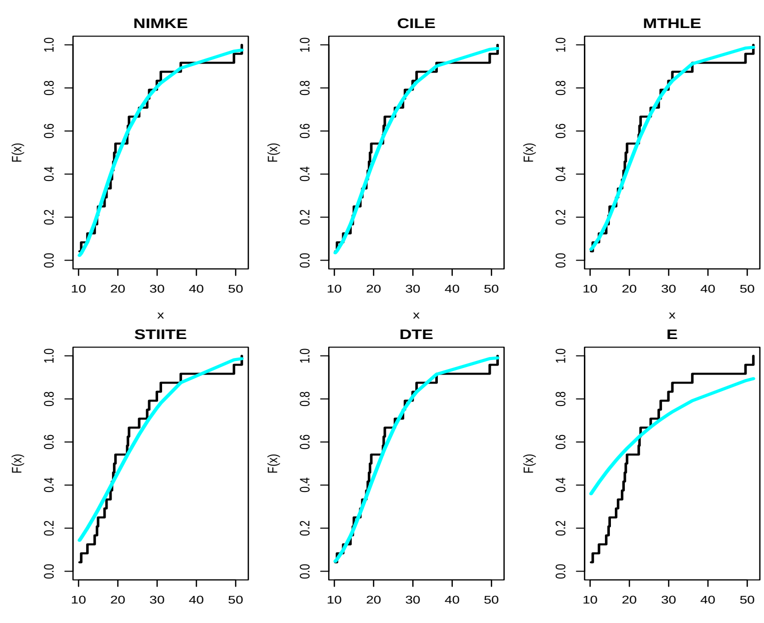

FIGURE 3. Fitted CDF plots of the models for the dataset

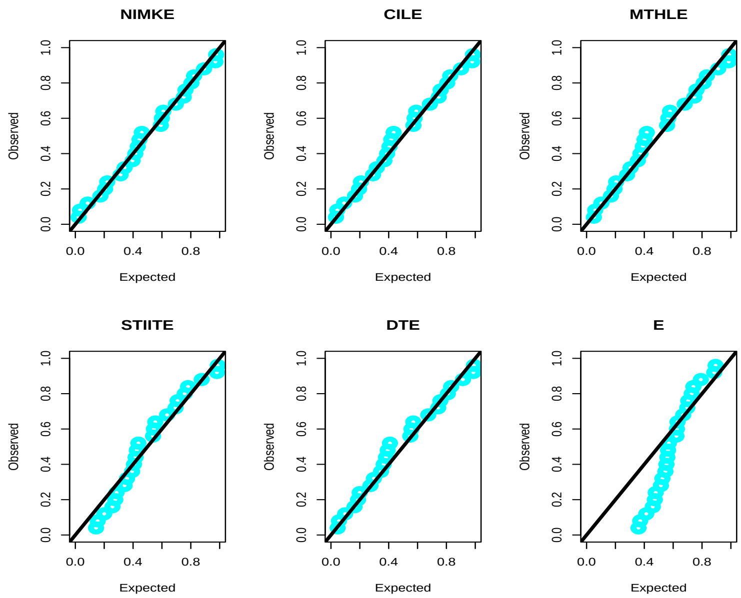

FIGURE 4. Fitted P–P plots of the models for the dataset.

Overall, the results illustrate that the generalized families reviewed in this study, when combined with the exponential baseline, provide improved flexibility and goodness-of-fit in modeling lifetime data, reinforcing their relevance in reliability analysis.

4. CONCLUDING REMARKS

This review highlighted several generalized families of distributions developed between 2023 and 2025, emphasizing their construction mechanisms, statistical properties, estimation techniques, and applications. The review showed that these families offer greater flexibility than their baseline distribution, enabling them to model complex data behaviors such as skewness, heavy tails, and non-monotonic hazard rates. The comparative analysis, which fitted five of the discussed families, each combined with the exponential distribution, to a real dataset, revealed that all extended models provided an adequate fit, while the exponential model failed to do so. This finding confirms the practical advantage of generalized families in capturing real-world data patterns. Overall, the study shows that generalized families enrich distribution theory and enhance applied statistical modeling. Future research may explore new generator structures and transformation mechanisms, improved estimation and inference techniques, and potential extensions to graph-structured data, where flexible distributions could be used to model stochastic attributes such as node- or edge-level lifetimes, weights, or inter-event times. Investigating such connections may broaden the applicability of generalized families to modern data-analytic settings.

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