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SOLUTION OF THE MILLENNIUM PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. One of the millennium problems can be stated as follows: can there be smooth, not vanishing identically initial data for the Navier-Stokes equations in $\mathbb{R}^3$ such that the corresponding solution to the NSP (Navier-Stokes problem) exists for all times $t \geq 0$? We prove that such a solution does not exist.

1. INTRODUCTION

Consider the Navier-Stokes problem (NSP) in $\mathbb{R}^3$:

$$v' + (v \cdot \nabla)v = -\nabla p + \nu \Delta v + f, \quad x \in R^3, \ t \geq 0, \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

$$v(x,0) = v_0(x), \tag{1.3}$$

(see, for example, books [4] and [5]). Here v = v(x, t) is the velocity of incompressible viscous fluid, a vector function, $v' := v_t$, $p = p(x,t)$ is the pressure, a scalar function, $f = f(x,t)$ is the exterior force, $\nu = const > 0$ is the viscosity coefficient, $v_0 = v_0(x) = v(x,0)$ is the initial velocity,

$$\nabla \cdot v_0 = 0. \tag{1.4}$$

The data $v_0(x)$ and $f(x,t)$ are given, the v and p are to be found. The fluid's density ρ is assumed to be constant, namely $\rho = 1$. We assume, for simplicity only, that $f(x,t) = 0$. We also assume that $v(x,0) \not\equiv 0$ is a smooth rapidly decaying function.

Let us assume that the solution to NSP (1.1)-(1.3) exists for all $t > 0$..

Theorem 1. Under these assumptions $v(x,0) = 0$.

The statement of Theorem 1 (the NSP paradox, see [1]–[5]) contradicts the assumption $v(x,0) \not\equiv 0$. Therefore the solution $v(x,t)$ to NSP cannot exist for all $t > 0$ under our assumptions.

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The millennium problem concerning the Navier-Stokes equations can be formulated as follows:

A millennium problem:

Assuming that the data f and $v(x,0)$ are smooth and rapidly decaying, can a solution to NSP exist for all times $t > 0$?

Theorem 1 gives a negative answer to the above millennium problem.

2. PROOF

This millennium problem was solved in monographs [1], [2]. This solution is sketched in papers [3], [4]. The NSP paradox is stated in paper [5].

The aim of this short paper is to outline the steps of the proof of Theorem 1.

Step 1. Equivalence of the NSP to the integral equation:

$$\tilde{v} = \tilde{F} - (2\pi)^3 \int_0^t \tilde{G}(\xi, t-s)\tilde{v} \star (i\xi\tilde{v})ds. \tag{2.1}$$

This equivalence was proved in [2], p.18.

Step 2. Derivation of an integral inequality for the solution to NSP:

$$b(t) \leq b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} b(s)q(s)ds, \tag{2.2}$$

where $c > 0$ is a constant, and

$$b_0(t) := \||\xi|\tilde{F}(\xi,t)\|, \quad b(t) := \||\xi|\tilde{v}(\xi,t)\|. \tag{2.3}$$

The norm is $L^2(\mathbb{R}^3)$ norm. A derivation of integral inequality (2.2) is given in [2] and in [4], p. 3730.

The integral in (2.2) diverges from the classical point of view. We have to define this integral (see [7] and [8]).

Together with inequality (2.2) we study integral equation

$$q(t) = b_0(t) + c\int_0^t (t-s)^{-\frac{5}{4}} q(s)ds. \tag{2.4}$$

The reason to consider (2.2) and (2.4) together is this: we prove that

$$0 < b(t) \leq q(t), \tag{2.5}$$

and

$$\sup_{t\geq 0} |q(t)| < \infty, \ q(0) = 0. \tag{2.6}$$

From (2.5) and (2.6) it follows that $b(t) = 0$.

This implies $\tilde{v}(x,0) = 0$ even if we assume that $v(x,0)$ is not equal to zero identically, smooth, rapidly decaying, and the solution to (1.1)–(1.3) exists for all $t > 0$. This is the *NSP paradox* discovered in [5]. It solves the millennium problem related to the Navier-Stokes problem.

It is also proved in [2], p. 33, that

$$\sup_{t\geq 0}[\|v(x,t)\| + \|\nabla v(x,t)\|] < \infty. \tag{2.7}$$

Step 3. Theory of some divergent integrals.

Let us define $\Phi(t) := \frac{t^{\lambda-1}}{\Gamma(\lambda)}$, where $t^{\lambda-1} := 0$ if $t < 0$, and $\Gamma(\lambda)$ is the Gamma-function. Consider the Laplace transform of $\Phi(t)$:

$$L\Phi = \int_0^\infty e^{-pt}\Phi(t)dt. \tag{2.8}$$

One checks that for Re$p > 0$ and for any complex λ:

$$L\Phi = \frac{1}{p^\lambda}. \tag{2.9}$$

For $\lambda > 0$ the $L\Phi$ is defined classically, and for Re$\lambda < 0$ it is defined *by analytic continuation* (see [6] and [7]).

Let us solve analytically equation (2.4). Take the Laplace transform of equation (2.4) and get

$$Lq = Lb_0 + c\Gamma(-\frac{1}{4})p^{\frac{1}{4}}Lq, \tag{2.10}$$

so

$$Lq = \frac{Lb_0}{1 + 4c\Gamma(\frac{3}{4})p^{\frac{1}{4}}}. \tag{2.11}$$

One has $c > 0$ and $\Gamma(-\frac{1}{4}) = -4\Gamma(\frac{3}{4})$, where we have used the well-known formula $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ for $z = -\frac{1}{4}$.

Since $b_0(t) > 0$, $b(t) > 0$, $c > 0$, $\Gamma(\frac{3}{4}) > 0$, formula (2.11) implies that $Lq > 0$. We have assumed that $v(x,0) \not\equiv 0$ is smooth and rapidly decaying at infinity together with its derivatives. Therefore, its Fourier transform is smooth and rapidly decaying at infinity. Consequently, $Lb_0 = O(\frac{1}{p})$ as $p \to \infty$, and $Lq = O(\frac{1}{|p|^{\frac{5}{4}}})$ as $p \to \infty$. This implies (see [6], p.271, Lemma 1) that $q(0) = 0$.

This proves the NSP paradox and Theorem 1.

3. Conclusion

Author's solution to the millennium problem related to the Navier-Stokes equations is discussed. It is proved that the Navier-Stokes problem in $\mathbb{R}^3$ with a smooth rapidly decaying not identically zero initial data does not have a solution defined for all $t > 0$. The NSP paradox is formulated. An outline of the author's proof is given so that the reader can see the basic steps.

4. Disclosure statement.

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