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ON SOME TRIGONOMETRIC AND INVERSE TRIGONOMETRIC INTEGRAL FORMULAS

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ABSTRACT. This paper presents a new collection of trigonometric and inverse trigonometric integral formulas based on a known integral result. Some of these formulas yield zero, while others are notable for their connection to well-known mathematical constants, such as π, $\sqrt{2}$, and the Catalan constant. Comprehensive proofs are provided for all results, and an open problem is posed to inspire further investigation.

1. INTRODUCTION

1.1. **Context.** Integral formulas are indispensable tools for solving problems in a variety of disciplines, including mathematics, physics and engineering. They often represent key stages in the resolution of complex analytical challenges. Comprehensive collections of these formulas can be found in well-established reference works, particularly in the books [4, 12, 14, 13]. Despite the extensive body of known results, the development of broader and more versatile integral formulas remains a dynamic field of research, as evidenced by recent contributions in [16, 17, 18, 19, 6, 7, 8, 11, 1]. Such formulas are also important for validating or invalidating integral inequalities. Further information on this topic can be found in the books [2, 3, 15, 20] and recent studies in [9, 10].

1.2. **Motivation.** Among the numerous results presented in [13], one encounters the following elegant trigonometric and inverse trigonometric integral formula:

$$\int_0^{\pi} \arctan[\cos(x)]dx = 0.$$

See [13, Entry 4.512]. It is stated directly, without proof. This naturally raises several intriguing questions: Can this formula be proved by a direct and elementary argument? Moreover, can it be generalized to a wider class of integral formulas involving integrals of

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the form

$$\int_0^\pi f(x)\arctan[g(x)]dx,$$

where f and g are suitably chosen functions? Such generalizations could uncover deeper structural properties of trigonometric and inverse trigonometric integrals, and might lead to new formulas of independent interest.

1.3. **Organization.** The structure of the paper is as follows: In Section 2, we introduce a family of integral formulas of the form

$$\int_0^\pi f(x)\arctan[g(x)]dx = 0,$$

where f and g satisfy certain symmetry or functional conditions. Section 3 extends this discussion by presenting related examples in which the integrals do not vanish. Some of these evaluations lead to expressions involving well-known mathematical constants. An open problem is posed in Section 4, and final remarks are given in Section 5.

2. First series of integral formulas

The proposition below provides a generalization of the integral formula in [13, Entry 4.512], introducing an auxiliary function f.

Proposition 2.1. *Let $f : (0,\pi) \to \mathbb{R}$ be a function such that, for any $x \in (0,\pi)$, $f(\pi - x) = f(x)$. Then we have*

$$\int_0^\pi f(x)\arctan[\cos(x)]dx = 0,$$

provided that the integral exists (or converges).

Proof. Making the change of variables $x = \pi - y$, and using $\cos(\pi - y) = -\cos(y)$, the fact that $\arctan$ is an odd function and the property $f(\pi - y) = f(y)$, we have

$$\begin{aligned} I &= \int_0^\pi f(x)\arctan[\cos(x)]dx = \int_\pi^0 f(\pi - y)\arctan[\cos(\pi - y)](-dy) \\ &= \int_0^\pi f(y)\arctan[-\cos(y)]dy = -\int_0^\pi f(y)\arctan[\cos(y)]dy = -I. \end{aligned}$$

Since I exists, we necessarily have $I = 0$, concluding the proof. □

Thus, a key part of the proof is the symmetry assumption made on f.

Some examples of applications are given below.

- Selecting $f(x) = 1$ in Proposition 2.1, we obtain

$$\int_0^\pi \arctan[\cos(x)]dx = 0.$$

  We rediscover [13, Entry 4.512]. Proposition 2.1 thus provides a generalization and a mathematical explanation of this relatively complex result.
- Selecting $f(x) = \sin(x)$ in Proposition 2.1, we have

$$\int_0^\pi \sin(x)\arctan[\cos(x)]dx = 0.$$

- Selecting $f(x) = \cos(2x)$ in Proposition 2.1, we get
$$\int_0^{\pi} \cos(2x) \arctan[\cos(x)] dx = 0.$$
- Selecting $f(x) = \cos^2(x)$ in Proposition 2.1, we have
$$\int_0^{\pi} \cos^2(x) \arctan[\cos(x)] dx = 0.$$
- Selecting $f(x) = \sin^2(x)$ in Proposition 2.1, we obtain
$$\int_0^{\pi} \sin^2(x) \arctan[\cos(x)] dx = 0.$$
- Selecting $f(x) = (x - \pi/2)^2$ in Proposition 2.1, we get
$$\int_0^{\pi} \left(x - \frac{\pi}{2}\right)^2 \arctan[\cos(x)] dx = 0.$$

The proposition below completes Proposition 2.1 by considering $\arctan(\sin)$ instead of $\arctan(\cos)$, and another symmetric assumption on f.

Proposition 2.2. *Let $f : (0, \pi) \to \mathbb{R}$ be a function such that, for any $x \in (0, \pi)$, $f(\pi - x) = -f(x)$. Then we have*
$$\int_0^{\pi} f(x) \arctan[\sin(x)] dx = 0,$$
provided that the integral exists (or converges).

Proof. Making the change of variables $x = \pi - y$, and using $\sin(\pi - y) = \sin(y)$ and the property $f(\pi - y) = -f(y)$, we have
$$J = \int_0^{\pi} f(x) \arctan[\sin(x)] dx = \int_{\pi}^{0} f(\pi - y) \arctan[\sin(\pi - y)](-dy)$$
$$= -\int_0^{\pi} f(y) \arctan[\sin(y)] dy = -J.$$
Since J exists, we necessarily have $J = 0$, ending the proof. □

Some examples of applications are given below.

- Selecting $f(x) = \cos(x)$ in Proposition 2.2, we have
$$\int_0^{\pi} \cos(x) \arctan[\sin(x)] dx = 0.$$
- Selecting $f(x) = \text{cotan}(x)$ in Proposition 2.2, we obtain
$$\int_0^{\pi} \text{cotan}(x) \arctan[\sin(x)] dx = 0.$$
- Selecting $f(x) = (x - \pi/2)^3$ in Proposition 2.2, we get
$$\int_0^{\pi} \left(x - \frac{\pi}{2}\right)^3 \arctan[\sin(x)] dx = 0.$$

The proposition below also completes Proposition 2.1 by considering $\arctan(\text{cotan})$ instead of $\arctan(\cos)$, under the same assumption on f.

Proposition 2.3. *Let* $f : (0, \pi) \to \mathbb{R}$ *be a function such that, for any* $x \in (0, \pi)$, $f(\pi - x) = f(x)$. *Then we have*

$$\int_0^{\pi} f(x) \arctan[\operatorname{cotan}(x)] dx = 0,$$

provided that the integral exists (or converges).

Proof. Making the change of variables $x = \pi - y$, and using $\operatorname{cotan}(\pi - y) = -\operatorname{cotan}(y)$, the fact that $\arctan$ is an odd function and the property $f(\pi - y) = f(y)$, we have

$$K = \int_0^{\pi} f(x) \arctan[\operatorname{cotan}(x)] dx = \int_{\pi}^{0} f(\pi - y) \arctan[\operatorname{cotan}(\pi - y)](-dy)$$
$$= \int_0^{\pi} f(y) \arctan[-\operatorname{cotan}(y)] dy = -\int_0^{\pi} f(y) \arctan[\operatorname{cotan}(y)] dy = -K.$$

Since K exists, we necessarily have $K = 0$, completing the proof. □

Some examples of applications are given below.

- Selecting $f(x) = 1$ in Proposition 2.3, we have
$$\int_0^{\pi} \arctan[\operatorname{cotan}(x)] dx = 0.$$
- Selecting $f(x) = \sin(x)$ in Proposition 2.3, we get
$$\int_0^{\pi} \sin(x) \arctan[\operatorname{cotan}(x)] dx = 0.$$
- Selecting $f(x) = \cos(2x)$ in Proposition 2.3, we obtain
$$\int_0^{\pi} \cos(2x) \arctan[\operatorname{cotan}(x)] dx = 0.$$
- Selecting $f(x) = \cos^2(x)$ in Proposition 2.3, we have
$$\int_0^{\pi} \cos^2(x) \arctan[\operatorname{cotan}(x)] dx = 0.$$
- Selecting $f(x) = \sin^2(x)$ in Proposition 2.3, we get
$$\int_0^{\pi} \sin^2(x) \arctan[\operatorname{cotan}(x)] dx = 0.$$
- Selecting $f(x) = (x - \pi/2)^2$ in Proposition 2.3, we obtain
$$\int_0^{\pi} \left(x - \frac{\pi}{2}\right)^2 \arctan[\operatorname{cotan}(x)] dx = 0.$$

3. Some notable integral formulas

We now focus on notable integral formulas for integrals of the form

$$\int_0^{\pi} f(x) \arctan[g(x)] dx$$

and having a result different to 0. The proposition below is the first result of this kind, involving the mathematical constants $\sqrt{2}$ and π.

Proposition 3.1. *We have*

$$\int_0^{\pi} \cos(x) \arctan[\cos(x)] dx = \left[\sqrt{2} - 1\right] \pi.$$

Proof. Using standard primitive techniques, we get

$$\begin{aligned}&\int_0^{\pi} \cos(x)\arctan[\cos(x)]dx\\&= \Bigg[-x-\sqrt{2}\arctan\left[1-\sqrt{2}\tan\left(\frac{x}{2}\right)\right]\\&+\sqrt{2}\arctan\left[\sqrt{2}\tan\left(\frac{x}{2}\right)+1\right]+\sin(x)\arctan[\cos(x)]\Bigg]_{x=0}^{x=\pi}\\&=\left[\sqrt{2}-1\right]\pi.\end{aligned}$$

The proof is completed. □

The proposition below is about a trigonometric and inverse trigonometric integral formula involving π.

Proposition 3.2. *We have*

$$\int_0^{\pi} \sin(2x)\arctan[\cos(x)]dx = \pi - 2.$$

Proof. Using standard primitive techniques, we get

$$\begin{aligned}&\int_0^{\pi} \sin(2x)\arctan[\cos(x)]dx\\&=\left[\cos(x)-\frac{1}{2}\cos(2x)\arctan[\cos(x)]-\frac{3}{2}\arctan[\cos(x)]\right]_{x=0}^{x=\pi}\\&=\pi-2.\end{aligned}$$

This concludes the proof. □

The proposition below presents another interesting integral formula depending on the Catalan constant.

Proposition 3.3. *We have*

$$\int_0^{\pi} \tan(x)\arctan[\cos(x)]dx = 2G,$$

where G denotes the Catalan constant.

Proof. By the Chasles integral relation and the change of variables $x = \pi - y$, $\tan(\pi-y) = -\tan(y)$, $\cos(\pi - y) = -\cos(y)$ and the fact that $\arctan$ is an odd function, we have

$$\begin{aligned}&\int_0^{\pi} \tan(x)\arctan[\cos(x)]dx\\&=\int_0^{\pi/2} \tan(x)\arctan[\cos(x)]dx+\int_{\pi/2}^{\pi} \tan(x)\arctan[\cos(x)]dx\\&=\int_0^{\pi/2} \tan(x)\arctan[\cos(x)]dx+\int_{\pi/2}^{0} \tan(\pi-y)\arctan[\cos(\pi-y)](-dy)\\&=\int_0^{\pi/2} \tan(x)\arctan[\cos(x)]dx+\int_0^{\pi/2} [-\tan(y)]\{-\arctan[\cos(y)]\}dy\\&=2\int_0^{\pi/2} \tan(x)\arctan[\cos(x)]dx. \qquad (3.1)\end{aligned}$$

Making the change of variables $z = \cos(x)$ and using a well-known integral representation of the Catalan constant (see [5, Equation (1)]), we obtain

$$\int_0^{\pi/2} \tan(x) \arctan[\cos(x)]dx = -\int_1^0 \frac{1}{z} \arctan(z)dz = \int_0^1 \frac{1}{z} \arctan(z)dz = G. \tag{3.2}$$

It follows from Equations (3.1) and (3.2) that

$$\int_0^{\pi} \tan(x) \arctan[\cos(x)]dx = 2G.$$

This ends the proof. □

In particular, we derive the following integral representation of the Catalan constant, not listed in [5]:

$$G = \frac{1}{2} \int_0^{\pi} \tan(x) \arctan[\cos(x)]dx.$$

The proposition below presents another integral of interest whose result is simply 2.

Proposition 3.4. *We have*

$$\int_0^{\pi} \cos(x) \arctan[\cotan(x)]dx = 2.$$

Proof. Using standard primitive techniques, we get

$$\int_0^{\pi} \cos(x) \arctan[\cotan(x)]dx = [\sin(x) \arctan[\cotan(x)] - \cos(x)]_{x=0}^{x=\pi}$$
$$= 2.$$

The proof is concluded. □

The proposition below offers another integral formula depending on π^3.

Proposition 3.5. *We have*

$$\int_0^{\pi} x \arctan[\cotan(x)]dx = -\frac{\pi^3}{12}.$$

Proof. Using standard primitive techniques, we get

$$\int_0^{\pi} x \arctan[\cotan(x)]dx = \left[\frac{1}{6}x^2 \{x + 3\arctan[\cotan(x)]\}\right]_{x=0}^{x=\pi}$$
$$= -\frac{\pi^3}{12}.$$

The proof is completed. □

4. Open problem

We can numerically confirm the validity of the following integral formula:

$$\int_0^{\pi} \tan(x) \arctan[\cot(x)]dx = \pi \log(2).$$

Remarkably, it involves the product of two fundamental mathematical constants: π and $\log(2)$. However, the absence of a closed form for $\arctan[\cot(x)]$ poses an obstacle to a rigorous proof. This remains to be discovered, making it an intriguing challenge for further investigation.

5. Conclusion

In conclusion, this paper presents new integral formulas involving trigonometric and inverse trigonometric functions. Some of these formulas are connected to well-known constants, such as π and the Catalan constant. One possible direction for future research is to extend the class of admissible functions f and g in order to uncover broader families of vanishing or non-vanishing integrals. Another approach would be to explore possible generalizations to higher dimensions, for example by studying analogous double or multiple integrals. Examining connections with the Fourier transform, where similar structures often arise, would also be of interest. Finally, addressing the open problem posed in Section 4 may reveal deeper structural properties underlying the phenomena discussed here.

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