ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 4 (2025), 459-471
DOI: 10.62072/acm.2025.080404
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

ON LOGARITHMIC CORDIAL LABELING OF SOME GRAPHS

JASON D. ANDOYO

ABSTRACT. Let $\eta \geq 3$ be an integer with primitive root ϖ. For a simple connected graph G of order n, a bijective function $f : V(G) \to \{1, 2, \ldots, n\}$ is called a logarithmic cordial labeling to the base ϖ modulo η if the induced function $f^*_{\varpi,\eta} : E(G) \to \{0, 1\}$, defined by $f^*_{\varpi,\eta}(ab) = 0$ if $\text{ind}_{\varpi,\eta}(f(a)+f(b)) \equiv 0 (\text{mod } 2)$ or $\gcd(f(a)+f(b), \eta) \neq 1$, and $f^*_{\varpi,\eta}(ab) = 1$ if $\text{ind}_{\varpi,\eta}(f(a) + f(b)) \equiv 1 (\text{mod } 2)$, satisfies the condition $|e_{f^*_{\varpi,\eta}}(0) - e_{f^*_{\varpi,\eta}}(1)| \leq 1$ where $e_{f^*_{\varpi,\eta}}(i)$ is the number of edges with label $i (i = 0, 1)$. In this paper, we study the logarithmic cordial labeling of various classes of graphs, including path graphs, cycle graphs, star graphs, and complete graphs.

1. INTRODUCTION

A graph $G = (V, E)$ has a *vertex set* $V = V(G)$ and an *edge set* $E = E(G)$. The elements of V and E are called *vertices* and *edges*, respectively. If G has n vertices and m edges, then G has *order* n and *size* m. Let $v \in V(G)$. The *degree* of v, denoted by $\deg(v)$, is the number of vertices in G that are adjacent to v. One notable property of graphs is the *Fundamental Theorem of Graph Theory*, which states that the sum of the degrees of the vertices of G is twice its size.

Graph labeling is a broad area of graph theory in which numbers or sets are assigned to the vertices and edges of a graph subject to specific conditions, and it has found numerous applications in diverse fields such as communication networks, coding theory, database design, and X-ray crystallography. Over the years, various types of graph labeling have been introduced, each highlighting different structural properties of graphs. One important and widely studied family is *cordial labeling*, first introduced by I. Cahit in 1987 [5]. A binary labeling of the vertex set of a graph is called *cordial* if the numbers of vertices labeled 0 and 1 differ by at most one, and if the induced edge labeling—defined by assigning to each edge the absolute difference of the labels of its endpoints modulo 2—also satisfies the condition that the numbers of edges labeled 0 and 1 differ by at most one. This notion has inspired many variants such as edge k-product cordial labeling [8], Tribonacci cordial

2020 *Mathematics Subject Classification.* 11A07, 05C78.
Key words and phrases. primitive root; modulo; logarithmic cordial labeling.
Received: October 08, 2025. Accepted: December 25, 2025. Published: December 31, 2025.

labeling [7], and Euler cordial labeling [1]. For more information on graph labeling and variants of cordial labeling, see [6].

In recent years, number-theoretic functions have been incorporated into the framework of cordial labeling, giving rise to new variants that connect algebraic properties with combinatorial structures. For instance, the Möbius function, which is fundamental in multiplicative number theory, and the Legendre symbol, a classical quadratic residue character, have been used to define novel labeling schemes. These extensions—referred to as Möbius cordial labeling [3], Legendre cordial labeling [2], and Legendre product cordial labeling [9]—illustrate how arithmetic functions can be applied to graph labeling, further broadening the scope of cordial labeling and strengthening the connection between graph theory and number theory.

These developments motivate the introduction of logarithmic cordial labeling, which incorporates the discrete logarithm into the framework of cordial labeling. The discrete logarithm has significant applications in computational number theory and modern cryptography, serving as the foundation of protocols such as the Diffie–Hellman key exchange, digital signatures, and various public-key cryptosystems. Its importance lies in its computational difficulty, as determining discrete logarithms for large primes is widely regarded as an intractable problem. By linking this number-theoretic function to cordial labeling, logarithmic cordial labeling not only broadens the scope of graph labeling but also establishes new connections between combinatorial structures and problems of cryptographic relevance.

In this paper, we investigate the logarithmic cordial labeling of graphs such as path graphs, cycle graphs, complete graphs, and additional classes of graphs.

2. Basic Concepts

2.1. Special Graphs.

Definition 2.1. A path graph P_n of order n and size $n-1$ is a graph with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

Definition 2.2. A complete graph K_n of order n and size $\frac{n(n-1)}{2}$ is a graph where every two distinct vertices of K_n are adjacent.

2.2. Discrete Logarithm and Logarithmic Cordial Labeling.

Definition 2.3. [10] The *Euler phi function* $\phi(n)$ of a positive integer n is the number of positive integers not exceeding n that are relatively prime to n.

Remark 1. [10] *For each positive integer $n \geq 3$, $\phi(n)$ is even.*

Definition 2.4. [10] Suppose that $\gcd(\varpi, \eta) = 1$ with $\varpi \neq 0$ and $\eta > 0$. Then the *order of ϖ modulo η* is $\text{ord}_\eta(\varpi) = x$ if x is the least positive integer such that $\varpi^x \equiv 1 (\text{mod } \eta)$. If $\text{ord}_\eta(\varpi) = \phi(\eta)$, then ϖ is a *primitive root* of η. In addition, suppose that η has a primitive root, say ϖ. Let a be an integer relatively prime to η. Then the *discrete logarithm (or index) of a to the base ϖ modulo η* is $\text{ind}_{\varpi,\eta}(a) = y$ where y is a unique integer with $1 \leq y \leq \phi(\eta)$ and $\varpi^y \equiv a (\text{mod } \eta)$.

Remark 2. [10] *Let a and b be relatively prime to η, where η has a primitive root, say ϖ. If $a \equiv b (\text{mod } \eta)$, then* $\text{ind}_{\varpi,\eta}(a) = \text{ind}_{\varpi,\eta}(b)$.

Theorem 2.1. [10] *A positive integer η has a primitive root if and only if $\eta \in \{2, 4, p^k, 2p^k\}$ where p is an odd prime and k is a positive integer.*

Theorem 2.2. [4] *For an odd prime p and positive integer k, every primitive root of p^k is also a primitive root of p.*

Definition 2.5. Suppose that $\eta \geq 3$ is an integer with primitive root ϖ. For a simple connected graph G of order n, a bijective function $f : V(G) \to \{1, 2, \ldots, n\}$ is called a *logarithmic cordial labeling to the base ϖ modulo η* if the induced function $f^*_{\varpi,\eta}$: $E(G) \to \{0, 1\}$, defined by

$$f^*_{\varpi,\eta}(ab) = \begin{cases} 0 & \text{if } \mathrm{ind}_{\varpi,\eta}(f(a) + f(b)) \equiv 0(\mathrm{mod}\ 2) \text{ or } \gcd(f(a) + f(b), \eta) \neq 1 \\ 1 & \text{if } \mathrm{ind}_{\varpi,\eta}(f(a) + f(b)) \equiv 1(\mathrm{mod}\ 2), \end{cases}$$

satisfies the condition $|e_{f^*_{\varpi,\eta}}(0) - e_{f^*_{\varpi,\eta}}(1)| \leq 1$ where $e_{f^*_{\varpi,\eta}}(i)$ is the number of edges with label i $(i = 0, 1)$. A graph that admits a logarithmic cordial labeling to the base ϖ modulo η is called *logarithmic cordial graph to the base ϖ modulo η*. Also, if $\gcd(f(a)+f(b), \eta) = 1$ for each $ab \in E(G)$, then f is called *pure logarithmic cordial labeling to the base ϖ modulo η* and a graph that admits this labeling is called *pure logarithmic cordial graph to the base ϖ modulo η*.

Definition 2.6. Let η be a positive integer with a primitive root ϖ. A *discrete logarithm graph* $DL^k_{\varpi,\eta}(f, n)$ of order n is a graph with edge set

$$E(DL^k_{\varpi,\eta}(f, n)) = \{ab : \mathrm{ind}_{\varpi,\eta}(f(a) + f(b)) \equiv k(\mathrm{mod}\ 2)\}$$

where $f : V(DL^k_{\varpi,\eta}(n)) \to \{1, 2, \ldots, n\}$ is a bijective function and $k \in \{0, 1\}$.

2.3. **Legendre Symbol.**

Definition 2.7. [4] Let p be an odd prime and a be an integer relatively prime to p. Then the *Legendre symbol* (a/p) is defined by

$$(a/p) = \begin{cases} 1 & \text{if } x^2 \equiv a(\mathrm{mod}\ p) \text{ has a solution} \\ -1 & \text{otherwise.} \end{cases}$$

Theorem 2.3 (Euler's Criterion)**.** [4] *If p is an odd prime and a is relatively prime to p, then*

$$a^{(p-1)/2} \equiv (a/p)(\mathrm{mod}\ p).$$

Theorem 2.4. [10] *Let p be an odd prime. Then*

$$(2/p) = \begin{cases} 1 & \text{if } p \equiv \pm 1(\mathrm{mod}\ 8) \\ -1 & \text{if } p \equiv \pm 3(\mathrm{mod}\ 8). \end{cases}$$

Theorem 2.5. [4] *If p is an odd prime and a is relatively prime to p, then the congruence $x^2 \equiv a(\mathrm{mod}\ p^k)$, $k \geq 1$, has a solution if and only if $(a/p) = 1$.*

3. Preliminary Results

Remark 3. *Let η be a positive integer with primitive root ϖ. In addition, suppose that $\varphi(\eta) = \{\xi : \gcd(\xi, \eta) = 1,\ 0 < \xi < \eta\}$. Then*

$$\{\mathrm{ind}_{\varpi,\eta}(\xi) : \xi \in \varphi(\eta)\} = \{1, 2, \ldots, \phi(\eta)\}.$$

So, if

$$A = \{\xi \in \varphi(\eta) : \mathrm{ind}_{\varpi,\eta}(\xi) \equiv 1(\mathrm{mod}\ 2)\}$$

and

$$B = \{\xi \in \varphi(\eta) : \mathrm{ind}_{\varpi,\eta}(\xi) \equiv 0(\mathrm{mod}\ 2)\},$$

then $|A| = |B| = \frac{\phi(\eta)}{2}$.

Proof. Note that every $\xi \in \varphi(\eta)$ can be written uniquely as $\xi \equiv \varpi^k (\text{mod } \eta)$ where k is an integer with $1 \leq k \leq \phi(\eta)$. As a result, $\text{ind}_{\varpi,\eta}(\xi) = k$ for $k = 1, 2, \ldots, \phi(\eta)$. Thus, the conclusion follows. □

Lemma 3.1. *Let p be an odd prime and k be a positive integer. If ϖ is a primitive root of p^k, then $\text{ind}_{\varpi,p^k}(2) \equiv 0(\text{mod } 2)$ if and only if $(2/p) = 1$. Equivalently, $\text{ind}_{\varpi,p^k}(2) \equiv 1(\text{mod } 2)$ if and only if $(2/p) = -1$.*

Proof. Let $r = \text{ind}_{\varpi,p^k}(2)$.

($\Rightarrow$) Suppose that $r \equiv 0(\text{mod } 2)$. So, $r = 2s$ for some integer s. Also, note that $\varpi^r \equiv 2(\text{mod } p^k)$, that is, $(\varpi^s)^2 \equiv \varpi^r \equiv 2(\text{mod } p^k)$. Clearly, by Theorem 2.5, $(2/p) = 1$.

($\Leftarrow$) Conversely, let $(2/p) = 1$ and assume to the contrary that $r \equiv 1(\text{mod } 2)$. Because $\varpi^r \equiv 2(\text{mod } p^k)$, we have $\varpi^r \equiv 2(\text{mod } p)$ and so, in accordance with Theorem 2.3,

$$(\varpi^{(p-1)/2})^r \equiv 2^{(p-1)/2} \equiv 1(\text{mod } p).$$

According to Theorem 2.2, ϖ is a primitive root of p. Consequently, $\text{ord}_p(\varpi) = p - 1$ and $\varpi^{p-1} \equiv 1(\text{mod } p)$. Thus,

$$(\varpi^{(p-1)/2})^2 \equiv \varpi^{p-1} \equiv 1(\text{mod } p).$$

It is immediate that $\varpi^{(p-1)/2} \equiv -1(\text{mod } p)$ since the congruence $\varpi^{(p-1)/2} \equiv 1(\text{mod } p)$ contradicts the fact that $\text{ord}_p(\varpi) = p - 1$. Since $r \equiv 1(\text{mod } 2)$, we have

$$(\varpi^{(p-1)/2})^r \equiv (-1)^r \equiv -1(\text{mod } p).$$

This is a contradiction. Therefore, $r \equiv 0(\text{mod } 2)$. □

Lemma 3.2. *Suppose that $DL^k_{\varpi,p^z}(f, p^z)$ is a discrete logarithm graph of order p^z where p is an odd prime and z is a positive integer. If $v \in V(DL^k_{\varpi,p^z}(f, p^z))$, then*

$$\deg(v) = \begin{cases} \frac{p^z(1-p^{-1})}{2} - 1 & \text{if } \text{ind}_{\varpi,p^z}(2f(v)) \equiv k(\text{mod } 2) \\ \frac{p^z(1-p^{-1})}{2} & \text{if } \text{ind}_{\varpi,p^z}(2f(v)) \equiv (1-k)(\text{mod } 2) \\ & \text{or } \gcd(2f(v), p^z) \neq 1. \end{cases} \tag{3.1}$$

Consequently, the size of $DL^k_{\varpi,p^z}(f, p^z)$ is $\frac{p^z(1-p^{-1})(p^z-1)}{4}$.

Proof. Let $k = 1$ and $n = p^z$. Assume that $u, v \in V(DL^1_{\varpi,n}(f, n))$ with $u \neq v$. Note that $uv \in E(DL^1_{\varpi,n}(f, n))$ if and only if $\text{ind}_{\varpi,n}(f(u) + f(v)) \equiv 1(\text{mod } 2)$. Define the set

$$Y_v = \bigcup_{\substack{u \in V(DL^1_{\varpi,n}(f,n)) \\ u \neq v}} \{\xi : f(u) + f(v) \equiv \xi(\text{mod } n),\ 0 \leq \xi < n\}.$$

If $f(u) \equiv \theta(\text{mod } n)$ and $2f(v) \equiv \sigma(\text{mod } n)$, $0 \leq \theta, \sigma < n$, then

$$Y_v = \{1 + \theta, 2 + \theta, \ldots, n - 1, 0, 1, \ldots, \theta\} - \{\sigma\} = \{0\} \cup \{1, 2, \ldots, n\} - \{\sigma\}.$$

Observe that $\text{ind}_{\varpi,n}(2f(v)) = \text{ind}_{\varpi,n}(\sigma)$ since $2f(v) \equiv \sigma(\text{mod } n)$. Using Remark 3, v is adjacent to $\frac{\phi(n)}{2} - 1$ vertices whenever $\text{ind}_{\varpi,n}(\sigma) \equiv 1(\text{mod } n)$; otherwise v is adjacent to $\frac{\phi(n)}{2}$ vertices. Since $\phi(p^z) = p^z(1 - p^{-1})$, (3.1) follows immediately.

For the size of $DL^1_{\varpi,n}(f, n)$, consider the set

$$\vartheta = \bigcup_{v \in V(DL^1_{\varpi,n}(f,n))} \{\xi : 2f(v) \equiv \xi(\text{mod } n),\ 0 < \xi < n\}.$$

Clearly,

$$\vartheta = \{2, 4, \ldots, n-1, 1, 3, \ldots, n-2\} = \{1, 2, \ldots, n-1\}.$$

Applying Remark 3, we obtain the following

$$\begin{aligned}\sum_{v \in V(DL^1_{\varpi,n}(f,n))} \deg(v) &= \sum_{\text{ind}_{\varpi,n}(2f(v)) \equiv 1 (\text{mod } 2)} \left(\frac{\phi(n)}{2} - 1\right) \\ &\quad + \sum_{\text{ind}_{\varpi,n}(2f(v)) \equiv 0 (\text{mod } 2)} \frac{\phi(n)}{2} + \sum_{\gcd(2f(v),n) \neq 1} \frac{\phi(n)}{2} \\ &= \frac{\phi(n)}{2}\left[\frac{\phi(n)}{2} - 1\right] + \frac{\phi(n)}{2}\left[\frac{\phi(n)}{2}\right] + [n - \phi(n)]\left[\frac{\phi(n)}{2}\right] \\ &= \frac{\phi(n)^2}{2} - \frac{\phi(n)}{2} + \frac{n\phi(n)}{2} - \frac{\phi(n)^2}{2} \\ &= \frac{\phi(n)(n-1)}{2}.\end{aligned}$$

Note that $\phi(p^z) = p^z(1 - p^{-1})$. By the Fundamental Theorem of Graph Theory, the size of $DL^1_{\varpi,p^z}(f, p^z)$ is $\frac{p^z(1-p^{-1})(p^z-1)}{4}$.

For $k = 0$, the proof is analogous. □

4. Main Results

Theorem 4.1. *The path graph P_{4k} is a pure logarithmic cordial graph to the base* 3 *modulo* 4 *for $k \geq 1$.*

Proof. Let $V(P_{4k}) = \{v_1, v_2, \ldots, v_{4k}\}$ and let $f : V(P_{4k}) \to \{1, 2, \ldots, 4k\}$ be a bijective function defined by

$$\begin{aligned} f(v_{1+4(j-1)}) &= 1 + 4(j-1) \\ f(v_{2+4(j-1)}) &= 4 + 4(j-1) \\ f(v_{3+4(j-1)}) &= 3 + 4(j-1) \\ f(v_{4+4(j-1)}) &= 2 + 4(j-1) \end{aligned}$$

for $j = 1, 2, \ldots, k$.

For the edges of P_{4k},

$$\begin{aligned} f(v_{1+4(j-1)}) + f(v_{2+4(j-1)}) &\equiv 1 (\text{mod } 4) \\ f(v_{2+4(j-1)}) + f(v_{3+4(j-1)}) &\equiv 3 (\text{mod } 4) \\ f(v_{3+4(j-1)}) + f(v_{4+4(j-1)}) &\equiv 1 (\text{mod } 4) \\ f(v_{4+4(j-1)}) + f(v_{1+4j}) &\equiv 3 (\text{mod } 4) \qquad (j \neq k) \end{aligned}$$

for $j = 1, 2, \ldots, k$. Since $\text{ind}_{3,4}(1) = 0$ and $\text{ind}_{3,4}(3) = 1$, we have

$$\begin{aligned} f^*_{3,4}(v_{1+4(j-1)}v_{2+4(j-1)}) = f^*_{3,4}(v_{3+4(j-1)}v_{4+4(j-1)}) &= 0 \\ f^*_{3,4}(v_{2+4(j-1)}v_{3+4(j-1)}) &= 1 \\ f^*_{3,4}(v_{4+4(j-1)}v_{1+4j}) &= 1 \qquad (j \neq k) \end{aligned}$$

for $j = 1, 2, \ldots, k$. Consequently,

$$e_{f^*_{3,4}}(0) = 2k \text{ and } e_{f^*_{3,4}}(1) = 2k - 1$$

and so $|e_{f^*_{3,4}}(0) - e_{f^*_{3,4}}(1)| = 1$. Therefore, P_{4k} is a pure logarithmic cordial graph to the base 3 modulo 4. □

Theorem 4.2. *The path graph P_{6k} is a pure logarithmic cordial graph to the base 5 modulo 6 for $k \geq 1$.*

Proof. Suppose that $V(P_{6k}) = \{v_1, v_2, \ldots, v_{6k}\}$ and define the bijective function $f : V(P_{6k}) \to \{1, 2, \ldots, 6k\}$ as

$$\begin{aligned} f(v_{1+6(j-1)}) &= 1 + 6(j-1) \\ f(v_{2+6(j-1)}) &= 4 + 6(j-1) \\ f(v_{3+6(j-1)}) &= 3 + 6(j-1) \\ f(v_{4+6(j-1)}) &= 2 + 6(j-1) \\ f(v_{5+6(j-1)}) &= 5 + 6(j-1) \\ f(v_{6+6(j-1)}) &= 6 + 6(j-1) \end{aligned}$$

for $j = 1, 2, \ldots, k$.

For the edges of P_{6k}, we have

$$\begin{aligned} f(v_{1+6(j-1)}) + f(v_{2+6(j-1)}) &\equiv 5 (\text{mod } 6) \\ f(v_{2+6(j-1)}) + f(v_{3+6(j-1)}) &\equiv 1 (\text{mod } 6) \\ f(v_{3+6(j-1)}) + f(v_{4+6(j-1)}) &\equiv 5 (\text{mod } 6) \\ f(v_{4+6(j-1)}) + f(v_{5+6(j-1)}) &\equiv 1 (\text{mod } 6) \\ f(v_{5+6(j-1)}) + f(v_{6+6(j-1)}) &\equiv 5 (\text{mod } 6) \\ f(v_{1+6(j-1)}) + f(v_{2+6j}) &\equiv 1 (\text{mod } 6) \qquad (j \neq k) \end{aligned}$$

for $j = 1, 2, \ldots, k$. Because $\text{ind}_{5,6}(1) = 0$ and $\text{ind}_{5,6}(5) = 1$, we obtain

$$\begin{aligned} f^*_{5,6}(v_{1+6(j-1)}v_{2+6(j-1)}) &= 1 \\ f^*_{5,6}(v_{3+6(j-1)}v_{4+6(j-1)}) = f^*_{5,6}(v_{5+6(j-1)}v_{6+6(j-1)}) &= 1 \\ f^*_{5,6}(v_{2+6(j-1)}v_{3+6(j-1)}) = f^*_{5,6}(v_{4+6(j-1)}v_{5+6(j-1)}) &= 0 \\ f^*_{5,6}(v_{1+6(j-1)}v_{2+6j}) &= 0 \qquad (j \neq k) \end{aligned}$$

for $j = 1, 2, \ldots, k$. This means

$$e_{f^*_{5,6}}(0) = 3k \text{ and } e_{f^*_{5,6}}(1) = 3k - 1.$$

So, $|e_{f^*_{5,6}}(0) - e_{f^*_{5,6}}(1)| = 1$. Hence, P_{6k} is a pure logarithmic cordial graph to the base 5 modulo 6. □

Theorem 4.3. *Let G be a connected graph of order n and size $m\phi(\eta)$. If there is a bijective function $f : V(G) \to \{1, 2, \ldots, n\}$ with the following conditions:*

i.
$$\{h(ab) : ab \in E(G)\} = \bigcup_{i=0}^{m-1} \{j + i\eta : j \in \varphi(\eta)\}$$
where $\varphi(\eta) = \{\xi : \gcd(\xi, \eta) = 1,\ 0 < \xi < \eta\}$;

ii. $h(uv) \neq h(ab)$ *for every* $uv \neq ab$,

where $h(uv) = f(u) + f(v)$ for $uv \in E(G)$, then G is a pure logarithmic cordial graph to the base ϖ modulo η for any primitive root ϖ.

Proof. It is obvious that $\gcd(f(a)+f(b),\eta)=1$ for all $ab\in E(G)$. Let

$$\vartheta_t=\{j+t\eta : j\in\varphi(\eta)\}$$

for $t=0,1,\ldots,m-1$. Thus,

$$\{h(ab) : ab\in E(G)\}=\bigcup_{t=0}^{m-1}\vartheta_t.$$

Suppose that $\vartheta_t(\text{mod }\eta)=\{\xi : \xi\equiv\rho(\text{mod }\eta),\ \rho\in\vartheta_t,\ 0<\xi<\eta\}$. Hence,

$$\vartheta_t(\text{mod }\eta)=\varphi(\eta)$$

for $t=0,1,\ldots,m-1$. The condition (ii) ensures that each edge label of G is unique under h. Therefore, by Remark 3,

$$e_{f_{\varpi,\eta}}(0)=e_{f_{\varpi,\eta}}(1)=m\left(\frac{\phi(\eta)}{2}\right).$$

Hence, $|e_{f_{\varpi,\eta}}(0)-e_{f_{\varpi,\eta}}(1)|=0$. Consequently, G is a pure logarithmic cordial graph to the base ϖ modulo η for any primitive root ϖ. □

In the following results, let p denote an odd prime. In addition, suppose that $\varphi(\eta)=\{\xi : \gcd(\xi,\eta)=1,\ 0<\xi<\eta\}$. Note that $\phi(2p)=p-1$ and $\phi(p^k)=p^k(1-p^{-1})$ for any positive integer k. It is clear that

$$\varphi(2p)=\{1,3,\ldots,p-2\}\cup\{p+2,p+4\ldots,2p-1\}$$

and

$$\varphi(p^k)=\{1,2,\ldots,p^k\}-\{p,2p,\ldots,p^k\}.$$

Furthermore, we denote ϖ as a primitive root of p, $2p$, or p^k.

Theorem 4.4. *The following graphs are logarithmic cordial graphs to the base ϖ modulo p.*

i. Path graph P_p
ii. Cycle graph C_p
iii. Star graph S_p
iv. Bistar graph $B_{p,p}$
v. Fan graph F_{p+1}

In particular, (i) and (iii) are pure logarithmic cordial graphs to the base ϖ modulo p.

For the definitions of cycle, star, bistar, and fan graphs, refer to [2].

Proof. For each of the graphs listed, a corresponding bijective function is constructed in [2]. Furthermore, Theorem 2.9 of [2], formulated in terms of the Legendre symbol, is directly analogous to Remark 3 with $\eta=p$, noting that $\phi(p)=p-1$. It follows that the proofs in [2] extend directly with the role of Theorem 2.9 in [2] is assumed here by Remark 3. Accordingly, each of the listed graphs is a logarithmic cordial graph to the base ϖ modulo p. Also, the bijective function constructed for S_p satisfies Theorem 4.3 (i) and (ii) and so S_p is a pure logarithmic cordial graph to the base ϖ modulo p. □

Theorem 4.5. *The path graph P_{2p-1} is a pure logarithmic cordial graph to the base ϖ modulo $2p$.*

Proof. Let $V(P_{2p-1}) = \{v_1, v_2, \ldots, v_{2p-1}\}$ and let $f : V(P_{2p-1}) \to \{1, 2, \ldots, 2p-1\}$ be a bijective function defined by

$$f(v_i) = \begin{cases} i & \text{for } i = 1, 2, \ldots, \frac{p-1}{2}, p + \frac{p+1}{2}, p + \frac{p+3}{2}, \ldots, 2p-1 \\ 2p - i & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p + \frac{p-1}{2}. \end{cases}$$

For $v_i v_{i+1} \in E(P_{2p-1})$,

$$f(v_i) + f(v_{i+1}) \equiv \begin{cases} (2i+1)(\text{mod } 2p) & \text{for } i = 1, 2, \ldots, \frac{p-3}{2} \\ (2p-1)(\text{mod } 2p) & \text{for } i = \frac{p-1}{2} \\ (2p-2i-1)(\text{mod } 2p) & \text{for } i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1 \\ (4p-2i-1)(\text{mod } 2p) & \text{for } i = p, p+1, \ldots, p + \frac{p-3}{2} \\ 1(\text{mod } 2p) & \text{for } i = p + \frac{p-1}{2} \\ (2i+1-2p)(\text{mod } 2p) & \text{for } i = p + \frac{p+1}{2}, p + \frac{p+3}{2}, \ldots, 2p-2. \end{cases}$$

With this, let

$$\vartheta_1 = \bigcup_{i \in \Gamma} \{\xi : f(v_i) + f(v_{i+1}) \equiv \xi (\text{mod } 2p),\ 0 \leq \xi < 2p\}$$

where $\Gamma = \left\{1, 2, \ldots, \frac{p-1}{2}\right\} \cup \left\{p + \frac{p-1}{2}, p + \frac{p+1}{2}, \ldots, 2p-2\right\}$ and

$$\vartheta_2 = \bigcup_{i = \frac{p+1}{2}}^{p + \frac{p-3}{2}} \{\xi : f(v_i) + f(v_{i+1}) \equiv \xi (\text{mod } 2p),\ 0 \leq \xi < 2p\}.$$

Thus,

$$\begin{aligned} \vartheta_1 &= \left\{2i+1 : i = 1, 2, \ldots, \frac{p-3}{2}\right\} \cup \{2p-1\} \cup \{1\} \\ &\quad \cup \left\{2i+1-2p : i = p + \frac{p+1}{2}, p + \frac{p+3}{2}, \ldots, 2p-2\right\} \\ &= \{3, 5, \ldots, p-2\} \cup \{1\} \cup \{2p-1\} \cup \{p+2, p+4, \ldots, 2p-3\} \\ &= \{1, 3, \ldots, p-2\} \cup \{p+2, p+4, \ldots, 2p-1\}, \text{ and} \\ \vartheta_2 &= \left\{2p-2i-1 : i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p-1\right\} \\ &\quad \cup \left\{4p-2i-1 : i = p, p+1, \ldots, p + \frac{p-3}{2}\right\} \\ &= \{p-2, p-4, \ldots, 1\} \cup \{2p-1, 2p-3, \ldots, p+2\} \\ &= \{1, 3, \ldots, p-2\} \cup \{p+2, p+4, \ldots, 2p-1\}. \end{aligned}$$

Therefore, $\vartheta_1 = \vartheta_2 = \varphi(2p)$ and by Remark 3,

$$e_{f^*_{\varpi,2p}}(0) = e_{f^*_{\varpi,2p}}(1) = 2(p-1).$$

Consequently, $|e_{f^*_{\varpi,2p}}(0) - e_{f^*_{\varpi,2p}}(1)| = 0$ and it follows that P_{2p-1} is a pure logarithmic cordial graph to the base ϖ modulo $2p$. □

Theorem 4.6. *The path graph P_{p+1} is a logarithmic cordial graph to the base ϖ modulo $2p$.*

Proof. Let $V(P_{p+1}) = \{v_1, v_2, \ldots, v_{p+1}\}$ and let $f : V(P_{p+1}) \to \{1, 2, \ldots, p+1\}$ be a bijective function defined by $f(v_i) = i$ for $i = 1, 2, \ldots, p+1$.

For $v_iv_{i+1} \in E(P_{p+1})$, we have

$$f(v_i) + f(v_{i+1}) \equiv (2i+1)(\text{mod } 2p)$$

for $i = 1, 2, \ldots, p-1$ and $f(v_pv_{p+1}) \equiv 1(\text{mod } 2p)$. Define the set

$$\vartheta = \bigcup_{\substack{i=1 \\ i \neq \frac{p-1}{2}}}^{p} \{\xi : f(v_i) + f(v_{i+1}) \equiv \xi(\text{mod } 2p),\ 0 \leq \xi < 2p\}$$

and so

$$\begin{aligned}\vartheta &= \left\{2i+1 : i = 1, 2, \ldots, \frac{p-3}{2}\right\} \cup \left\{2i+1 : i = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p\right\} \cup \{1\} \\ &= \{3, 5, \ldots, p-2\} \cup \{p+2, p+4, \ldots, 2p-1\} \cup \{1\} \\ &= \{1, 3, \ldots, p-2\} \cup \{p+2, p+4, \ldots, 2p-1\}.\end{aligned}$$

Thus, $\vartheta = \varphi(2p)$. Also, since $\gcd(p, 2p) \neq 1$, $f^*_{\varpi,2p}(v_{(p-1)/2}v_{(p+1)/2}) = 0$. Using Remark 3 together with the fact that $f^*_{\varpi,2p}(v_{(p-1)/2}v_{(p+1)/2}) = 0$, we obtain

$$e_{f^*_{\varpi,2p}}(0) = \frac{p-1}{2} + 1 \text{ and } e_{f^*_{\varpi,2p}}(1) = \frac{p-1}{2}.$$

So, $|e_{f^*_{\varpi,2p}}(0) - e_{f^*_{\varpi,2p}}(1)| = 1$ and it follows that P_{p+1} is a logarithmic cordial graph to the base ϖ modulo $2p$. □

Theorem 4.7. *The complete graph K_{p^z}, z is a positive integer, is a logarithmic cordial graph to the base ϖ modulo p^z if and only if $\varpi = 2$, $z = 1$, and $p = 3$.*

Proof. Let $f : V(K_n) \to \{1, 2, \ldots, p^z\}$ be an arbitrary bijective function. Suppose that $DL^k_{\varpi,p^z}(f, p^z)$ is a discrete logarithm graph of order p^z with $V(DL^k_{\varpi,p^z}(f, p^z)) = V(K_{p^z})$. Obviously, $DL^0_{\varpi,p^z}(f, p^z)$ and $DL^1_{\varpi,p^z}(f, p^z)$ are subgraphs of K_{p^z}. Let G be a subgraph of K_{p^z} with vertex set $V(G) = V(K_{p^z})$ and edge set

$$E(G) = E(K_{p^z}) - [E(DL^0_{\varpi,p^z}(f, p^z)) \cup E(DL^1_{\varpi,p^z}(f, p^z))].$$

It is clear that for any $ab \in E(G)$, we have $\gcd(f(a) + f(b), p^z) \neq 1$ and so any edge of G has label 0 under f^*_{ϖ,p^z}. Also, it is clear that all edges of $E(DL^0_{\varpi,p^z}(f, p^z))$ and $E(DL^1_{\varpi,p^z}(f, p^z))$ are labeled by 0 and 1, respectively, under f^*_{ϖ,p^z}. Hence,

$$\begin{aligned}e_{f^*_{\varpi,p^z}}(0) &= |E(DL^0_{\varpi,p^z}(f, p^z))| + |E(G)| \\ &= |E(DL^0_{\varpi,p^z}(f, p^z))| + |E(K_{p^z})| \\ &\quad - [|E(DL^0_{\varpi,p^z}(f, p^z))| + |E(DL^1_{\varpi,p^z}(f, p^z))|] \\ &= |E(K_{p^z})| - |E(DL^1_{\varpi,p^z}(f, p^z))|\end{aligned}$$

and $e_{f^*_{\varpi,p^z}}(1) = |E(DL^1_{\varpi,p^z}(f, p^z))|$. Thus, by Lemma 3.2,

$$\begin{aligned}e_{f^*_{\varpi,p^z}}(0) - e_{f^*_{\varpi,p^z}}(1) &= |E(K_{p^z})| - 2|E(DL^1_{\varpi,p^z}(f, p^z))| \\ &= \frac{p^z(p^z-1)}{2} - 2\left[\frac{p^z(1-p^{-1})(p^z-1)}{4}\right] \\ &= \frac{p^z(p^z-1)(1-1+p^{-1})}{2} \\ &= \frac{p^{z-1}(p^z-1)}{2}\end{aligned}$$

for all bijective function f.

If K_{p^z} is a logarithmic cordial graph, then

$$\begin{aligned} |e_{f^*_{\varpi,p^z}}(0) - e_{f^*_{\varpi,p^z}}(1)| &\leq 1 \\ \frac{p^{z-1}(p^z-1)}{2} &\leq 1 \\ p^{z-1}(p^z-1) &\leq 2. \end{aligned}$$

If $z = 1$, then $p \leq 3$ and so $p = 3$. Also, if $z \geq 2$ and $p \geq 3$, then

$$p^{z-1}(p^z - 1) \geq 3(3^2 - 1) = 24.$$

This is not possible.

Since the only primitive root of 3 is 2, $\varpi = 2$. Therefore, $z = 1$, $p = 3$, and $\varpi = 2$.

The converse is trivial. □

Theorem 4.8. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_{p^k}\}$ and edge set*

$$E(G) = \{v_i v_{p^k} : i = 1, 2, \ldots, p^k - 1\} \cup \{v_{mp} v_{p^k - mp+2} : m = 1, 2, \ldots, p^{k-1}\}.$$

If $p \equiv \pm 3 (\text{mod } 8)$, then G is a logarithmic cordial graph to the base ϖ modulo p^k.

Proof. Since $p \equiv \pm 3(\text{mod } 8)$, by Theorem 2.4 and Lemma 3.1, $\text{ind}_{\varpi,p^k}(2) \equiv 1(\text{mod } 2)$. Let $f : V(G) \to \{1, 2, \ldots, p^k\}$ be a bijective function defined by $f(v_i) = i$ for $i = 1, 2, \ldots, p^k$.

For the edges of the form $v_i v_{p^k}$, we have

$$f(v_i) + f(v_{p^k}) \equiv i (\text{mod } p^k)$$

for $i = 1, 2, \ldots, p^k - 1$. Clearly, for $i = jp$, $f^*_{\varpi,p^k}(v_{jp} v_{p^k}) = 0$ since $\gcd(jp, p^k) \neq 1$ for $j = 1, 2, \ldots, p^{k-1}$. In addition, let

$$\vartheta = \bigcup_{\substack{i=1 \\ \gcd(i,p^k)=1}}^{p^k-1} \{\xi : f(v_i) + f(v_{p^k}) \equiv \xi (\text{mod } p^k),\ 0 \leq \xi < p^k\}.$$

Thus,

$$\begin{aligned} \vartheta &= \left\{i : i = 1, 2, \ldots, p^k - 1 \text{ and } \gcd(i, p^k) = 1\right\} \\ &= \{1, 2, \ldots, p^k\} - \{p, 2p, \ldots, p^k\} \end{aligned}$$

and evidently, $\vartheta = \varphi(p^k)$.

For the edges of the form $v_{mp} v_{p^k - mp+2}$,

$$f(v_{mp}) + f(v_{p^k - mp+2}) \equiv 2 (\text{mod } p^k)$$

which means $f^*_{\varpi,p^k}(v_{mp} v_{p^k - mp+2}) = 1$ for $m = 1, 2, \ldots, p^{k-1}$.

Combining all arguments above and apply Remark 3, we conclude that

$$e_{f^*_{\varpi,p^k}}(0) = e_{f^*_{\varpi,p^k}}(1) = \frac{p^k(1 - p^{-1})}{2} + p^{k-1}.$$

It follows that $|e_{f^*_{\varpi,p^k}}(0) - e_{f^*_{\varpi,p^k}}(1)| = 0$. Therefore, G is a pure logarithmic cordial graph to the base ϖ modulo p^k. □

Theorem 4.9. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_{p^2+p-1}\}$ and edge set*

$$E(G) = \bigcup_{m=1}^{p} \bigcup_{i=1}^{p-1} \{v_{(m-1)p+i}v_{mp}, v_{mp}v_{p^2+i}\}.$$

Then G is a pure logarithmic cordial graph to the base ϖ modulo p^2.

Proof. Define the bijective function $f : V(G) \to \{1, 2, \ldots, p^2\}$ as $f(v_i) = i$ for $i = 1, 2, \ldots, p^2 + p - 1$. For the edges of the form $v_{(m-1)p+i}v_{mp}$, we have

$$f(v_{(m-1)p+i}) + f(v_{mp}) \equiv \begin{cases} (2mp + i - p)(\text{mod } p^2) & \text{for } m = 1, 2, \ldots, \frac{p-1}{2} \\ (2mp + i - p - p^2)(\text{mod } p^2) & \text{for } m = \frac{p+1}{2}, \frac{p+3}{2}, \ldots, p \end{cases}$$

for $i = 1, 2, \ldots, p-1$. Set

$$\vartheta_1 = \bigcup_{i=1}^{p-1} \bigcup_{m=1}^{p} \{\xi : f(v_{(m-1)p+i}) + f(v_{mp}) \equiv \xi(\text{mod } p^2),\ 0 \le \xi < p^2\}.$$

So,

$$\begin{aligned} \vartheta_1 &= \left[\bigcup_{m=1}^{\frac{p-1}{2}} \{2mp + i - p : i = 1, 2, \ldots, p-1\}\right] \\ &\quad \cup \left[\bigcup_{m=\frac{p+1}{2}}^{p} \{2mp + i - p - p^2 : i = 1, 2, \ldots, p-1\}\right] \\ &= \{p+1, p+2, \ldots, 2p-1\} \cup \{3p+1, 3p+2, \ldots, 4p-1\} \cup \cdots \\ &\quad \cup \{(p-2)p+1, (p-2)p+2, \ldots, (p-1)p-1\} \\ &\quad \cup \{1, 2, \ldots, p-1\} \cup \{2p+1, 2p+2, \ldots, 3p-1\} \cup \cdots \\ &\quad \cup \{(p-1)p+1, (p-1)p+2, \ldots, p^2-1\} \\ &= \{1, 2, \ldots, p^2\} - \{p, 2p, \ldots, p^2\}. \end{aligned}$$

Clearly, $\vartheta_1 = \varphi(p^2)$.

For the edges of the form $v_{mp}v_{p^2+i}$, observe that

$$f(v_{mp}) + f(v_{p^2+i}) \equiv \begin{cases} (mp + i)(\text{mod } p^2) & \text{for } m = 1, 2, \ldots, p-1 \\ (mp + i - p^2)(\text{mod } p^2) & \text{for } m = p \end{cases}$$

for $i = 1, 2, \ldots, p-1$. Assume that

$$\vartheta_2 = \bigcup_{i=1}^{p-1} \bigcup_{m=1}^{p} \{\xi : f(v_{mp}) + f(v_{p^2+i}) \equiv \xi(\text{mod } p^2),\ 0 \le \xi < p^2\}.$$

Hence,

$$\begin{aligned} \vartheta_2 &= \left[\bigcup_{m=1}^{p-1} \{mp + i : i = 1, 2, \ldots, p-1\}\right] \cup \{i : i = 1, 2, \ldots, p-1\} \\ &= \{1, 2, \ldots, p-1\} \cup \{p+1, p+2, \ldots, 2p-1\} \cup \cdots \\ &\quad \cup \{(p-1)p+1, (p-1)p+2, \ldots, p^2-1\} \\ &= \{1, 2, \ldots, p^2\} - \{p, 2p, \ldots, p^2\} \end{aligned}$$

and it follows that $\vartheta_2 = \varphi(p^2)$.

By Remark 3,

$$e_{f^*_{\varpi,p^2}}(0) = e_{f^*_{\varpi,p^2}}(1) = 2\left[\frac{p^2(1-p^{-1})}{2}\right] = p^2(1-p^{-1}).$$

Thus, $|e_{f^*_{\varpi,p^2}}(0) - e_{f^*_{\varpi,p^2}}(1)| = 0$ and it follows that G is a pure logarithmic cordial graph to the base ϖ modulo p^2. □

5. Conclusion

This paper has extended the study of cordial labeling by introducing logarithmic cordial labeling and examining its behavior on several fundamental graph classes, including path graphs, cycle graphs, and complete graphs. By incorporating discrete logarithms into the labeling process, this work highlights a novel intersection among number theory, cryptography, and graph labeling theory, thereby enriching the existing family of cordial labelings. Beyond the results obtained for basic graph classes, this framework opens several promising directions for future research. One natural extension is to investigate the behavior of logarithmic cordial labeling under graph operations such as graph union, join, Cartesian product, corona product, lexicographic product, tensor product, and strong product. Understanding whether logarithmic cordiality is preserved or transformed under these operations could lead to broader structural characterizations. Additionally, further study may focus on extending logarithmic cordial labeling to other graph families, including trees, bipartite graphs, planar graphs, and Eulerian graphs, as well as exploring necessary and sufficient conditions for the existence of such labelings. Another potential avenue is the interaction between logarithmic cordial labeling and other labeling schemes, such as Legendre cordial labeling, Legendre product cordial labeling, and Euler cordial labeling, which may yield generalized labeling concepts.

Acknowledgement

The author gratefully acknowledges the support of the Department of Science and Technology – Science Education Institute (DOST-SEI) under the STRAND Scholarship Program.

Conflict of Interest

The author declares that there is no conflict of interest.

References

[1] Andoyo, J. D. (2025). *Euler cordial graphs*, Asian Journal of Pure and Applied Mathematics, 7(1), 728–742. DOI:10.56557/ajpam/2025/v7i1237

[2] Andoyo, J. D., & Rulete, R. F. (2025). *On legendre cordial labeling of some graphs*, European Journal of Mathematics and Statistics, 6(5), 1–7. DOI:10.24018/ejmath.2025.6.5.409

[3] Asha Rani, A., Thirusangu K., & Balamurugan, B. J. (2022). *Möbius cordial labeling of graphs*. Advances in Mathematical Modeling and Scientific Computing, Trends in Mathematics, 2, 865–877.

[4] Burton, D. M. (2011). *Elementary number theory* (7th ed). McGraw-Hill.

[5] Cahit, I. (1987). Cordial graphs: a weaker version of graceful and harmonious graphs. Ars Combinatoria, 23, 201–207.

[6] Gallian, J. A. (2022). *A dynamic survey of graph labeling* (25th ed), The Electronic Journal of Mathematics.

[7] Gudin, M. V. S., & Pedrano, A. C. (2025). *Tribonacci cordial labeling of some snake graphs*, Annals of Communication in Mathematics, 8(3), 393–405. DOI:10.62072/acm.2025.080306

[8] NourEldeen, N. M., Jenisha, J., Jeya Daisy, K., Jeyanthi, P., & Abdel-Aal, M. E. (2025). *Edge k-product cordial labeling of graphs*, European Journal of Pure and Applied Mathematics, 18(2), 1–21. DOI:10.29020/nybg.ejpam.v18i2.5887

[9] Prudencio, J. C. C., & Rulete, R. F. (2025). *On legendre product cordial labeling of some special graphs*, European Journal of Mathematics and Statistics, 6(5), 1–12. DOI:10.24018/ejmath.2025.6.6.418

[10] Rosen, K. H. (2011). Elementary *Number Theory and its Application* (6th ed). Addison Wesley.

JASON D. ANDOYO

MATHEMATICS AND STATISTICS DEPARTMENT, UNIVERSITY OF SOUTHEASTERN PHILIPPINES, DAVAO CITY, 8000, PHILIPPINES.

ORCID: 0009-0002-0540-9895

Email address: jasonandoyo8000@gmail.com