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TRIBONACCI CORDIAL LABELING OF SOME SNAKE GRAPHS

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ABSTRACT. An injective function $f : V(G) \to \{T_0, T_1, T_2, \ldots, T_n\}$, where $n = |V(G)| - 1$, is said to be a Tribonacci cordial labeling if the induced function $f^* : E(G) \to \{0, 1\}$ defined by $f^*(uv) = (f(u) + f(v)) \mod 2$ satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ is the number of edges with label 0 and $e_f(1)$ is the number of edges with label 1. A graph that admits a tribonacci cordial labeling is called a Tribonacci cordial graph. In this paper, we determined the Tribonacci Cordial Labeling of Triangular Snake Graph TS_n, Double Triangular Snake Graph $D(TS_n)$, Quadrilateral Snake Graph QS_n, Double Quadrilateral Snake Graph $D(QS_n)$, and Cycle Quadrilateral Snake Graph $C(QS_n)$.

1. INTRODUCTION

The earliest known reference for graph labeling is a 1967 paper by Alex Rosa [5]. In this pioneering work, Rosa introduced concepts like "graceful labeling" and established fundamental principles for assigning labels to graph vertices under specific constraints, which would later influence much of the graph labeling theory. Graph labeling involves assigning integers to vertices, edges, or both, under specific conditions. One of the ways to label graphs is by using number sequences such as Tribonacci and Fibonacci numbers. Fibonacci and Tribonacci numbers are special integer sequences that arise from simple recurrence relations but carry deep mathematical properties. The Fibonacci sequence is formed by adding the two preceding terms, while the Tribonacci sequence extends this idea by summing the three preceding terms. These sequences, known for their natural growth patterns, have found important applications beyond pure mathematics, particularly in graph theory.

Numerous types of graph labeling exist, with cordial labeling being one example. The concept of cordial labeling in graphs was introduced by Cahit [2] in 1987. A graph is considered cordial if its vertices can be labeled with 0s and 1s, such that labeling the edges based on the differences between the endpoint labels results in a balance where the count

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of vertices (or edges) labeled with 1s and 0s differs by no more than one [3]. Several variants of cordial labeling exist such as Fibonacci Cordial Labeling [6], Topological Cordial Labeling [7], Legendre Cordial Labeling [1] and others.

The study on cordial labeling of graphs using tribonacci numbers was conducted by Mitra and Bhoumik [4]. An injective function $f : V(G) \to \{T_0, T_1, T_2, \ldots, T_n\}$ where $n = |V(G)| - 1$ is said to be a *tribonacci cordial labeling* if the induced function $f^* : E(G) \to \{0, 1\}$ defined by $f^*(uv) = (f(u) + f(v)) \mod 2$ satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ is the number of edges with label 0 and $e_f(1)$ is the number of edges with label 1. A graph that admits a tribonacci cordial labeling is called a *Tribonacci cordial graph*.

Tribonacci numbers play a significant role in graph labeling because they extend the structure and complexity found in Fibonacci-based labelings. While Fibonacci numbers follow a two-term recurrence relation, tribonacci numbers are generated using a three-term recurrence, which produces faster growth and richer combinatorial patterns. This allows researchers to design more flexible and varied labeling schemes for graphs, especially when dealing with problems that require higher diversity in vertex or edge labels. Compared to Fibonacci numbers, which are widely used in graceful, harmonious, or cordial labeling, tribonacci numbers introduce an additional dimension that can help in constructing new classes of graph labelings, broadening applications in network design, coding theory, and mathematical modeling. In essence, Fibonacci labelings capture pairwise relationships, while tribonacci labelings can better model more complex, three-way interdependencies within graph structures as can be seen on the succeeding sections.

2. Preliminaries

Definition 2.1. [4] The sequence T_n of Tribonacci numbers is defined by the third-order linear recurrence relation (for $n \geq 0$):

$$T_{n+3} = T_n + T_{n+1} + T_{n+2};.$$

where the first 4 Tribonacci numbers are $T_0 = 0$, $T_1 = 1$, $T_2 = 1\, T_3 = 2$ and $T_4 = 4$.
Definition 2.2. [4] An injective function $f : V(G) \to \{T_0, T_1, T_2, \ldots, T_{n-1}\}$ where $n = |V(G)|$ is said to be a *Tribonacci cordial labeling* if the induced function $f^* : E(G) \to \{0, 1\}$ defined by $f^*(uv) \equiv (f(u)+f(v)) \mod 2$ satisfies the condition $|e_f(0)-e_f(1)| \leq 1$. A graph that admits a Tribonacci cordial labeling is called a *Tribonacci cordial graph*.

3. Tribonacci Cordial Graphs

This section present the results of tribonacci cordial labeling on some snake graphs.
Theorem 3.1. The Tribonnaci Sequence T_n is even if $n \equiv 3 \pmod 4$ or $0 \pmod 4$, $n \in \mathbb{W}$.

Proof. To show that for all $n \in \mathbb{W}$, the tribonnaci number T_n is even when $n \equiv 3 \pmod 4$ or $0 \pmod 4$, we use the recurrence relation for the Tribonnaci sequence in Definition 2.1 for $n \geq 0$. **Case 1:** $n \equiv 0 \pmod 4$
Assume T_k is even when $k \equiv 0 \pmod 4$ for some $k \in \mathbb{W}$. We need to show that the assumption holds up to T_{k+4} where $k + 4 \equiv 0 \pmod 4$. Now,

$$\begin{aligned} T_{k+4} &= T_{k+1} + T_{k+2} + T_{k+3} \\ &= T_{k+1} + T_{k+2} + T_k + T_{k+1} + T_{k+2} \\ &= 2T_{k+1} + 2T_{k+2} + T_k. \end{aligned}$$

Since T_k is even by the inductive hypothesis and both $2T_{k+1}$ and $2T_{k+2}$ are even, T_{k+4} is also even. Thus, for $k \equiv 0 \pmod 4$, T_{k+4} remains even.
Case 2: $n \equiv 3 \pmod 4$
Assume T_k is even when $k \equiv 3 \pmod 4$ for some $k \in \mathbb{W}$. We need to show that the assumption holds up to T_{k+4} where $k+4 \equiv 3 \pmod 4$. Now,

$$\begin{aligned} T_{k+4} &= T_{k+1} + T_{k+2} + T_{k+3} \\ &= T_{k+1} + T_{k+2} + T_k + T_{k+1} + T_{k+2} \\ &= 2T_{k+1} + 2T_{k+2} + T_k. \end{aligned}$$

Since T_k is even by the inductive hypothesis and both $2T_{k+1}$ and $2T_{k+2}$ are even, T_{k+4} is also even. Thus, for $k \equiv 3 \pmod 4$, T_{k+4} remains even.

Therefore, by Principle of Mathematical Induction, T_n is even if $n \equiv 3 \pmod 4$ or $0 \pmod 4$. □

Theorem 3.2. The Tribonnaci Sequence T_n is odd if $n \equiv 1 \pmod 4$ or $2 \pmod 4$, $n \in \mathbb{W}$.

Proof. To show that for all $n \in \mathbb{W}$, the tribonnaci number T_n is even when $n \equiv 1 \pmod 4$ or $2 \pmod 4$, we use the recurrence relation for the Tribonnaci sequence in Definition 2.1 for $n \geq 0$. **Case 1:** $n \equiv 1 \pmod 4$
Assume T_k is odd when $k \equiv 1 \pmod 4$ for some $k \in \mathbb{W}$. We need to show that the assumption holds up to T_{k+4} where $k+4 \equiv 1 \pmod 4$. Now,

$$\begin{aligned} T_{k+4} &= T_{k+1} + T_{k+2} + T_{k+3} \\ &= T_{k+1} + T_{k+2} + T_k + T_{k+1} + T_{k+2} \\ &= 2T_{k+1} + 2T_{k+2} + T_k. \end{aligned}$$

Since T_k is odd by the inductive hypothesis and both $2T_{k+1}$ and $2T_{k+2}$ are even, T_{k+4} odd. Thus, for $k \equiv 1 \pmod 4$, T_{k+4} remains odd.
Case 2: $n \equiv 2 \pmod 4$
Assume T_k is odd when $k \equiv 2 \pmod 4$ for some $k \in \mathbb{W}$. We need to show that the assumption holds up to T_{k+4} where $k+4 \equiv 2 \pmod 4$. Now,

$$\begin{aligned} T_{k+4} &= T_{k+1} + T_{k+2} + T_{k+3} \\ &= T_{k+1} + T_{k+2} + T_k + T_{k+1} + T_{k+2} \\ &= 2T_{k+1} + 2T_{k+2} + T_k. \end{aligned}$$

Since T_k is odd by the inductive hypothesis and both $2T_{k+1}$ and $2T_{k+2}$ are even, T_{k+4} odd. Thus, for $k \equiv 2 \pmod 4$, T_{k+4} also odd.

Therefore, by Principle of Mathematical Induction, T_n is odd if $n \equiv 1 \pmod 4$ or $2 \pmod 4$. □

Theorem 3.3. The Triangular Snake Graph TS_n admits Tribonacci Cordial Labeling if $n = 4m$ for all $m \in \mathbb{N}$.

Proof. Let $V(TS_n) = \{v_i | 1 \leq i \leq n\} \cup \{u_i | 1 \leq i \leq n-1\}$ and $E(TS_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i, v_{i+1} u_i | 1 \leq i \leq n-1\}$ be the vertex set and edge set of the Triangular Snake Graph TS_n, respectively. Hence, the order and the size is $|V(TS_n)| = 2n-1$ and $|E(TS_n)| = 3n-3$, respectively. To prove the theorem, we consider the following cases.
Case 1: $n = 4$.
The order and the size of the Triangular Snake Graph TS_4 is $|V(TS_4)| = 7$ and $|E(TS_4)| =$

9, respectively. Let $f : V(TS_4) \to \{T_0, T_1, T_2, T_3, T_4, T_5, T_6\}$ be the function defined by:

$$f(v_1) = T_0 \qquad f(u_1) = T_3$$
$$f(v_2) = T_1 \qquad f(u_2) = T_4$$
$$f(v_3) = T_2 \qquad f(u_3) = T_5$$
$$f(v_4) = T_6.$$

From the established labeling, the induced edge labels are:

$$f^*(v_1v_2) = 1 \qquad f^*(v_1u_1) = 0 \qquad f^*(v_2u_1) = 1$$
$$f^*(v_2v_3) = 0 \qquad f^*(v_2u_2) = 1 \qquad f^*(v_3u_2) = 1$$
$$f^*(v_3v_4) = 0 \qquad f^*(v_3u_3) = 0 \qquad f^*(v_4u_3) = 0$$

Note that $e_f(0) = 5$ and $e_f(1) = 4$. Thus, $|e_f(0) - e_f(1)| = |5 - 4| = |1| = 1 \leq 1$. Therefore, the Triangular Snake Graph TS_4 is a Tribonacci Cordial Graph.

Case 2: $n = 4m$ for all $m \geq 2$.

Let $f : V(TS_n) \to \{T_0, T_1, T_2, \ldots, T_{2n-2}\}$ be the function defined by:

$$f(v_i) = \begin{cases} T_0, & \text{if } i = 1 \\ T_{2i-3}, & \text{if } i \text{ is even}, \quad 2 \leq i \leq \frac{n}{2} \\ T_{2i-4}, & \text{if } i \text{ is odd}, \quad 3 \leq i \leq \frac{n}{2} + 1 \\ T_{\frac{5n}{2}-i}, & \text{if } \quad \frac{n}{2} + 2 \leq i \leq n \end{cases}$$

$$f(u_i) = \begin{cases} T_{2i}, & \text{if } i \text{ is even}, \quad 2 \leq i \leq \frac{n}{2} \\ T_{2i+1}, & \text{if } i \text{ is odd}, \quad 1 \leq i \leq \frac{n}{2} - 1 \\ T_{\frac{n}{2}+i}, & \text{if } \quad \frac{n}{2} + 1 \leq i \leq n - 1 \end{cases}$$

From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if} \quad 2 \leq i \leq \frac{n}{2} + 1 \quad \text{or} \\ & i \text{ is even}, \quad \frac{n}{2} + 2 \leq i \leq n - 2 \\ 1, & \text{if} \quad i = 1 \quad \text{or} \\ & i \text{ is odd}, \quad \frac{n}{2} + 3 \leq i \leq n - 1 \end{cases}$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if} \quad i = 1, \frac{n}{2} + 1 \quad \text{or} \\ & i \text{ is even}, \quad \frac{n}{2} + 2 \leq i \leq n - 2 \\ 1, & \text{if} \quad 2 \leq i \leq \frac{n}{2} \quad \text{or} \\ & i \text{ is odd}, \quad \frac{n}{2} + 3 \leq i \leq n - 1 \end{cases}$$

$$f^*(v_{i+1}u_i) = \begin{cases} 0, & \text{if} \quad \frac{n}{2} + 1 \leq i \leq n - 1 \\ 1, & \text{if} \quad 1 \leq i \leq \frac{n}{2} \end{cases}$$

Note that $e_f(0) = \frac{n}{2} + \frac{n-4}{4} + 1 + 1 + \frac{n-4}{4} + \frac{n}{2} - 1 = \frac{3n}{2} - 1$ and $e_f(1) = 1 + \frac{n-4}{4} + \frac{n}{2} - 1 + \frac{n-4}{4} + \frac{n}{2} = \frac{3n}{2} - 2$. Thus, $|e_f(0) - e_f(1)| = |(\frac{3n}{2} - 1) - (\frac{3n}{2} - 2)| = |1| = 1 \leq 1$. Therefore, the Triangular Snake Graph TS_n is a Tribonacci Cordial Graph if $n = 4m$ for all $m \geq 2$.

Based on the above cases, it follows that the Triangular Snake Graph TS_n is a Tribonacci Cordial Graph if $n = 4m$ for all $m \in \mathbb{N}$. □

Theorem 3.4. The Double Triangular Snake Graph $D(TS_n)$ admits Tribonacci Cordial Labeling for all $n \geq 2$.

Proof. Let $V(D(TS_n)) = \{v_i | 1 \leq i \leq n\} \cup \{u_i, w_i | 1 \leq i \leq n-1\}$ and $E(D(TS_n)) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i, v_{i+1} u_i, v_i w_i, v_{i+1} w_i | 1 \leq i \leq n-1\}$ be the vertex set and edge set of the Double Triangular Snake Graph $D(TS_n)$, respectively. Hence, the order and the size of the Double Triangular Snake Graph $D(TS_n)$ is $|V(D(TS_n))| = 3n-2$ and $|E(D(TS_n))| = 5n-5$, respectively. To prove the theorem, we consider the following cases:

Case 1: n is even.

Subcase 1.1: $n = 2$.

Let $V(D(TS_2)) = \{v_1, v_2, u_1, w_1\}$ and $E(D(TS_2)) = \{v_1v_2, v_1u_1, v_2u_1, v_1w_1, v_2w_1\}$. The order and the size is $|V(D(TS_2))| = 4$ and $|E(D(TS_2))| = 5$, respectively. Let $f : V(D(TS_2)) \to \{T_0, T_1, T_2, T_3, T_4\}$ be the function defined by:

$$f(v_1) = T_0 \qquad f(u_1) = T_1$$
$$f(v_2) = T_2 \qquad f(w_1) = T_3$$

Hence, the induced edge labels are:

$$f^*(v_1v_2) = 1 \qquad f^*(v_1w_1) = 0$$
$$f^*(v_1u_1) = 1 \qquad f^*(v_2u_1) = 0$$
$$f^*(v_2w_1) = 1.$$

Note that $e_f(0) = 2$ and $e_f(1) = 3$. Thus, $|e_f(0) - e_f(1)| = |2 - 3| = |-1| = 1 \leq 1$. Therefore, the Double Triangular Snake Graph $D(TS_2)$ is a Tribonacci Cordial Graph.

Subcase 1.2: $n \geq 4$.

Let $f : V(D(TS_n)) \to \{T_0, T_1, T_2, \ldots, T_{3n-3}\}$ be the function defined by:

$$\begin{aligned} f(v_1) &= T_0 \\ f(v_{i+1}) &= T_{3i-1}, \quad 1 \leq i \leq n-1 \\ f(u_i) &= T_{3i-2}, \quad 1 \leq i \leq n-1 \\ f(w_i) &= T_{3i}, \quad 1 \leq i \leq n-1. \end{aligned}$$

From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$
$$f^*(v_iu_i) = 1, \qquad 1 \leq i \leq n-1$$
$$f^*(v_{i+1}u_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$
$$f^*(v_iw_i) = 0, \qquad 1 \leq i \leq n-1$$
$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$

Note that $e_f(0) = \frac{n-2}{2} + \frac{n}{2} + n - 1 + \frac{n-2}{2} = \frac{5n}{2} - 3$ and $e_f(1) = \frac{n}{2} + n - 1 + \frac{n-2}{2} + \frac{n}{2} = \frac{5n}{2} - 2$. Thus, $|e_f(0) - e_f(1)| = |(\frac{5n}{2} - 3) - (\frac{5n}{2} - 2)| = |-1| = 1 \leq 1$. Therefore, the Double Triangular Snake Graph $D(TS_n)$ is a Tribonacci Cordial Graph for all $n \geq 4$ and n is even.

Case 2: n is odd, $n \geq 3$.
Let $f : V(D(TS_n)) \to \{T_0, T_1, T_2, \ldots, T_{3n-3}\}$ be the function defined by:

$$f(v_1) = T_0$$
$$f(v_{i+1}) = T_{3i-1}, \quad 1 \leq i \leq n-1$$
$$f(u_i) = T_{3i-2}, \quad 1 \leq i \leq n-1$$
$$f(w_i) = T_{3i}, \quad 1 \leq i \leq n-1.$$

From the established labeling, the induced edge labels are:

$$f^*(v_i v_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \end{cases}$$
$$f^*(v_i u_i) = 1, \qquad 1 \leq i \leq n-1$$
$$f^*(v_{i+1} u_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$
$$f^*(v_i w_i) = 0, \qquad 1 \leq i \leq n-1$$
$$f^*(v_{i+1} w_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \end{cases}$$

Note that $e_f(0) = \frac{n-1}{2} + \frac{n-1}{2} + n - 1 + \frac{n-1}{2} = \frac{5n-5}{2}$ and $e_f(1) = \frac{n-1}{2} + n - 1 + \frac{n-1}{2} + \frac{n-1}{2} = \frac{5n-5}{2}$. Thus, $|e_f(0) - e_f(1)| = |(\frac{5n-5}{2}) - (\frac{5n-5}{2})| = |0| = 0 \leq 1$. Therefore, the Double Triangular Snake Graph $D(TS_n)$ is a Tribonacci Cordial Graph for all $n \geq 3$ and n is odd.

Based on the above cases, it follows that the Double Triangular Snake Graph $D(TS_n)$ is a Tribonacci Cordial Graph for all $n \geq 2$. □

Theorem 3.5. The Quadrilateral Snake Graph QS_n admits Tribonacci Cordial Labeling for all $n \geq 2$.

Proof. Let $V(QS_n) = \{v_i | 1 \leq i \leq n\} \cup \{u_i, w_i | 1 \leq i \leq n-1\}$ and $E(QS_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i u_i, v_{i+1} w_i, u_i w_i | 1 \leq i \leq n-1\}$ be the vertex set and edge set of the Quadrilateral Snake Graph QS_n, respectively. The order and the size is $|V(QS_n)| = 3n-2$ and $|E(QS_n)| = 4n-4$, respectively. To prove the theorem, we consider the following cases:
Case 1: n is even and $n \geq 2$.
Let $f : V(Q_n) \to \{T_0, T_1, T_2, \ldots, T_{3n-3}\}$ be the function defined by:

$$f(v_i) = T_{3i-3}, \quad 1 \leq i \leq n$$
$$f(u_i) = T_{3i-2}, \quad 1 \leq i \leq n-1$$
$$f(w_i) = T_{3i-1}, \quad 1 \leq i \leq n-1.$$

From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \le i \le n-1 \\ 1, & \text{if } i \text{ is even, } 2 \le i \le n-2 \end{cases}$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \le i \le n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \le i \le n-1 \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \le i \le n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \le i \le n-1 \end{cases}$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \le i \le n-1 \\ 1, & \text{if } i \text{ is even, } 2 \le i \le n-2 \end{cases}$$

Note that $e_f(0) = \frac{n}{2} + \frac{n-2}{2} + \frac{n-2}{2} + \frac{n}{2} = \frac{4n-4}{2} = 2n-2$ and $e_f(1) = \frac{n-2}{2} + \frac{n}{2} + \frac{n}{2} + \frac{n-2}{2} = \frac{4n-4}{2} = 2n-2$. Thus, $|e_f(0) - e_f(1)| = |(2n-2) - (2n-2)| = |0| = 0 \le 1$. Therefore, the Quadrilateral Snake Graph QS_n is a Tribonacci Cordial Graph for all $n \ge 2$ and n is even.

Case 2: n is odd and $n \ge 3$.

Let $f : V(Q_n) \to \{T_0, T_1, T_2, \ldots, T_{3n-2}\}$ be the function defined by:

$$f(v_i) = T_{3i-3}, \quad 1 \le i \le n$$
$$f(u_i) = T_{3i-2}, \quad 1 \le i \le n-1$$
$$f(w_i) = T_{3i-1}, \quad 1 \le i \le n-1.$$

From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \le i \le n-2 \\ 1, & \text{if } i \text{ is even, } 2 \le i \le n-1 \end{cases}$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \le i \le n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \le i \le n-2 \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \le i \le n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \le i \le n-2 \end{cases}$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \le i \le n-2 \\ 1, & \text{if } i \text{ is even, } 2 \le i \le n-1 \end{cases}$$

Note that $e_f(0) = \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{4n-4}{2} = 2n-2$ and $e_f(1) = \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{4n-4}{2} = 2n-2$. Thus, $|e_f(0) - e_f(1)| = |(2n-2) - (2n-2)| = |0| = 0 \le 1$. Therefore, the Quadrilateral Snake Graph QS_n is a Tribonacci Cordial Graph for all $n \ge 3$ and n is odd.

Based on the above cases, it follows that the Quadrilateral Snake Graph Q_n is a Tribonacci Cordial Graph for all $n \ge 2$. □

Theorem 3.6. The Cycle Quadrilateral Snake Graph $C(QS_n)$ admits Tribonacci Cordial Labeling for all $n \ge 3$.

Proof. Let $V(C(QS_n)) = \{v_i, u_i, w_i | 1 \le i \le n\}$ and $E(C(QS_n)) = \{v_iv_{i+1}, v_{i+1}w_i | 1 \le i \le n-1\} \cup \{v_1v_n\} \cup \{v_1w_n\} \cup \{u_iw_i, v_iu_i | 1 \le i \le n\}$ be the vertex set and edge set of the Cycle Quadrilateral Snake Graph $C(QS_n)$, respectively. Hence, the order and the size

is $|V(C(QS_n))| = 3n$ and $|E(C(QS_n))| = 4n$, respectively. To prove the theorem, we consider the following cases:
Case 1: n is even and $n \geq 4$.

Subcase 1.1: $n = 4m$ where $m \in \mathbb{N}$

Let $f : V(C(QS_n)) \to \{T_0, T_1, T_2, \ldots, T_{3n-1}\}$ be the function defined by:

$$f(v_i) = T_{3i-3}$$
$$f(u_i) = T_{3i-2}$$
$$f(w_i) = T_{3i-1}$$

where $1 \leq i \leq n$. From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0 & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1 & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$
$$f^*(v_1v_n) = 1$$
$$f^*(v_iu_i) = \begin{cases} 0 & \text{if } i \text{ is even, } 2 \leq i \leq n \\ 1 & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$
$$f^*(v_{i+1}w_i) = \begin{cases} 0 & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ 1 & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$
$$f^*(v_1w_n) = 0$$
$$f^*(u_iw_i) = \begin{cases} 0 & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1 & \text{if } i \text{ is even, } 2 \leq i \leq n. \end{cases}$$

Note that $e_f(0) = \frac{n}{2} + \frac{n}{2} + \frac{n-2}{2} + 1 + \frac{n}{2} = \frac{4n}{2} = 2n$ and $e_f(1) = \frac{n-2}{2} + 1 + \frac{n}{2} + \frac{n}{2} + \frac{n}{2} = \frac{4n}{2} = 2n$. Thus, $|e_f(0) - e_f(1)| = |2n - 2n| = |0| = 0 \leq 1$. Therefore, the Cycle Quadrilateral Snake Graph $C(QS_n)$ is a Tribonacci Cordial Graph for all $n = 4m$ where $m \in \mathbb{N}$.

Subcase 1.2: $n = 4m + 2$ where $m \in \mathbb{N}$.
Let $f : V(C(QS_n)) \to \{T_0, T_1, T_2, \ldots, T_{3n-1}\}$ be the function defined by:

$$f(v_i) = T_{3i-3}$$
$$f(u_i) = T_{3i-2}$$
$$f(w_i) = T_{3i-1},$$

where $1 \leq i \leq n$. From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$
$$f^*(v_1v_n) = 0$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even}, 2 \leq i \leq n \\ 1, & \text{if } i \text{ is odd}, 1 \leq i \leq n-1 \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is even}, 2 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is odd}, 1 \leq i \leq n-1 \end{cases}$$

$$f^*(v_1w_n) = 1$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd}, 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even}, 2 \leq i \leq n. \end{cases}$$

Note that $e_f(0) = \frac{n}{2}+1+\frac{n}{2}+\frac{n-2}{2}+\frac{n}{2} = \frac{4n}{2} = 2n$ and $e_f(1) = \frac{n-2}{2}+\frac{n}{2}+\frac{n}{2}+1+\frac{n}{2} = \frac{4n}{2} = 2n$. Thus, $|e_f(0) - e_f(1)| = |2n - 2n| = |0| = 0 \leq 1$. Therefore, the Cycle Quadrilateral Snake Graph $C(QS_n)$ is a Tribonacci Cordial Graph for all $n = 4m + 2$ where $m \in \mathbb{N}$.

Case 2: n is odd and $n \geq 3$

Subcase 2.1: $n = 4m - 1$ where $m \in \mathbb{N}$

Let $f : V(C(QS_n)) \to \{T_0, T_1, T_2, \ldots, T_{3n-1}\}$ be the function defined by:

$$f(v_i) = T_{3i-3}$$
$$f(u_i) = T_{3i-2}$$
$$f(w_i) = T_{3i-1}$$

where $1 \leq i \leq n$. From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd}, 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even}, 2 \leq i \leq n-1 \end{cases}$$

$$f^*(v_1v_n) = 1$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even}, 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd}, 1 \leq i \leq n \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is even}, 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd}, 1 \leq i \leq n-2 \end{cases}$$

$$f^*(v_1w_n) = 0$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd}, 1 \leq i \leq n \\ 1, & \text{if } i \text{ is even}, 2 \leq i \leq n-1 \end{cases}$$

Note that $e_f(0) = \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + 1 + \frac{n+1}{2} = \frac{4n}{2} = 2n$ and $e_f(1) = \frac{n-1}{2} + 1 + \frac{n+1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{4n}{2} = 2n$. Thus, $|e_f(0) - e_f(1)| = |2n - 2n| = |0| = 0 \leq 1$. Therefore, the Cycle Quadrilateral Snake Graph $C(QS_n)$ is a Tribonacci Cordial Graph for all $n = 4m - 1$ where $m \in \mathbb{N}$.

Subcase 2.2: $n = 4m + 1$ where $m \in \mathbb{N}$

Let $f : V(C(QS_n)) \to \{T_0, T_1, T_2, \ldots, T_{3n-1}\}$ be the function defined by:

$$f(v_i) = T_{3i-3}$$
$$f(u_i) = T_{3i-2}$$
$$f(w_i) = T_{3i-1}$$

where $1 \leq i \leq n$. From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

$$f^*(v_1v_n) = 0$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \end{cases}$$

$$f^*(v_1w_n) = 1$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1. \end{cases}$$

Note that $e_f(0) = \frac{n-1}{2} + 1 + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n+1}{2} = \frac{4n}{2} = 2n$ and $e_f(1) = \frac{n-1}{2} + \frac{n+1}{2} + \frac{n-1}{2} + 1 + \frac{n-1}{2} = \frac{4n}{2} = 2n$. Thus, $|e_f(0) - e_f(1)| = |2n - 2n| = |0| = 0 \leq 1$. Therefore, the Cycle Quadrilateral Snake Graph $C(QS_n)$ is a Tribonacci Cordial Graph for all $n = 4m + 1$ where $m \in \mathbb{N}$.

Based on the above cases, it follows that the Cycle Quadrilateral Snake Graph $C(QS_n)$ is a Tribonacci Cordial Graph for all $n \geq 3$. □

Theorem 3.7. The Double Quadrilateral Snake Graph $D(QS_n)$ admits Tribonacci Cordial Labeling for all $n \geq 2$.

Proof. Let $V(D(QS_n)) = \{v_i | 1 \leq i \leq n\} \cup \{u_i, w_i, x_i, y_i | 1 \leq i \leq n-1\}$ and $E(D(QS_n)) = \{v_iv_{i+1} | 1 \leq i \leq n-1\} \cup \{v_iu_i, v_{i+1}w_i, u_iw_i, v_ix_i, v_{i+1}y_i, x_iy_i | 1 \leq i \leq n-1\}$ be the vertex set and edge set of the Double Quadrilateral Snake Graph $D(QS_n)$, respectively. Hence, the order and the size of the Double Quadrilateral Snake Graph $D(QS_n)$ is $|V(QS_n)| = 5n - 4$ and $E(QS_n)| = 7n - 7$, respectively. To prove the theorem, we consider the following cases:

Case 1: n is even and $n \geq 2$

Let $f : V(D(QS_n)) \rightarrow \{T_0, T_1, T_2, \ldots, T_{5n-5}\}$ be the function defined by:

$$f(v_i) = T_{5i-5}, \quad 1 \leq i \leq n$$
$$f(u_i) = T_{5i-4}, \quad 1 \leq i \leq n-1$$
$$f(w_i) = T_{5i-3}, \quad 1 \leq i \leq n-1$$
$$f(x_i) = T_{5i-2}, \quad 1 \leq i \leq n-1$$
$$f(y_i) = T_{5i-1}, \quad 1 \leq i \leq n-1.$$

From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$

$$f^*(v_ix_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$

$$f^*(v_{i+1}y_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \end{cases}$$

$$f^*(x_iy_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-2 \end{cases}$$

Note that $e_f(0) = \frac{n-2}{2} + \frac{n-2}{2} + \frac{n}{2} + \frac{n}{2} + \frac{n}{2} + \frac{n-2}{2} + \frac{n}{2} = \frac{7n-6}{2}$ and $e_f(1) = \frac{n}{2} + \frac{n}{2} + \frac{n-2}{2} + \frac{n-2}{2} + \frac{n-2}{2} + \frac{n}{2} + \frac{n-2}{2} = \frac{7n-8}{2}$. Thus, $|e_f(0) - e_f(1)| = |(\frac{7n-6}{2}) - (\frac{7n-8}{2})| = |1| = 1 \leq 1$. Therefore, the Double Quadrilateral Snake Graph $D(QS_n)$ is a Tribonacci Cordial Graph for all $n \geq 2$ and n is even.

Case 2: n is odd and $n \geq 3$

Let $f : V(D(QS_n)) \to \{T_0, T_1, T_2, \ldots, T_{5n-5}\}$ be the function defined by:

$$\begin{aligned} f(v_i) &= T_{5i-5}, \quad 1 \leq i \leq n \\ f(u_i) &= T_{5i-4}, \quad 1 \leq i \leq n-1 \\ f(w_i) &= T_{5i-3}, \quad 1 \leq i \leq n-1 \\ f(x_i) &= T_{5i-2}, \quad 1 \leq i \leq n-1 \\ f(y_i) &= T_{5i-1}, \quad 1 \leq i \leq n-1. \end{aligned}$$

From the established labeling, the induced edge labels are:

$$f^*(v_iv_{i+1}) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \end{cases}$$

$$f^*(v_iu_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \end{cases}$$

$$f^*(v_{i+1}w_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

$$f^*(u_iw_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

$$f^*(v_ix_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

$$f^*(v_{i+1}y_i) = \begin{cases} 0, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \\ 1, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \end{cases}$$

$$f^*(x_iy_i) = \begin{cases} 0, & \text{if } i \text{ is odd, } 1 \leq i \leq n-2 \\ 1, & \text{if } i \text{ is even, } 2 \leq i \leq n-1 \end{cases}$$

Note that $e_f(0) = \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{7n-7}{2}$ and $e_f(1) = \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} + \frac{n-1}{2} = \frac{7n-7}{2}$. Thus, $|e_f(0) - e_f(1)| = |(\frac{7n-7}{2}) - (\frac{7n-7}{2})| = |0| = 0 \leq 1$. Therefore, the Double Quadrilateral Snake Graph $D(QS_n)$ is a Tribonacci Cordial Graph for all $n \geq 3$ and n is odd.

Based on the above cases, it follows that the Double Quadrilateral Snake Graph $D(QS_n)$ is a Tribonacci Cordial Graph for all $n \geq 2$.

□

4. Conclusion

The study shows that Tribonacci cordial labeling effectively extends sequence-based graph labeling to snake graphs, enriching the theory beyond Fibonacci-based approaches. Future work may apply this method to broader graph families, compare with other number sequence labelings, and explore algorithmic and practical applications in areas such as network design and coding theory.

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