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FUZZY BI-QUASI INTERIOR IDEALS OF SEMIRINGS

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ABSTRACT. In this paper, we introduce the notion of a fuzzy bi-quasi interior ideal as a generalization of fuzzy ideals, fuzzy bi-quasi ideals, fuzzy quasi-interior ideals and fuzzy bi-interior ideals of a semiring. We prove that every fuzzy right quasi-interior ideal of a semiring is a fuzzy bi-quasi interior ideal, and a fuzzy bi-quasi interior ideal is a fuzzy right tri-ideal of a semiring. We characterize the regular semiring in terms of fuzzy bi-quasi interior ideals and study some of the properties.

1. INTRODUCTION

Algebraic structures play a prominent role in mathematics. Generalizing ideals of algebraic and ordered algebraic structures plays a remarkable role and is necessary for further advanced studies and applications of ideals in algebraic structures. The theory of rings and semigroups considerably impacts the development of the theory of semirings. Semirings play an important role in studying determinants, theoretical computer science, solution of graph theory, optimization theory, automata, coding theory and formal languages. The notion of ideals introduced by Dedekind for the theory of algebraic numbers was generalized by E. Noether for associative rings. The one and two-sided ideals presented by her are still central concepts in ring theory. We know that the notion of a one-sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi-ideals are the generalization of left and right ideals, whereas the bi-ideals are generalization of quasi-ideals. The notion of bi-ideals in semigroups introduced by Lajos [6]. Iseki[3, 4] introduced the concept of quasi ideal for a semiring. M. Henriksen[2] studied ideals in semirings. As a further generalization of ideals, Steinfeld[33] first introduced the notion of quasi-ideals for semigroups and then for rings. We know that the notion of the bi-ideal in semirings is a special case of (m, n) ideal introduced by S. Lajos. R. A. Good and D. R. Hughes[1] introduced the concept of bi-ideals for a semigroup. Lajos and Szasz[7] introduced the concept of bi-ideals for rings.

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Uncertainty is a key factor that makes many real-world problems difficult to handle, and conventional methods are not always the most effective for such cases. To overcome these challenges, alternative frameworks such as randomness, rough set theory, and fuzzy set theory were developed. Among these, fuzzy set theory was introduced by L. A. Zadeh[34] in 1965, and it quickly gained prominence through numerous studies demonstrating its wide-ranging applications in logic, set theory, group theory, ring theory, real analysis, topology, and measure theory. The concept of fuzzy subgroups was introduced by Rosenfeld[32], and their structural properties were subsequently investigated. Building on this foundation, N. Kuroki[5] contributed further by studying fuzzy interior ideals in semigroups.

The concept of bi-ideals, quasi-ideals, and interior-ideals of various algebraic structures were introduced and studied by mathematicians. Mandal[8] introduced and studied fuzzy ideals and fuzzy interior ideals of ordered semirings. M.Murali Krishna Rao[9, 10, 11, 12, 13, 14, 15, 16, 17, 18] introduced and studied bi-quasi ideals, bi-interior ideals, quasi-interior ideals, tri-ideals, tri-quasi ideals, and weak interior ideals of various algebraic structures, as a generalization of bi-ideals, quasi-ideals, interior ideals and characterized regular as well as simple algebraic structures using these ideals. M.Murali Krishna Rao et al.[19, 20, 21, 22, 23, 24, 25, 26, 27, 31] introduced and studied fuzzy(soft) bi-quasi ideals, bi-interior ideals, quasi-interior ideals, tri-quasi ideals of various algebraic structures. Fuzzy bi-quasi interior ideals of semirings are motivated by the need to extend classical ideal theory to accommodate uncertainty and partial membership, which is common in real-world applications involving algebraic structures like semirings. Their importance lies in algebraic analysis, where elements do not have crisp inclusion, supporting applications in computer science, optimization theory, and information sciences. Fuzzy(soft) algebraic structures have applications in decision making, artificial intelligence, data mining, machine learning, and image processing. This paper aims to introduce the notion of a fuzzy bi-quasi interior ideal of a semiring. We prove that every fuzzy bi-quasi interior ideal of a regular semiring if and only if it is a fuzzy quasi ideal of a semiring. Regular semiring is characterized in terms of a fuzzy bi-quasi interior ideal of a semiring.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. [10] A set M together with two associative binary operations addition and multiplication (denoted by + and $\cdot$ respectively) will be called semiring provided

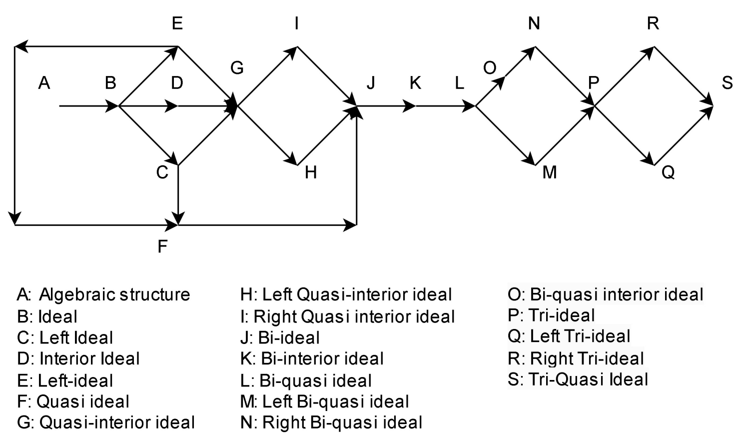
- (i) addition is a commutative operation.
- (ii) multiplication distributes over addition both from the left and from the right.
- (iii) there exists $0 \in M$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

Definition 2.2. [12, 13, 14, 17]
A non-empty subset B of a semiring M is called

- (i) a *subsemiring* of M, if $(B, +)$ is a subsemigroup of $(M, +)$ and $BB \subseteq B$,
- (ia) a quasi ideal of M if B is a subsemiring of M and $BM \cap MB \subseteq B$.
- (ii) a bi-ideal of M if B is a subsemiring of M and $BMB \subseteq B$.
- (iii) an interior ideal of M if B is a subsemiring of M and $MBM \subseteq B$.
- (iv) an ideal if B is a subsemiring of $M, BM \subseteq B$ and $MB \subseteq B$.
- (v) a bi-interior ideal of M if B is a subsemiring of M and $MBM \cap BMB \subseteq B$.

(vi) a left bi-quasi ideal (right bi-quasi ideal) of M if B is a subsemiring of $(M,+)$ and $MB \cap BMB \subseteq B$ $(BM \cap BMB \subseteq B)$.
(vii) a bi- quasi ideal of M if B is a left bi- quasi ideal and a right bi- quasi ideal of M
(viii) a left quasi-interior ideal (right quasi-interior ideal) of M if B is a subsemiring of M and $MBMB \subseteq B$ $(BMBM \subseteq B)$.
(ix) a quasi-interior ideal of M if B is a subsemiring of M and $B$is a left quasi-interior ideal and a right quasi-interior ideal of M.
(x) a bi-quasi interior ideal of M if B is a subsemiring of M and $BMBMB \subseteq B$.
(xi) a left tri- ideal (right tri- ideal) of M if B is a subsemiring of M and $BMBB \subseteq B$ $(BBMB \subseteq B)$.
(xii) a tri-ideal of M if B is a subsemiring of M and $BMBB \subseteq B$ and $BBMB \subseteq B$.
(xiii) a tri-quasi ideal of M if B is a subsemiring of M and $BBMBB \subseteq B$.

Figure(1): The inter relationships between some generalization of ideal mentioned before are visualized in figure(1). (Arrows indicates proper inclusions. That is if X and Y are ideals then $X \to Y$ means $X \subset Y$.)



Definition 2.3. [10] An element a of a semiring M is called a regular element if there exists an element x of M such that $a = axa$. A semiring M is called a regular semiring if every element of M is a regular element.

Definition 2.4. [34] Let M be a non-empty set. A mapping $\mu : M \to [0,1]$ is called a fuzzy subset of a semiring M. If μ is not a constant function then μ is called a non-empty fuzzy subset.

Definition 2.5. [22] Let μ be a fuzzy subset of a non-empty set M, for $t \in [0,1]$ the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a level subset of M with respect to f.

Definition 2.6. [22] For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in M$.

Definition 2.7. [22] Let f and g be fuzzy subsets of a semiring M. Then $f \circ g, f + g, f \cup g, f \cap g$, are defined by: for all $z \in M$,

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy} \{min\{f(x), g(y)\}\}, \\ 0, \ \ \text{otherwise.} \end{cases} ; f + g(z) = \begin{cases} \sup\limits_{z=x+y} \{min\{f(x), g(y)\}\}, \\ 0, \ \ \text{otherwise} \end{cases}$$

$$f \cup g(z) = max\{f(z), g(z)\} \; ; \; f \cap g(z) = min\{f(z), g(z)\},$$
where $x, y \in M$.

Definition 2.8. [22] Let A be a non-empty subset of M. The *characteristic function* of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.9. [22, 23, 24, 25, 26]
Let M be a semiring. A fuzzy subset μ of M is called a
(1) fuzzy quasi-ideal of M, if
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}, \mu(xy) \geq min\ \{\mu(x), \mu(y)\}$,
(ii) $\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$, for all $x, y \in M$.
(2) fuzzy left (right) bi-quasi ideal of M, if
(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}, \mu(xy) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.
(ii) $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu\big(\mu \circ \chi_M \circ \mu \circ \chi_M \subseteq \mu\big)$.
(3) fuzzy tri-quasi ideal of M, if
(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu \circ \mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu,\ for\ all\ x, y \in M.$
(4) fuzzy bi-interior ideal of M, if
(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu,\ for\ all\ x, y \in M.$
(5) fuzzy tri-ideal of M, if
(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu,\ for\ all\ x, y \in M.$
(6) A fuzzy quasi-interior ideal of M, if
(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu,\ for\ all\ x, y \in M.$
(7) fuzzy weak-interior ideal of M, if
(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu \circ \mu \circ \chi_M \subseteq \mu,\ for\ all\ x, y \in M.$

3. Fuzzy bi-quasi interior ideals of semirings

In this section, we introduce the notion of fuzzy bi-quasi interior ideals as a generalization of fuzzy ideals, fuzzy bi-interior ideals, fuzzy quasi-ideals, fuzzy bi-quasi ideals of a semiring and study the properties of fuzzy bi-quasi interior ideals.

Definition 3.1. A fuzzy subset μ of a semiring M is called a fuzzy bi-quasi-interior ideal if

(i) $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in M$,
(iii) $\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.

Example 3.2. Let $M = \{a, b, c, d, e\}$. Define the binary operations " $+$ " on M and " $\cdot$ " with the following Cayleys tables

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	a	a	a	a
c	c	a	a	a	a
d	d	a	a	a	a
e	e	a	a	a	a

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Then $(M, +)$ is a commutative semigroup.
And M is a semiring. Let $B = \{a, c\}$ then B is a bi-quasi interior ideal of M.
(i) Define a fuzzy subset $\mu : M \to [0, 1]$ of M, by $\mu(a) = 1, \mu(b) = 0.9$, $\mu(c) = 0.4, \mu(d) = 0.6, \mu(e) = 0.8$. Then μ is a fuzzy bi-quasi interior ideal of M.
(ii) Define a fuzzy subset $\mu : M \to [0, 1]$ of M by

$$\mu(x) = \begin{cases} 1, & \text{if } x \in B; \\ 0, & \text{if } x \notin B. \end{cases}$$

Then μ is a fuzzy bi-quasi interior ideal of M.

Theorem 3.1. *Let M be a semiring, and μ be a nonempty fuzzy subset of M. Every fuzzy right ideal of a semiring M is a fuzzy right-quasi interior ideal of M.*

Proof. Let μ be a fuzzy right ideal of a semiring M and $x \in M$. Then

$$\begin{aligned} \mu \circ \chi_M(x) &= \sup_{x=ab} \{\min\{\mu(a), \chi_M(b)\}\},\ a, b \in M. \\ &= \sup_{x=ab} \{\mu(a)\} \\ &\leq \sup_{x=ab} \{\mu(ab)\} \\ &= \mu(x). \end{aligned}$$

$$Hence, \mu \circ \chi_M(x) \leq \mu(x)$$

$$\begin{aligned} \text{Now } \mu \circ \chi_M \circ \mu \circ \chi_M(x) &= \sup_{x=uv} \{\min\{\mu \circ \chi_M(u), \mu \circ \chi_M(v)\}\} \\ &\leq \sup_{x=uv} \{\min\{\mu(u), \mu(v)\}\} \\ &= \mu \circ \mu(x) \\ &\leq \mu(x). \end{aligned}$$

Hence μ is a fuzzy right-quasi interior ideal of M. □

Corollary 3.2. *Let M be a semiring, and μ be a nonempty fuzzy subset of M. Every fuzzy (left) ideal of a semiring M is a fuzzy (left)-quasi interior ideal of M.*

Theorem 3.3. *Let M be a semiring, and μ be a nonempty fuzzy subset of M. If μ is a fuzzy left-ideal of M, then μ is a fuzzy bi-ideal of M.*

Proof. Suppose, μ is a fuzzy left-ideal of M. Let $z \in M$,

$$\begin{aligned}\chi_M \circ \mu(z) &= \sup_{z=lm}\{\min\{\chi_M(l), \mu(m)\}\}\\ &= \sup_{z=lm}\{\min\{1, \mu(m)\}\}\\ &= \sup_{z=lm}\{\mu(m)\}\\ &\leq \sup_{z=lm}\{\mu(lm)\}\\ &= \mu(z)\end{aligned}$$

$$\begin{aligned}\mu \circ \chi_M \circ \mu(z) &= \sup_{z=lm}\{\min\{\mu(l), \chi_M \circ \mu(m)\}\}\ l, m \in M.\\ &\leq \sup_{z=lm}\{\min\{\mu(l), \mu(m)\}\}\\ &= \mu(z)\end{aligned}$$

$$\Rightarrow \mu \circ \chi_M \circ \mu(z) \leq \mu(z)$$

Therefore, $\mu \circ \chi_M \circ \mu \subseteq \mu$. Hence, μ is a fuzzy bi-ideal of M. □

Theorem 3.4. *Let M be a semiring, and μ be a nonempty fuzzy subset of M. Every fuzzy right quasi-interior ideal of a semiring M is a fuzzy bi-quasi interior ideal of M.*

Proof. Let μ be a fuzzy right quasi-interior ideal of M and $x \in M$. Then

$$\mu \circ \chi_M \circ \mu \circ \chi_M \subseteq \mu.$$

$$\begin{aligned}\text{Now } \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=ab}\{\min\{\mu \circ \chi_M \circ \mu \circ \chi_M(a), \mu(b)\}\}\\ &\leq \sup_{x=ab}\{\min\{\mu(a), \mu(b)\}\}\\ &= \mu \circ \mu(x)\\ &\leq \mu(x).\end{aligned}$$

Hence, μ is a fuzzy bi-quasi interior ideal of M. □

Corollary 3.5. *Every fuzzy (left) quasi-interior ideal of a semiring M is a fuzzy bi-quasi-interior ideal of M.*

Theorem 3.6. *Let M be a semiring and μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy bi-quasi interior ideal of a semiring M if and only if the level subset μ_t of μ is a bi-quasi interior ideal of a semiring M for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy bi-quasi interior ideal of a semiring M, $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t$. Then

$$\begin{aligned}&\mu(a) \geq t, \mu(b) \geq t\\ &\Rightarrow \mu(a + b) \geq \min\{\mu(a), \mu(b)\} \geq t, \mu(ab) \geq \min\{\mu(a), \mu(b)\} \geq t\\ &\Rightarrow a + b \in \mu_t, ab \in \mu_t.\end{aligned}$$

Let $x \in \mu_t M \mu_t M \mu_t$. Then $x = abcde$, where $c \in M, a, b, d \in \mu_t$, then

$$\mu_t M \mu_t M \mu_t \geq t$$

Therefore $x \in \mu_t$.

Hence, μ_t is a bi-quasi interior ideal of M.

Conversely, suppose that μ_t is a bi-quasi interior ideal of M, for all $t \in Im(\mu)$. Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$.

$$\Rightarrow x + y \in \mu_{t_2} \text{ and } xy \in \mu_{t_2},$$
$$\Rightarrow \mu(x + y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$$

Therefore $\mu(x + y) \geq t_2 = \min\{\mu(x), \mu(y), \mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

We have $\mu_l M \mu_l M \mu_l \subseteq \mu_t$, for all $l \in Im(\mu)$.
Suppose, $t = \min\{Im(\mu)\}$. Then $\mu_t M \mu_t M \mu_t \subseteq \mu_t$.
Therefore, $\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.
Hence, μ is a fuzzy bi-quasi interior ideal of M. □

Theorem 3.7. *Let M be a semiring. If μ is a fuzzy bi-ideal of M, then μ is a fuzzy bi-quasi interior ideal of M.*

Proof. Suppose, μ is a fuzzy bi-ideal of M. Then
$\mu \circ \chi_M \circ \mu \subseteq \mu$. Let $z \in M$,

$$\begin{aligned}\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu(z) &= \sup_{z=lm} \{\min\{\mu \circ \chi_M \circ \mu(l), \chi_M \circ \mu(m)\}\}\ l, m \in M.\\ &\leq \sup_{z=lm} \{\min\{\mu(l), \chi_M \circ \mu(m)\}\}\\ &= \mu \circ \chi_M \circ \mu(z)\\ &\leq \mu(z).\end{aligned}$$

Therefore, μ is a fuzzy bi-quasi interior of M. □

Theorem 3.8. *Let M be a semiring, if μ is a fuzzy right ideal of M, then μ is a fuzzy left tri-ideal of M.*

Proof. Let μ be a fuzzy right ideal of a semiring M. Then $\mu \circ \chi_M \subseteq \mu$. Let $z \in M$.

$$\begin{aligned}\mu \circ \chi_M \circ \mu \circ \mu(z) &= \sup_{z=lm} \{\min\{\mu \circ \chi_M(l), \mu \circ \mu(m)\}\}\\ &\leq \sup_{z=lm} \{\min\{\mu(l), \mu(m)\}\}\\ &= \mu \circ \mu(z)\\ &\leq \mu(z).\end{aligned}$$

Hence, μ is a fuzzy left tri-ideal of M. □

Corollary 3.9. *Let M be a semiring. If μ is a fuzzy (right) ideal of M, then μ is a fuzzy (right) tri-ideal of M.*

Theorem 3.10. *Let I be a non-empty subset of a semiring M and χ_I be the characteristic function of I. Then I is a bi-quasi interior ideal of a semiring M if and only if χ_I is a fuzzy bi-quasi interior ideal of a semiring M.*

Proof. Let I be a non-empty subset of a semiring M and χ_I be the characteristic function of I. Suppose, I is a bi-quasi interior ideal of M.

Obviously, χ_I is a fuzzy subsemiring of M. We have $IMIMI \subseteq I$. Then

$$\begin{aligned}\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \circ \chi_I &= \chi_{IMIMI}\\ &\subseteq \chi_I.\end{aligned}$$

Therefore, χ_I is a fuzzy bi-quasi interior ideal of M. Conversely, suppose that χ_I is a fuzzy bi-quasi interior ideal of M.

Then, I is a subsemiring of M. We have

$$\chi_I \circ \chi_M \circ \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I$$
$$\Rightarrow \chi_{IMIMI} \subseteq \chi_I$$
$$\text{Therefore } \ IMIMI \subseteq I.$$

Hence, I is a bi-quasi interior ideal of M. □

Theorem 3.11. *If μ and λ are fuzzy bi-quasi interior ideals of a semiring M, then $\mu \cup \lambda$ is a fuzzy bi-quasi interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy bi-quasi interior ideals of semiring M. Then

$$\begin{aligned}\mu \cup \lambda(x+y) &= \max\{\mu(x+y), \lambda(x+y)\}\\ &\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}\\ &= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y).\}\end{aligned}$$

$$\begin{aligned}\mu \cup \lambda(xy) &= \max\{\mu(xy), \lambda(xy)\}\\ &\geq \max\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}\\ &= \min\{\mu \cup \lambda(x), \mu \cup \lambda(y).\}\end{aligned}$$

Then $\mu \cup \lambda$ is a fuzzy subsemiring. And

$$\begin{aligned}\chi_M \circ \mu \cup \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cup \lambda(b)\}\\ &= \sup_{x=ab} \min\{\chi_M(a), \max\{\mu(b), \lambda(b)\}\}\\ &= \sup_{x=ab} \max\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\\ &= \max\{\sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\}\\ &= \max\{\chi_M \circ \mu(x), \chi_M \circ \lambda(x)\}\\ &= (\chi_M \circ \mu \cup \chi_M \circ \lambda)(x).\end{aligned}$$

Therefore $\chi_M \circ \mu \cup \chi_M \circ \lambda = \chi_M \circ (\mu \cup \lambda)$.

$$\begin{aligned}&\mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda(x)\\ &= \sup_{x=ab} \min\{\mu \cup \lambda(a), \chi_M \circ \mu \cup \lambda(b)\}\\ &= \sup_{x=ab} \min\{\max\{\mu(a), \lambda(a)\},\ \max\{\chi_M \circ \mu(b), \chi_M \circ \lambda(b)\}\}\\ &= \sup_{x=ab} \min\{\max\{\mu(a), \chi_M \circ \mu(b)\},\ \max\{\lambda(a), \chi_M \circ \lambda(b)\}\}\\ &= \max\{\sup_{x=ab} \min\{\mu(a), \chi_M \circ \mu(b)\}, \sup_{x=ab} \min\{\lambda(a), \chi_M \circ \lambda(b)\}\}\\ &= \max\{\mu \circ \chi_M \circ \mu(x), \lambda \circ \chi_M \circ \lambda(x)\}\\ &= \mu \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \lambda(x).\end{aligned}$$

$\mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda(x) =$
$= \sup_{x=ab} \min\{\mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda(a), \chi_M \circ \mu \cup \lambda(b)\}$
$= \sup_{x=ab} \min\{\mu \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \lambda(a), \chi_M \circ \mu \cup \chi_M \circ \lambda(b)\}$
$= \sup_{x=ab} \max\{\min\{\mu \circ \chi_M \circ \mu(a), \lambda \circ \chi_M \circ \lambda(a)\}, \min\{\chi_M \circ \mu(b), \chi_M \circ \lambda(b)\}\}$
$= \max\{\sup_{x=ab} \min\{\mu \circ \chi_M \circ \mu(a), \chi_M \circ \mu(b)\}, \sup_{x=a\alpha b} \min\{\lambda \circ \chi_M \circ \lambda(a), \chi_M \circ \lambda(b)\}\}$
$= \max\{\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu(z), \lambda \circ \chi_M \circ \lambda \circ \chi_M \circ \lambda(z)\}$
$= \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \lambda \circ \chi_M \circ \lambda(z)$

Therefore,
$\mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \lambda \circ \chi_M \circ \lambda \subseteq \mu \cup \lambda$.
Hence, $\mu \cup \lambda$ is a fuzzy bi-quasi interior ideal of M. □

Theorem 3.12. *If μ and λ are fuzzy bi-quasi interior ideals of a semiring M, then $\mu \cap \lambda$ is a fuzzy bi-quasi interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy bi-quasi interior ideals of a semiring M, $x, y \in M$, then

$$\begin{aligned}\mu \cap \lambda(x+y) &= \min\{\mu(x+y), \lambda(x+y)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\}\\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\}\end{aligned}$$

$$\begin{aligned}\mu \cap \lambda(xy) &= \min\{\mu(xy), \lambda(xy)\}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\}\\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\}\\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y).\}\end{aligned}$$

$$\begin{aligned}\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=ab} \min\{\chi_M(a), \mu \cap \lambda(b)\}\\ &= \sup_{x=ab} \min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\}\\ &= \sup_{x=ab} \min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\}\\ &= \min\{\sup_{x=ab} \min\{\chi_M(a), \mu(b)\}, \sup_{x=ab} \min\{\chi_M(a), \lambda(b)\}\}\\ &= \min\{\chi_M \circ \mu(x), \chi_M \circ \lambda(x)\}\\ &= \chi_M \circ \mu \cap \chi_M \circ \lambda(x)\end{aligned}$$

Therefore $\chi_M \circ \mu \cap \chi_M \circ \lambda = \chi_M \circ (\mu \cap \lambda)$.

$$\begin{aligned}
&\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda(x)\\
&=\sup_{x=ab}\min\{\mu\cap\lambda(a),\chi_M\circ\mu\cap\lambda(b)\}\\
&=\sup_{x=ab}\min\{\mu\cap\lambda(a),\chi_M\circ\mu\cap\chi_M\circ\lambda(b)\}\\
&=\sup_{x=ab}\min\{\min\{\mu(a),\lambda(a)\},\min\{\chi_M\circ\mu(b),\chi_M\circ\lambda(b)\}\}\\
&=\sup_{x=ab}\min\{\min\{\mu(a),\chi_M\circ\mu(b)\},\min\{\lambda(a),\chi_M\circ\lambda(b)\}\}\\
&=\min\{\sup_{x=ab}\min\{\mu(a),\chi_M\circ\mu(b)\},\ \sup_{x=a\alpha b}\min\{\lambda(a),\chi_M\circ\mu(b)\}\}\\
&=\min\{\mu\circ\chi_M\circ\mu(x),\lambda\circ\chi_M\circ\lambda(x)\}\\
&=\mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda(x).
\end{aligned}$$

$$\begin{aligned}
&\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda(x)=\\
&=\sup_{x=ab}\min\{\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda(a),\chi_M\circ\mu\cap\lambda(b)\}\\
&=\sup_{x=ab}\min\{\mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda(a),\chi_M\circ\mu\cap\chi_M\circ\lambda(b)\}\\
&=\sup_{x=ab}\min\{\min\{\mu\circ\chi_M\circ\mu(a),\lambda\circ\chi_M\circ\lambda(a)\},\min\{\chi_M\circ\mu(b),\chi_M\circ\lambda(b)\}\}\\
&=\min\{\sup_{x=ab}\min\{\mu\circ\chi_M\circ\mu(a),\chi_M\circ\mu(b)\},\ \sup_{x=a\alpha b}\min\{\lambda\circ\chi_M\circ\lambda(a),\chi_M\circ\lambda(b)\}\}\\
&=\min\{\mu\circ\chi_M\circ\mu\circ\chi_M\circ\mu(z),\lambda\circ\chi_M\circ\lambda\circ\chi_M\circ\lambda(z)\}\\
&=\mu\circ\chi_M\circ\mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda\circ\chi_M\circ\lambda(z)
\end{aligned}$$

Therefore,
$\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\mu\cap\lambda(x)=\mu\circ\chi_M\circ\mu\circ\chi_M\circ\mu\cap\lambda\circ\chi_M\circ\lambda\circ\chi_M\circ\lambda\subseteq\mu\cap\lambda$.
Hence, $\mu\cap\lambda$ is a fuzzy bi-quasi interior ideal of M. □

Theorem 3.13. *Let M be a regular semiring. If μ is a fuzzy subset set of M, then*

$$\mu\circ\mu=\chi_M\circ\mu=\mu\circ\chi_M=\mu$$

Proof. Suppose M is a regular semiring, and $z\in M$. Then there exist $x\in M$, such that $z=zxz$.

$$\begin{aligned}
(i)\ \mu\circ\mu(z)&=\sup_{z=zxz}\{\{\min\{\mu(zx),\mu(z)\}\}\\
&=\sup_{z=zxz}\{\mu(z)\}\\
&=\mu(z).
\end{aligned}$$

Therefore, $\mu\circ\mu=\mu$.

$$\begin{aligned}
(ii)\ \chi_M\circ\mu(z)&=\sup_{z=zxz}\{\min\{\chi_M(zx),\mu(z)\}\}\\
&=\sup_{z=zxz}\{\min\{1,\mu(z)\}\}\\
&=\sup_{z=zxz}\mu(z)\\
&=\mu(z).
\end{aligned}$$

Therefore, $\chi_M\circ\mu=\mu$.
Similarly, $\mu\circ\chi_M=\mu$. □

Theorem 3.14. *Let M be a semiring and μ be a nonempty fuzzy subset of a regular semiring M. Then M is a regular if and only if $\mu = \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy bi-quasi interior ideal μ of a semiring M.*

Proof. Let μ be a fuzzy bi-quasi-interior ideal of M and $x, y \in M$. Then $\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.

$$\begin{aligned} \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu(x) &= \sup_{x=xyx} \{\min\{\mu \circ \chi_M \circ \mu(x), \chi_M \circ \mu(yx)\}\} \\ &\geq \sup_{x=xyx} \{\min\{\mu(x), \mu(x)\}\} \\ &= \mu(x). \end{aligned}$$

Therefore, $\mu \subseteq \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu$. Hence $\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu = \mu$.

Conversely, suppose that $\mu = \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu$, for any fuzzy bi-quasi interior ideal μ of M. Let B be a bi-quasi interior ideal of M.
Then χ_B be a fuzzy bi-quasi interior ideal of M. Therefore

$$\begin{aligned} \chi_B &= \chi_B \circ \chi_M \circ \chi_B \circ \chi_M \circ \chi_B \\ &= \chi_{BMBMB} \\ B &= BMBMB. \end{aligned}$$

suppose that $BMBMB = B$, for all bi-quasi-interior ideals B of M. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M. Then B is a bi- quasi interior ideal of M. Therefore $(R \cap L)M(R \cap L)M(R \cap L) = R \cap L$

$$\begin{aligned} R \cap L &= (R \cap L)M(R \cap L)M(R \cap L) \\ &\subseteq RMLML \\ &\subseteq RL \\ &\subseteq R \cap L (since\ RL \subseteq L\ and\ RL \subseteq R). \end{aligned}$$

Therefore, $R \cap L = RL$. Hence M is a regular semiring. □

Theorem 3.15. *Let M be a regular semiring. If μ is a fuzzy bi-quasi interior ideal of M, then μ is a fuzzy right tri-ideal of M.*

Proof. Suppose, μ is a fuzzy bi-quasi interior ideal of a regular semiring M. Then

$$\begin{aligned} \mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu(z) &= \sup_{z=xy} \{\min\{\mu \circ \chi_M(x), \mu \circ \chi_M \circ \mu(y)\}\} \\ &= \sup_{z=xy} \{\min\{\mu(x), \mu \circ \chi_M \circ \mu(y)\} \\ &= \mu \circ \mu \circ \chi_M \circ \mu. \end{aligned}$$

Therefore, $\mu \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.

By Theorem 3.13, we have

$$\begin{aligned} \mu \circ \chi_M(z) &= \sup_{z=zxz} \{\min\{\mu(z), \chi_M(xz), \}\} \\ &= \sup_{z=zxz} \{\min\{\mu(z), 1\}\} \\ &= \sup_{z=zxz} \mu(z) \\ &= \mu(z). \end{aligned}$$

Hence, $\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu = \mu \circ \mu \circ \chi_M \circ \mu \subseteq \mu$.
Therefore, μ is a fuzzy right tri-ideal of M. □

Corollary 3.16. *Let M be a regular semiring. If μ is a fuzzy bi-quasi interior ideal of M, then μ is a fuzzy (left) tri-ideal of M.*

Theorem 3.17. *Let M be a regular semiring, and μ be a fuzzy subset of M. If μ is a fuzzy left tri-ideal of M, then μ is a fuzzy left ideal of M.*

Proof. Let μ be a fuzzy left tri-ideal of a regular semiring M, and $x \in M$,then $\mu \circ \chi_M \circ \mu \circ \mu \subseteq \mu$. Let $z \in M$, then there exists $x \in M$, such that $z = zxz$. Now, by Theorem3.13, we have

$$\mu \circ \chi_M \circ \mu \subseteq \mu$$
$$\Rightarrow \chi_M \circ \mu \circ \mu \subseteq \mu$$
$$\Rightarrow \chi_M \circ \mu \subseteq \mu.$$

Hence, μ is a fuzzy left-ideal of M. □

Corollary 3.18. *Let M be a regular semiring, then μ is a fuzzy(right) tri-ideal if and only if μ is a fuzzy (right) ideal of M.*

Theorem 3.19. *Let M be a regular semiring. If μ is a fuzzy right tri-ideal of M, then μ is a fuzzy left quasi-interior ideal of M.*

Proof. Suppose μ is a fuzzy right tri-ideal of a regular semiring M. Then, $\mu \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. By Theorem3.13, we have

$$\chi_M \circ \mu \circ \chi_M \circ \mu = \mu \circ \mu \circ \chi_M \circ \mu \subseteq \mu.$$

Therefore, μ is a fuzzy left quasi-interior ideal of M. □

Theorem 3.20. *Let M be a regular semiring. If μ is a fuzzy (right)tri-ideal of M, then μ is a fuzzy (right) quasi-interior ideal of M.*

Theorem 3.21. *Let M be a (regular) semiring. If μ is a fuzzy bi-quasi interior ideal of M, then μ is a fuzzy tri-quasi ideal of M.*

Proof. Suppose μ is a fuzzy bi-quasi interior ideal of a regular semiring M. Then $\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu \subseteq \mu$. Let $z \in M$. Then there exist $x \in M$, such that $z = zxz$.

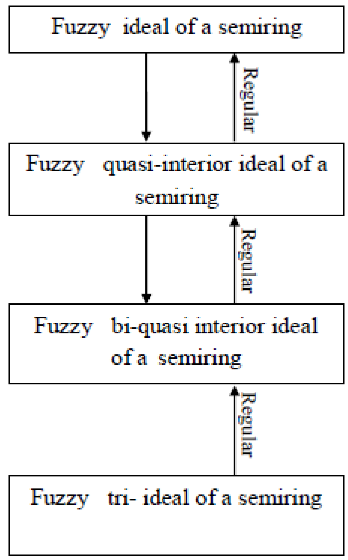
$$\mu \circ \chi_M \circ \mu \circ \chi_M \circ \mu(z) \leq \mu(z). \textit{ By Theorem3.13, we have}$$
$$\mu \circ \mu \circ \chi_M \circ \mu \circ \mu(z) \leq \mu(z).$$

Hence, μ is a fuzzy tri-quasi ideal of M. □

Theorem 3.22. *Let M be a (regular) semiring. If μ is a fuzzy subsemiring then the following are equivalent,*
(i) μ is a fuzzy ideal of M.
(ii) μ is a fuzzy quasi-interior ideal of M.
(iii) μ is a fuzzy bi-quasi interior ideal of M.
(iv) μ is a fuzzy tri-ideal of M.

Proof. $(i) \Rightarrow (ii)$ *The proof folllows from Corollary: 3.2.*
$(ii) \Rightarrow (iii)$ *The proof folllows from Corollary: 3.5.*
$(iii) \Rightarrow (iv)$ *The proof folllows from Corollary: 3.16.*
$(iv) \Rightarrow (i)$ *The proof folllows from Corollary: 3.18.* □

Relation between these fuzzy generalization of ideals are illustrated by the following diagram .



4. Conclusion

Semirings constitute a fundamental framework within theoretical computer science, graph theory, and optimization theory. They are particularly significant in the analysis of automata, coding theory, formal language theory, and cryptography. In this paper, we discussed the algebraic properties of fuzzy bi-quasi interior ideal of a semiring. We proved, that a fuzzy bi-ideal is a fuzzy bi-quasi interior ideal of a semiring. Regular semiring is characterized in terms of fuzzy quasi interior ideals and proved that for a regular semiring a fuzzy bi-quasi interior ideal is a fuzzy right tri-ideal of a semiring. We further wish to study ternary and soft algebraic structures.

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