ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 4 (2025), 431-441
DOI: 10.62072/acm.2025.080401
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

# ON FIVE ORIGINAL INTEGRAL INEQUALITIES OF THE HARDY-HILBERT TYPE

CHRISTOPHE CHESNEAU

ABSTRACT. This article is devoted to five integral inequalities of the Hardy-Hilbert type. Each one has its own unique features. In particular, we present new trigonometric Hardy-Hilbert-type integral inequalities that yield precise upper bounds involving weighted integral norms of the main functions. Complete and rigorous proofs are provided.

## 1. INTRODUCTION

The study of inequalities has always played a central role in mathematical analysis. It has been both a subject in its own right and a powerful tool in areas such as harmonic analysis, functional analysis and differential equations. One of the most notable results in this field is the Hardy-Hilbert integral inequality. It is renowned for its elegance, versatility, and wide range of applications. The inequality, in its most celebrated form, is described below. Let $p > 1$, $q = p/(p-1)$ and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}. \tag{1.1}$$

This inequality is known to be particularly sharp. It can be traced back to the pioneering work of G.H. Hardy and D. Hilbert. It is a natural extension of the classical Hilbert double series inequality to the framework of integral operators. Further details may be found in the monographs [5, 13, 14], the comprehensive survey [1], and in recent contributions such as [6, 7, 10, 11, 12, 9, 8, 2, 3, 4].

2020 *Mathematics Subject Classification.* 26D15.
*Key words and phrases.* Integral inequalities; Hardy-Hilbert-type integral inequalities; Trigonometric functions.
Received: September 27, 2025. Accepted: December 20, 2025. Published: December 31, 2025.

This article advances this line of research by establishing five new integral inequalities of the Hardy-Hilbert type. Adopting standard notations (to be specified later), these inequalities are associated with the integrals described below.

**First double integral:**

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}f(x)g(y)dxdy.$$

**Second double integral:**

$$\int_0^{\pi}\int_0^{\pi}\frac{1}{1+\cos(x)\cos(y)}f(x)g(y)dxdy.$$

**Third double integral:**

$$\int_0^{+\infty}\int_0^{+\infty}\frac{|\sin(x+y)|}{x+y}f(x)g(y)dxdy.$$

**Fourth double integral:**

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^3}F(x)G(y)dxdy.$$

**First triple integral:**

$$\int_0^{1}\int_0^{1}\int_0^{1}\frac{1}{1-xyz}f(x)g(y)h(z)dxdydz.$$

Each inequality preserves the spirit of the classical Hardy-Hilbert integral inequality while introducing new refinements. In particular, the trigonometric variants derived from the first three double integrals expand the theory by incorporating sine and cosine functions. These lead to original upper bounds involving weighted integral norms with trigonometric weight functions. The variant associated with the fourth integral innovates through its use of primitives of the main functions. The three-dimensional variant derived from the first triple integral introduces an original integrand with a singularity at $xyz = 1$. Detailed proofs are included for clarity and accessibility.

The rest of the article is organized as follows: The main results are presented in Section 2, followed by concluding remarks in Section 3.

## 2. Main results

The main results are presented as theorems, each with its own subsection containing the statement and proof.

### 2.1. First theorem.

The first integral inequality of the Hardy-Hilbert type, presented in the theorem below, is of a trigonometric nature and involves a sharp constant equal to $\pi$.

**Theorem 2.1.** *Let $p > 1$, $q = p/(p-1)$ and $f, g : [0, \pi/2] \to [0, +\infty)$ be two functions such that*

$$\int_0^{\pi/2}\frac{1}{\sin(2x)}f^p(x)dx < +\infty, \quad \int_0^{\pi/2}\frac{1}{\sin(2y)}g^q(y)dy < +\infty.$$

*Then we have*

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}f(x)g(y)dxdy$$
$$\leq \pi\left(\int_0^{\pi/2}\frac{1}{\sin(2x)}f^p(x)dx\right)^{1/p}\left(\int_0^{\pi/2}\frac{1}{\sin(2y)}g^q(y)dy\right)^{1/q}.$$

*Proof.* It is clear that, for any $x, y \in (0, \pi/2)$,

$$\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y) \geq 0.$$

A suitable decomposition of the integrand and the Hölder integral inequality give

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}f(x)g(y)dxdy\\
&=\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{(\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y))^{1/p}}f(x)\\
&\times\frac{1}{(\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y))^{1/q}}g(y)dxdy\\
&\leq\left(\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}f^p(x)dxdy\right)^{1/p}\\
&\times\left(\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}g^q(y)dxdy\right)^{1/q}.
\end{aligned}\tag{2.1}$$

We recall the following integral result:

$$\int_0^{\pi/2}\frac{1}{\cos^2(t)+\alpha\sin^2(t)}dt=\left[\frac{1}{\sqrt{\alpha}}\arctan\left[\sqrt{\alpha}\tan(t)\right]\right]_{t\to 0}^{t\to\pi/2}=\frac{\pi}{2\sqrt{\alpha}}.$$

Using the Fubini-Tonelli integral theorem, the above integral result under a suitable configuration and the standard trigonometric formula $\sin(2a) = 2\cos(a)\sin(a)$ for any $a \in \mathbb{R}$, we get

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}f^p(x)dxdy\\
&=\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}dy\right)f^p(x)dx\\
&=\int_0^{\pi/2}\frac{1}{\cos^2(x)}\left(\int_0^{\pi/2}\frac{1}{\cos^2(y)+\tan^2(x)\sin^2(y)}dy\right)f^p(x)dx\\
&=\int_0^{\pi/2}\frac{1}{\cos^2(x)}\times\frac{\pi}{2\sqrt{\tan^2(x)}}f^p(x)dx\\
&=\pi\int_0^{\pi/2}\frac{1}{2\sin(x)\cos(x)}f^p(x)dx\\
&=\pi\int_0^{\pi/2}\frac{1}{\sin(2x)}f^p(x)dx.
\end{aligned}\tag{2.2}$$

Similarly, we have

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)} g^q(y)dxdy \\
&= \int_0^{\pi/2}\left(\int_0^{\pi/2} \frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)}dx\right) g^q(y)dy \\
&= \int_0^{\pi/2}\frac{1}{\cos^2(y)}\left(\int_0^{\pi/2} \frac{1}{\cos^2(x)+\tan^2(y)\sin^2(x)}dx\right) g^q(y)dy \\
&= \int_0^{\pi/2}\frac{1}{\cos^2(y)}\times\frac{\pi}{2\sqrt{\tan^2(y)}} g^q(y)dy \\
&= \pi\int_0^{\pi/2}\frac{1}{2\sin(y)\cos(y)} g^q(y)dy \\
&= \pi\int_0^{\pi/2}\frac{1}{\sin(2y)} g^q(y)dy. \qquad (2.3)
\end{aligned}$$

Joining Equations (2.1), (2.2) and (2.3), we derive

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)} f(x)g(y)dxdy \\
&\leq \left(\pi\int_0^{\pi/2}\frac{1}{\sin(2x)}f^p(x)dx\right)^{1/p}\left(\pi\int_0^{\pi/2}\frac{1}{\sin(2y)}g^q(y)dy\right)^{1/q} \\
&= \pi\left(\int_0^{\pi/2}\frac{1}{\sin(2x)}f^p(x)dx\right)^{1/p}\left(\int_0^{\pi/2}\frac{1}{\sin(2y)}g^q(y)dy\right)^{1/q}.
\end{aligned}$$

The desired inequality is obtained. □

The result seems to be a new contribution to the Hardy-Hilbert-type integral inequalities, characterized by its involvement of trigonometric functions

Note that the ratio term $\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y)$ can be expressed in several ways. In particular, we have

$$\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y) = 1-\cos^2(x)-\cos^2(y)+2\cos^2(x)\cos^2(y),$$
$$\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y) = 1-\sin^2(x)-\sin^2(y)+2\sin^2(x)\sin^2(y)$$

and

$$\cos^2(x)\cos^2(y)+\sin^2(x)\sin^2(y) = \frac{1}{2}\left(1+\cos(2x)\cos(2y)\right).$$

This last concise expression has inspired the second theorem formulated below.

### 2.2. **Second theorem.**

The second integral inequality of the Hardy-Hilbert type, presented in the theorem below, is of a trigonometric nature and involves a sharp constant equal to $\pi$.

**Theorem 2.2.** *Let $p>1$, $q=p/(p-1)$ and $f,g:[0,\pi]\to[0,+\infty)$ be two functions such that*

$$\int_0^{\pi}\frac{1}{\sin(x)}f^p(x)dx<+\infty,\quad \int_0^{\pi}\frac{1}{\sin(y)}g^q(y)dy<+\infty.$$

*Then we have*

$$\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} f(x)g(y)dxdy \leq \pi \left(\int_0^\pi \frac{1}{\sin(x)} f^p(x)dx\right)^{1/p} \left(\int_0^\pi \frac{1}{\sin(y)} g^q(y)dy\right)^{1/q}.$$

*Proof.* We obviously have, for any $x, y \in (0,\pi)$,

$$1+\cos(x)\cos(y) \geq 0.$$

A suitable decomposition of the integrand and the Hölder integral inequality give

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} f(x)g(y)dxdy \\
&= \int_0^\pi \int_0^\pi \frac{1}{(1+\cos(x)\cos(y))^{1/p}} f(x) \times \frac{1}{(1+\cos(x)\cos(y))^{1/q}} g(y)dxdy \\
&\leq \left(\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} f^p(x)dxdy\right)^{1/p} \\
&\times \left(\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} g^q(y)dxdy\right)^{1/q}. \qquad (2.4)
\end{aligned}$$

We recall the following integral result:

$$\int_0^\pi \frac{1}{1+\cos(\alpha)\cos(t)} dt = \left[\frac{2}{\sin(\alpha)} \arctan\left[\tan\left(\frac{t}{2}\right)\tan\left(\frac{\alpha}{2}\right)\right]\right]_{t\to 0}^{t\to\pi} = \frac{\pi}{\sin(\alpha)}.$$

Using the Fubini-Tonelli integral theorem and the above integral result under a suitable configuration, we get

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} f^p(x)dxdy = \int_0^\pi \left(\int_0^\pi \frac{1}{1+\cos(x)\cos(y)} dy\right) f^p(x)dx \\
&= \int_0^\pi \frac{\pi}{\sin(x)} f^p(x)dx = \pi \int_0^\pi \frac{1}{\sin(x)} f^p(x)dx. \qquad (2.5)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} g^q(y)dxdy = \int_0^\pi \left(\int_0^\pi \frac{1}{1+\cos(x)\cos(y)} dx\right) g^q(y)dy \\
&= \int_0^\pi \frac{\pi}{\sin(y)} g^q(y)dy = \pi \int_0^\pi \frac{1}{\sin(y)} g^q(y)dy. \qquad (2.6)
\end{aligned}$$

Joining Equations (2.4), (2.5) and (2.6), we derive

$$\begin{aligned}
&\int_0^\pi \int_0^\pi \frac{1}{1+\cos(x)\cos(y)} f(x)g(y)dxdy \\
&\leq \left(\pi \int_0^\pi \frac{1}{\sin(x)} f^p(x)dx\right)^{1/p} \left(\pi \int_0^\pi \frac{1}{\sin(y)} g^q(y)dy\right)^{1/q} \\
&= \pi \left(\int_0^\pi \frac{1}{\sin(x)} f^p(x)dx\right)^{1/p} \left(\int_0^\pi \frac{1}{\sin(y)} g^q(y)dy\right)^{1/q}.
\end{aligned}$$

This concludes the proof. □

This appears to be a new Hardy-Hilbert-type integral inequality in the literature, which combines simplicity with the use of trigonometric functions. The associated weighted integral norms are also original. These, when combined, may lead to further extensions and applications in the study of related inequalities. We also note that alternative representations of the trigonometric integrand are possible, highlighting the versatility of the inequality.

2.3. **Third theorem.** The third integral inequality of the Hardy-Hilbert type, presented in the theorem below, is of a trigonometric nature and involves an original upper bound.

**Theorem 2.3.** *Let* $p > 1$*,* $q = p/(p-1)$ *and* $f, g : [0, +\infty) \to [0, +\infty)$ *be two functions such that*

$$\int_0^{+\infty} f^p(x)|\cos(x)|^p dx < +\infty, \quad \int_0^{+\infty} f^p(x)|\sin(x)|^p dx < +\infty,$$
$$\int_0^{+\infty} g^q(y)|\cos(y)|^q dy < +\infty, \quad \int_0^{+\infty} g^q(y)|\sin(y)|^q dy < +\infty.$$

*Then we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{|\sin(x+y)|}{x+y} f(x)g(y)dxdy \\
&\leq \frac{\pi}{\sin(\pi/p)}\left[\left(\int_0^{+\infty} f^p(x)|\cos(x)|^p dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)|\sin(y)|^q dy\right)^{1/q}\right. \\
&\left. + \left(\int_0^{+\infty} f^p(x)|\sin(x)|^p dx\right)^{1/p}\left(\int_0^{+\infty} g^q(y)|\cos(y)|^q dy\right)^{1/q}\right].
\end{aligned}$$

*Proof.* Using the standard trigonometric formula $\sin(a+b) = \cos(a)\sin(b)+\cos(b)\sin(a)$ for any $a, b \in \mathbb{R}$ and the triangle inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{|\sin(x+y)|}{x+y} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{|\cos(x)\sin(y) + \cos(y)\sin(x)|}{x+y} f(x)g(y)dxdy \\
&\leq \int_0^{+\infty}\int_0^{+\infty} \frac{|\cos(x)\sin(y)|}{x+y} f(x)g(y)dxdy \\
&+ \int_0^{+\infty}\int_0^{+\infty} \frac{|\cos(y)\sin(x)|}{x+y} f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f_\dagger(x)g_\dagger(y)dxdy + \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f_\diamond(x)g_\diamond(y)dxdy, \qquad (2.7)
\end{aligned}$$

where

$$f_\dagger(x) = f(x)|\cos(x)|, \quad g_\dagger(y) = g(y)|\sin(y)|$$

and

$$f_\diamond(x) = f(x)|\sin(x)|, \quad g_\diamond(y) = g(y)|\cos(y)|.$$

Applying the Hardy-Hilbert integral inequality in Equation (1.1) twice to the functions $f_\dagger, g_\dagger$ and $f_\diamond, g_\diamond$ yields

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f_\dagger(x) g_\dagger(y) dxdy + \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f_\diamond(x) g_\diamond(y) dxdy \\
&\leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f_\dagger^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g_\dagger^q(y)dy\right)^{1/q} \\
&+ \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f_\diamond^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g_\diamond^q(y)dy\right)^{1/q} \\
&= \frac{\pi}{\sin(\pi/p)} \left[\left(\int_0^{+\infty} f^p(x)|\cos(x)|^p dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)|\sin(y)|^q dy\right)^{1/q}\right. \\
&\left. + \left(\int_0^{+\infty} f^p(x)|\sin(x)|^p dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)|\cos(y)|^q dy\right)^{1/q}\right]. \qquad (2.8)
\end{aligned}$$

Joining Equations (2.7) and (2.8), we derive

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{|\sin(x+y)|}{x+y} f(x)g(y) dxdy \\
&\leq \frac{\pi}{\sin(\pi/p)} \left[\left(\int_0^{+\infty} f^p(x)|\cos(x)|^p dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)|\sin(y)|^q dy\right)^{1/q}\right. \\
&\left. + \left(\int_0^{+\infty} f^p(x)|\sin(x)|^p dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)|\cos(y)|^q dy\right)^{1/q}\right].
\end{aligned}$$

The desired inequality is established. □

This result should be regarded as a trigonometric variant of the Hardy-Hilbert integral inequality, yielding an original upper bound involving various weighted integral norms of $f$ and $g$.

2.4. **Fourth theorem.** The fourth integral inequality of the Hardy-Hilbert type, presented in the theorem below, is of a primitive nature and involves a sharp constant.

**Theorem 2.4.** *Let $p > 1$, $q = p/(p-1)$ and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions such that*

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dx < +\infty,$$

*and $F, G : [0, +\infty) \to [0, +\infty)$ be the corresponding primitives, i.e., for any $t \geq 0$,*

$$F(t) = \int_0^t f(x)dx, \quad G(t) = \int_0^t g(y)dy,$$

*provided that they exist.*

*Then we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^3} F(x)G(y) dxdy \\
&\leq \frac{\pi}{2\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}.
\end{aligned}$$

*Proof.* The proof relies on a careful manipulation of the main double integral. Applying the Fubini-Tonelli integral theorem and performing two well-chosen integrations by parts based on $\lim_{x\to 0} F(x) = 0$, $\lim_{x\to+\infty} F(x)/(x+y)^2 = 0$ for any $y > 0$, $\lim_{y\to 0} G(y) = 0$ and $\lim_{y\to+\infty} G(y)/(x+y) = 0$ for any $x > 0$, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^3} F(x)G(y)dxdy \\
&= \int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{(x+y)^3} F(x)dx\right) G(y)dy \\
&= \int_0^{+\infty} \left(\left[-\frac{1}{2(x+y)^2}F(x)\right]_{x\to 0}^{x\to+\infty} - \int_0^{+\infty} \left(-\frac{1}{2(x+y)^2}\right) f(x)dx\right) G(y)dy \\
&= \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^2} f(x)G(y)dxdy \\
&= \frac{1}{2}\int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{(x+y)^2} G(y)dy\right) f(x)dx \\
&= \frac{1}{2}\int_0^{+\infty} \left(\left[-\frac{1}{x+y}G(y)\right]_{y\to 0}^{y\to+\infty} - \int_0^{+\infty} \left(-\frac{1}{x+y}\right) g(y)dy\right) f(x)dx \\
&= \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy. \qquad (2.9)
\end{aligned}$$

It follows from the Hardy-Hilbert integral inequality in Equation (1.1) applied to $f$ and $g$ that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \\
&\leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}. \qquad (2.10)
\end{aligned}$$

Joining Equations (2.9) and (2.10), we derive

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^3} F(x)G(y)dxdy \\
&\leq \frac{\pi}{2\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx\right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy\right)^{1/q}.
\end{aligned}$$

This ends the proof. □

This integral inequality is original in that the main double integral involves primitives, while the classical Hardy-Hilbert integral inequality is not employed in this case. Instead, the strategy of working directly on the integrand of the double integral proves to be more effective here.

2.5. **Fifth theorem.** The fifth integral inequality of the Hardy-Hilbert type, presented in the theorem below, is of a three-dimensional nature and involves an original constant defined as a series.

**Theorem 2.5.** *Let $f, g, h : [0,1] \to [0,+\infty)$ be three functions such that*

$$\int_0^1 f^2(x)dx < +\infty, \quad \int_0^1 g^2(y)dy < +\infty, \quad \int_0^1 h^2(z)dz < +\infty.$$

*Then we have*

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}f(x)g(y)h(z)dxdydz$$
$$\leq \Omega\left(\int_0^1 f^2(x)dx\right)^{1/2}\left(\int_0^1 g^2(y)dy\right)^{1/2}\left(\int_0^1 h^2(z)dz\right)^{1/2},$$

*where*

$$\Omega=\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^{3/2}}\approx 1.689. \tag{2.11}$$

*Proof.* For any $x, y, z \in (0,1)$, it is clear that

$$\frac{1}{1-xyz}\geq 0.$$

Moreover, the following geometric series expansion holds:

$$\frac{1}{1-xyz}=\sum_{n=0}^{+\infty}(xyz)^n.$$

Using this and the exchange between integral and series by uniform convergence, we get

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}f(x)g(y)h(z)dxdydz$$
$$=\int_0^1\int_0^1\int_0^1 \sum_{n=0}^{+\infty}(xyz)^n f(x)g(y)h(z)dxdydz$$
$$=\sum_{n=0}^{+\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n f(x)g(y)h(z)dxdydz$$
$$=\sum_{n=0}^{+\infty}\left(\int_0^1 x^n f(x)dx\right)\left(\int_0^1 y^n g(y)dy\right)\left(\int_0^1 z^n h(z)dz\right). \tag{2.12}$$

The Cauchy-Schwarz integral inequality gives, for any $n\in\mathbb{N}$,

$$\int_0^1 x^n f(x)dx\leq\left(\int_0^1 x^{2n}dx\right)^{1/2}\left(\int_0^1 f^2(x)dx\right)^{1/2}=\frac{1}{\sqrt{2n+1}}\left(\int_0^1 f^2(x)dx\right)^{1/2}.$$

Similarly, we have

$$\int_0^1 y^n g(y)dy\leq\left(\int_0^1 y^{2n}dy\right)^{1/2}\left(\int_0^1 g^2(y)dy\right)^{1/2}=\frac{1}{\sqrt{2n+1}}\left(\int_0^1 g^2(y)dy\right)^{1/2}$$

and

$$\int_0^1 z^n h(z)dx\leq\left(\int_0^1 z^{2n}dz\right)^{1/2}\left(\int_0^1 h^2(z)dz\right)^{1/2}=\frac{1}{\sqrt{2n+1}}\left(\int_0^1 h^2(z)dz\right)^{1/2}.$$

Combining these inequalities, we obtain

$$\begin{aligned}
&\sum_{n=0}^{+\infty}\left(\int_0^1 x^n f(x)dx\right)\left(\int_0^1 y^n g(y)dy\right)\left(\int_0^1 z^n h(z)dz\right)\\
&\le \sum_{n=0}^{+\infty}\frac{1}{\sqrt{2n+1}}\left(\int_0^1 f^2(x)dx\right)^{1/2}\frac{1}{\sqrt{2n+1}}\left(\int_0^1 g^2(y)dy\right)^{1/2}\\
&\times \frac{1}{\sqrt{2n+1}}\left(\int_0^1 h^2(z)dz\right)^{1/2}\\
&= \Omega\left(\int_0^1 f^2(x)dx\right)^{1/2}\left(\int_0^1 g^2(y)dy\right)^{1/2}\left(\int_0^1 h^2(z)dz\right)^{1/2}, \qquad (2.13)
\end{aligned}$$

where $\Omega$ is given in Equation (2.11).

It follows from Equations (2.12) and (2.13) that

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz} f(x)g(y)h(z)dxdydz\\
&\le \Omega\left(\int_0^1 f^2(x)dx\right)^{1/2}\left(\int_0^1 g^2(y)dy\right)^{1/2}\left(\int_0^1 h^2(z)dz\right)^{1/2}.
\end{aligned}$$

We get the desired inequality. □

## 3. Conclusion

In this article, we present five distinct Hardy-Hilbert-type integral inequalities, each characterized by a specific integrand structure and a precise constant factor. These results extend existing inequalities in the literature, providing fresh perspectives on their trigonometric and analytic formulations. Proposed areas for future research include exploring multidimensional analogues. For instance, based on the framework of the trigonometric integral inequalities and logical notation, one could consider the triple integrals

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{\cos^2(x)\cos^2(y)\cos^2(z)+\sin^2(x)\sin^2(y)\sin^2(z)}\\
&\times f(x)g(y)h(z)dxdydz,
\end{aligned}$$

$$\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{1}{1+\cos(x)\cos(y)\cos(z)} f(x)g(y)h(z)dxdydz$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{|\sin(x+y+z)|}{x+y+z} f(x)g(y)h(z)dxdydz.$$

Other areas for investigation include connections with special functions and applying these inequalities to problems in functional analysis and operator theory.

## 4. Acknowledgements

The author would like to thank the reviewer for his constructive comments on the article.

## REFERENCES

[1] Q. Chen, and B.C. Yang. A survey on the study of Hilbert-type inequalities. Journal of Inequalities and Applications, **2015** (2015), 1-29.
[2] C. Chesneau. Study of two three-parameter non-homogeneous variants of the Hilbert integral inequality. Lobachevskii Journal of Mathematics, **45** (2024), 4931-4953.
[3] C. Chesneau. Theoretical results on new Hardy-Hilbert-type inequalities. Annals of Communications in Mathematics, **8** (2025), 253-274.
[4] C. Chesneau. Two new contributions to the three-dimensional Hardy-Hilbert-type integral inequalities. Earthline Journal of Mathematical Sciences, **16** (2026), 85-94.
[5] G.H. Hardy, J.E. Littlewood, and G. Pólya. Inequalities. Cambridge University Press, Cambridge, (1934).
[6] Z. Huang, and B.C. Yang. A new Hilbert-type integral inequality with the combination kernel. International Mathematical Forum, **6** (2011), 1363-1369.
[7] Y. Li, J. Wu, and B. He. A new Hilbert-type integral inequality and the equivalent form. International Journal of Mathematics and Mathematical Sciences, **8** (2006), 1-6.
[8] W.T. Sulaiman. Hardy-Hilbert's integral inequalities via homogeneous functions and some other generalizations. Acta et Commentationes Universitatis Tartuensis de Mathematica, **11** (2007), 23-32.
[9] W.T. Sulaiman. On two inequalities similar to Hardy-Hilbert's integral inequality. Applied Mathematical Sciences, **4** (2010), 133-138.
[10] B. Sun. Best generalization of a Hilbert-type inequality. Journal of Inequalities in Pure and Applied Mathematics, **7** (2006), 1-7.
[11] B.C. Yang. On Hardy-Hilbert's integral inequality. Journal of Mathematical Analysis and Applications, **261** (2001), 295-306.
[12] B.C. Yang. A basic Hilbert-type integral inequality with the homogeneous kernel of -1-degree and extensions. Journal of Guangdong Education Institute, **28(3)** (2008), 1-10.
[13] B.C. Yang. Hilbert-Type Integral Inequalities. Bentham Science Publishers, The United Arab Emirates, (2009).
[14] B.C. Yang. The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing, (2009).

CHRISTOPHE CHESNEAU
DEPARTMENT OF MATHEMATICS, LMNO, UNIVERSITY OF CAEN-NORMANDIE, 14032 CAEN, FRANCE.
*ORCID: 0000-0002-1522-9292*
*Email address*: christophe.chesneau@gmail.com