ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 3 (2025), 363-378
DOI: 10.62072/acm.2025.080303
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

ON SOME CONNECTIONS BETWEEN HILBERT AND HARDY TYPE INTEGRAL INEQUALITIES

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ABSTRACT. This article investigates some new connections between the Hilbert and Hardy integral inequalities. In particular, two general theorems are established, both based on integral terms derived from those used in these two famous inequalities. They have the property of depending on two functions and one modulable parameter. Applications and examples are given to specific cases combining Hilbert and Hardy type integral inequalities. Emphasis is placed on a particular weighted integral term, showing how our results can be used to improve what can be obtained with some classical integral inequalities in the literature.

1. INTRODUCTION

The study of integral inequalities is a central area of mathematical research. It is important in both theory and in applications. In fact, integral inequalities serve as fundamental tools in various branches of mathematics, facilitating the analysis of functions, operators, and systems. They also allow the modeling of physical phenomena. Among the well-known results, the Hilbert and Hardy integral inequalities stand out as key contributions. In addition to providing essential bounds for integrals, they have served developments in fields such as mathematical physics, functional analysis, optimization, and partial differential equations. More details on this topic, as well as comprehensive discussions, can be found in [16, 6, 23, 4, 27].

For the purposes of this article, we will describe these two famous inequalities. For the Hilbert integral inequality, we consider two functions $f, g : (0, +\infty) \mapsto (0, +\infty)$. Then, in its classical form, this inequality reads as follows:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \pi \sqrt{\int_0^{+\infty} f^2(x)dx} \sqrt{\int_0^{+\infty} g^2(x)dx}, \quad (1.1)$$

provided that the integrals introduced converge. Note that the factor π is the sharpest possible constant; it cannot be improved without more knowledge of the nature of f and

2020 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Integral inequalities; Hilbert integral inequality; Hardy integral inequality; Hölder integral inequality.

Received: July 12, 2025. Accepted: September 11, 2025. Published: September 30, 2025.

g. Among the notable studies that highlight the interest in this inequality, we refer to [28, 25, 26, 3, 5, 2, 1, 8, 9, 10]. For the Hardy integral inequality, we consider a parameter $p \in (1, +\infty)$, a function $f : (0, +\infty) \mapsto (0, +\infty)$, and its primitive operator defined, for any $x \in (0, +\infty)$, by

$$F(x) := \int_0^x f(t)dt.$$

Then, in its classical form, this inequality reads as follows:

$$\int_0^{+\infty} \frac{F^p(x)}{x^p} dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1.2}$$

provided that the integrals introduced converge. In this context, the factor $[p/(p-1)]^p$ is the sharpest possible constant. The Hardy integral inequality and its variants have attracted much attention, both in theory and in practice. We refer to the following notable studies: [22, 17, 21, 15, 24, 18, 14, 19, 7, 29, 11].

The Hilbert and Hardy integral inequalities have some connections, beyond their role in bounding integral operators and their sharp constants. This is also the case for all integral inequalities of the Hilbert and Hardy type. Some of these connections have been studied in [3]. In particular, general results have been established, of which the following is a special consequence:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{4} \sqrt{\int_0^{+\infty} \frac{F^2(x)}{x^2} dx} \sqrt{\int_0^{+\infty} \frac{G^2(x)}{x^2} dx}, \tag{1.3}$$

where G denotes the primitive operator associated with g (as F is associated with f), provided that the integrals introduced converge. More precisely, this simple integral inequality follows from [3, Equation (1.11) with $\lambda = 1$ and $p = q = 2$]. Thus we see a way of articulating the main terms of the Hilbert and Hardy type integral inequalities. The study in [3] extends this aspect by considering weighted L_p norms with $p \in (0, +\infty)$, not just $p \in (1, +\infty)$, and a variety of kernel functions, mainly of the form $h(x, y) = u(x) + v(y)$, where $u, v : (0, +\infty) \mapsto (0, +\infty)$. The motivation for investigating the further relations between Hilbert and Hardy type integral inequalities is to unify their applications and to understand deeper structural properties in analysis. Thus, the relation in Equation (1.3) is undeniably important, but perhaps more can be done on this subject; we can think of possible different directions, with innovative approaches or techniques.

In a sense, this article contributes to bridging the gap between Hilbert and Hardy type integral inequalities. More precisely, we provide new integral inequalities depending on their main terms. As an example, the following result is given as a special case:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \geq \frac{1}{2} \int_0^{+\infty} \frac{g(x)F(x) + f(x)G(x)}{x} dx. \tag{1.4}$$

Following the spirit of [3], this is extended by considering weighted L_p norms with $p \in (0, +\infty)$, not just $p \in (1, +\infty)$, and a variety of kernel functions, not necessarily of the form $h(x, y) = u(x) + v(y)$, where $u, v : (0, +\infty) \mapsto (0, +\infty)$; maximum or minimum kernels can be used, for example. Other specific results are derived, all of possibly independent interest. Numerous applications are also given, including bounds on the following integral term:

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p} dx,$$

which remains an original subject of work. For this term, we show that our results provide better bounds than those derived from classical integral inequalities. Since the optimality of our bounds is not established, some open problems are formulated, offering new and fresh perspectives of work on such a classical subject.

The remainder of the article is divided into four sections: Section 2 presents the main results, formally in the form of two theorems. Section 3 is devoted to the applications and examples. The proof of the first theorem is given in Section 4; the proof of the second theorem is omitted due to redundancy. The article concludes with some final remarks in Section 5.

2. Two theorems

2.1. **First theorem.** Our first main result is given in the theorem below, which is the integral inequality from which the simple example in Equation (1.4) is derived. Note that the parameter p satisfies $p \in (1,+\infty) \cup \{1\}$; the case $p \in (0,1)$ will be considered in the other theorem.

Theorem 2.1. *Let $p \in (1,+\infty)$, $f, g : (0,+\infty) \mapsto (0,+\infty)$ be functions and $h : (0,+\infty)^2 \mapsto (0,+\infty)$ be a bi-increasing function, i.e., for any $x, y, z \in (0,+\infty)$ such that $x \leq y$, we have $h(x,z) \leq h(y,z)$ and $h(z,x) \leq h(z,y)$. We set*

$$\alpha_p := \int_0^{+\infty} f^p(x)dx, \qquad \beta_p := \int_0^{+\infty} g^p(x)dx$$

and, for any $x \in (0,+\infty)$, we consider the following primitive operators:

$$F(x) := \int_0^x f(t)dt, \qquad G(x) := \int_0^x g(t)dt,$$

provided that the integrals introduced converge.

(1) Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy \geq \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx,$$

provided that the integrals introduced converge.

(2) Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\leq \int_0^{+\infty} \frac{\beta_p f^p(x) + \alpha_p g^p(x)}{h(x,x)}dx - \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx,$$

provided that the integrals introduced converge.

For the case $p = 1$, these inequalities still hold, and the corresponding proofs are simplified.

The detailed proof of Theorem 2.1 is given in Section 4. So, this theorem mainly shows how the following two integrals compare:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy, \qquad \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx,$$

where the first is related to a Hilbert type integral inequality and the second to a Hardy type integral inequality. This is of particular interest because several studies in the literature have already proposed bounds on these terms, often independently and under specific conditions. Bounds that we can use to evaluate one of the terms with a new perspective. In particular, based on the points 1 and 2 of Theorem 2.1, using an upper bound for the double integral term implies an upper bound for the simple integral term, and using a lower bound for the double integral term implies a lower bound for the simple integral term. The first aspect will be developed in Section 3, with various applications and examples.

We conclude this section with some remarks, mainly on special cases or extensions of Theorem 2.1.

- Taking $f = g$, the points 1 and 2 are reduced to
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)f^p(y)}{h(x,y)}dxdy \geq 2\int_0^{+\infty}\frac{f^p(x)F^p(x)}{x^{p-1}h(x,x)}dx$$
and
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)f^p(y)}{h(x,y)}dxdy \leq 2\left[\alpha_p\int_0^{+\infty}\frac{f^p(x)}{h(x,x)}dx - \int_0^{+\infty}\frac{f^p(x)F^p(x)}{x^{p-1}h(x,x)}dx\right].$$
Even in this case, there does not seem to be an exact equivalent of these inequalities in the literature.
- Analyzing the proof of Theorem 2.1, we can extend points 1 and 2 with little efforts, as, for any $p_1, p_2 \in (1, +\infty)$,
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^{p_1}(x)g^{p_2}(y)}{h(x,y)}dxdy \geq \int_0^{+\infty}\frac{1}{h(x,x)}\left[\frac{g^{p_2}(x)F^{p_1}(x)}{x^{p_1-1}} + \frac{f^{p_1}(x)G^{p_2}(x)}{x^{p_2-1}}\right]dx$$
and
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^{p_1}(x)g^{p_2}(y)}{h(x,y)}dxdy \leq \int_0^{+\infty}\frac{\beta_{p_2}f^{p_1}(x)+\alpha_{p_1}g^{p_2}(x)}{h(x,x)}dx - \int_0^{+\infty}\frac{1}{h(x,x)}\left[\frac{g^{p_2}(x)F^{p_1}(x)}{x^{p_1-1}} + \frac{f^{p_1}(x)G^{p_2}(x)}{x^{p_2-1}}\right]dx.$$
- Another almost immediate extension is to consider a bounded integration interval, say of the form (a, b), with $a, b \in \mathbb{R}$, where $a < b$. Redefining
$$\alpha_p := \int_a^b f^p(x)dx, \qquad \beta_p := \int_a^b g^p(x)dx$$
and, for any $x \in (a, b)$,
$$F(x) := \int_a^x f(t)dt, \qquad G(x) := \int_a^x g(t)dt,$$
with a little mathematical effort, we can establish that
$$\int_a^b\int_a^b\frac{f^p(x)g^p(y)}{h(x,y)}dxdy \geq \int_a^b\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{(x-a)^{p-1}h(x,x)}dx$$
and
$$\int_a^b\int_a^b\frac{f^p(x)g^p(y)}{h(x,y)}dxdy \leq \int_a^b\frac{\beta_p f^p(x)+\alpha_p g^p(x)}{h(x,x)}dx - \int_a^b\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{(x-a)^{p-1}h(x,x)}dx.$$
The details are omitted for brevity; the proof is almost identical to that of Theorem 2.1, except that the use of the Hölder integral inequality had to be adapted. This explains the presence of $(x-a)^{p-1}$ in some denominator terms.

The next subsection is devoted to our second general theorem.

2.2. **Second result.** The case $p \in (0,1)$, omitted in Theorem 2.1, is considered in the theorem below. The only change in the framework is the assumption that h is bi-decreasing instead of bi-increasing in Theorem 2.1.

Theorem 2.2. *Let $p \in (0,1)$, $f, g : (0,+\infty) \mapsto (0,+\infty)$ be functions and $h : (0,+\infty)^2 \mapsto (0,+\infty)$ be a bi-decreasing function, i.e., for any $x, y, z \in (0,+\infty)$ such that $x \le y$, we have $h(y,z) \le h(x,z)$ and $h(z,y) \le h(z,x)$. We set*

$$\alpha_p := \int_0^{+\infty} f^p(x)dx, \qquad \beta_p := \int_0^{+\infty} g^p(x)dx$$

and, for any $x \in (0,+\infty)$, we consider the following primitive operators:

$$F(x) := \int_0^x f(t)dt, \qquad G(x) := \int_0^x g(t)dt,$$

provided that the integrals introduced converge.

(1) Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy \le \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx,$$

provided that the integrals introduced converge.

(2) Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\ge \int_0^{+\infty} \frac{\beta_p f^p(x) + \alpha_p g^p(x)}{h(x,x)}dx - \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx,$$

provided that the integrals introduced converge.

The proof follows the same lines as that of Theorem 2.1, but instead of using convexity arguments and the Hölder inequality, concavity arguments and the reversed Hölder inequality are used. The details are thus omitted.

Thus, in comparison to Theorem 2.1, Theorem 2.2 considers the case $p \in (0,1)$ by adapting only the bi-monotonicity of h, and the inequalities in the points 1 and 2 in Theorem 2.1 are reversed.

Some remarks, mainly on special cases or extensions of Theorem 2.2, conclude this section.

- Taking $f = g$, the points 1 and 2 are reduced to

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)f^p(y)}{h(x,y)}dxdy \le 2\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^{p-1}h(x,x)}dx$$

and

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)f^p(y)}{h(x,y)}dxdy \ge 2\left[\alpha_p \int_0^{+\infty} \frac{f^p(x)}{h(x,x)}dx - \int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^{p-1}h(x,x)}dx\right].$$

Again, there seems to be no exact equivalent of these inequalities in the literature.

- Analyzing the proof of Theorem 2.2, we can extend points 1 and 2 with little efforts, as, for any $p_1, p_2 \in (1,+\infty)$,

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^{p_1}(x)g^{p_2}(y)}{h(x,y)}dxdy \le \int_0^{+\infty} \frac{1}{h(x,x)}\left[\frac{g^{p_2}(x)F^{p_1}(x)}{x^{p_1-1}} + \frac{f^{p_1}(x)G^{p_2}(x)}{x^{p_2-1}}\right]dx$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^{p_1}(x)g^{p_2}(y)}{h(x,y)}dxdy \geq \int_0^{+\infty}\frac{\beta_{p_2}f^{p_1}(x)+\alpha_{p_1}g^{p_2}(x)}{h(x,x)}dx$$
$$-\int_0^{+\infty}\frac{1}{h(x,x)}\left[\frac{g^{p_2}(x)F^{p_1}(x)}{x^{p_1-1}}+\frac{f^{p_1}(x)G^{p_2}(x)}{x^{p_2-1}}\right]dx.$$

- Another almost immediate extension is to consider a bounded integration interval, say of the form (a,b), with $a,b\in\mathbb{R}$, where $a<b$. Redefining
$$\alpha_p := \int_a^b f^p(x)dx, \qquad \beta_p := \int_a^b g^p(x)dx$$
and, for any $x\in(a,b)$,
$$F(x) := \int_a^x f(t)dt, \qquad G(x) := \int_a^x g(t)dt,$$
with a minor adaptation, we can see that
$$\int_a^b\int_a^b\frac{f^p(x)g^p(y)}{h(x,y)}dxdy \leq \int_a^b\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{(x-a)^{p-1}h(x,x)}dx$$
and
$$\int_a^b\int_a^b\frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\geq \int_a^b\frac{\beta_p f^p(x)+\alpha_p g^p(x)}{h(x,x)}dx - \int_a^b\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{(x-a)^{p-1}h(x,x)}dx.$$
Again, except for revisiting the use of the Hölder integral inequality, the proof is almost identical to that of Theorem 2.2.

Selected applications and examples of our main results are given in the next section.

3. Applications and examples

This section considers some general integral inequalities, provides examples, and studies an original integral inequality. For brevity, we focus only on the applications of Theorem 2.1, that is, for the case $p\in(1,+\infty)\cup\{1\}$ and h bi-increasing.

3.1. General integral inequalities.

For $p\in(1,+\infty)\cup\{1\}$, using the convexity inequality, i.e., $|a+b|^p\leq 2^{p-1}(|a|^p+|b|^p)$, $a,b\in\mathbb{R}$, with $a=g(x)F(x)$ and $b=f(x)G(x)$, we have
$$g^p(x)F^p(x)+f^p(x)G^p(x)\geq\frac{1}{2^{p-1}}\left[g(x)F(x)+f(x)G(x)\right]^p.$$
Based on this, the point 1 of Theorem 2.1 gives the following original integral inequality:
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy \geq \frac{1}{2^{p-1}}\int_0^{+\infty}\frac{x}{h(x,x)}\left[\frac{g(x)F(x)+f(x)G(x)}{x}\right]^p dx.$$
On the other hand, the point 2 of Theorem 2.1 implies that
$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\leq \int_0^{+\infty}\frac{\beta_p f^p(x)+\alpha_p g^p(x)}{h(x,x)}dx - \frac{1}{2^{p-1}}\int_0^{+\infty}\frac{x}{h(x,x)}\left[\frac{g(x)F(x)+f(x)G(x)}{x}\right]^p dx.$$

Other possible approaches in the same vein are possible. For instance, using the simple polynomial inequality $2|ab| \leq a^2 + b^2$, $a, b \in \mathbb{R}$, with $a = g^{p/2}(x)F^{p/2}(x)$ and $b = f^{p/2}(x)G^{p/2}(x)$, we get

$$g^p(x)F^p(x) + f^p(x)G^p(x) \geq 2g^{p/2}(x)F^{p/2}(x)f^{p/2}(x)G^{p/2}(x).$$

Based on this, the point 1 of Theorem 2.1 gives the following original integral inequality:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy \geq 2\int_0^{+\infty} \frac{1}{x^{p/2-1}h(x,x)}\left[\frac{g(x)F(x)f(x)G(x)}{x}\right]^{p/2} dx.$$

On the other hand, the point 2 of Theorem 2.1 implies that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\leq \int_0^{+\infty} \frac{\beta_p f^p(x) + \alpha_p g^p(x)}{h(x,x)}dx - 2\int_0^{+\infty} \frac{1}{x^{p/2-1}h(x,x)}\left[\frac{g(x)F(x)f(x)G(x)}{x}\right]^{p/2} dx.$$

Some examples of more direct connections between Hilbert and Hardy type integral inequalities are given in the next subsection.

3.2. **Examples.** Several variants of the Hilbert integral inequalities have been established in recent decades. The most important of these are proved and discussed in the book [27]. These bounds can be combined with our results in Theorem 2.1 (or Theorem 2.2) to produce innovative integral inequalities. This is explained below with some examples, working only with the L_2 norm for the bounds of the famous variants of the Hilbert integral inequalities considered, to simplify the situation.

Example 1: Applying the point 1 of Theorem 2.1 with the function $h(x, y) = x + y$, which is obviously bi-increasing, and the classical Hilbert integral inequality in Equation (1.1) with f^p instead of f and g^p instead of g, we get

$$\frac{1}{2}\int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^p}dx = \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx$$
$$\leq \int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{x+y}dxdy$$
$$\leq \pi\sqrt{\int_0^{+\infty} f^{2p}(x)dx}\sqrt{\int_0^{+\infty} g^{2p}(x)dx},$$

so that

$$\int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^p}dx \leq 2\pi\sqrt{\int_0^{+\infty} f^{2p}(x)dx}\sqrt{\int_0^{+\infty} g^{2p}(x)dx}. \quad (3.1)$$

As far as we know, this is a new integral inequality in the literature. It can be seen as a variant of the Hardy integral inequality. However, it is not optimal; the constant 2π can be improved, as we will see in the next point.

On the other hand, using the point 2 of Theorem 2.1, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{x+y}dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\leq\int_0^{+\infty}\frac{\beta_pf^p(x)+\alpha_pg^p(x)}{h(x,x)}dx-\int_0^{+\infty}\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx$$
$$=\frac{1}{2}\left[\int_0^{+\infty}\frac{\beta_pf^p(x)+\alpha_pg^p(x)}{x}dx-\int_0^{+\infty}\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{x^p}dx\right].$$

This provides an alternative to the Hilbert integral inequality. No equivalent in this form has been found in the literature.

Example 2: The following maximum variant of the Hilbert integral inequality is well-known:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{\max(x,y)}dxdy\leq 4\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(x)dx}. \tag{3.2}$$

It is proved that 4 is the best constant. See [16, 27]. Applying the point 1 of Theorem 2.1 with the function $h(x,y)=\max(x,y)$, which is bi-increasing, and Equation (3.2) with f^p instead of f and g^p instead of g, we get

$$\int_0^{+\infty}\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{x^p}dx=\int_0^{+\infty}\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx$$
$$\leq\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{\max(x,y)}dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty}f^{2p}(x)dx}\sqrt{\int_0^{+\infty}g^{2p}(x)dx}. \tag{3.3}$$

Since, obviously, $4<2\pi$, we have improved the precision of the inequality in Equation (3.1). On the other hand, using the point 2 of Theorem 2.1, we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{\max(x,y)}dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$
$$\leq\int_0^{+\infty}\frac{\beta_pf^p(x)+\alpha_pg^p(x)}{h(x,x)}dx-\int_0^{+\infty}\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx$$
$$=\int_0^{+\infty}\frac{\beta_pf^p(x)+\alpha_pg^p(x)}{x}dx-\int_0^{+\infty}\frac{g^p(x)F^p(x)+f^p(x)G^p(x)}{x^p}dx.$$

This provides an alternative to the maximum variant of the Hilbert integral inequality. To the best of our knowledge, it is new.

Example 3: The final example considers a function h of a different kind, such that $h(x,x)$ is not proportional to x. The starting point is the following exponential variant of the Hilbert integral inequality:

$$\int_0^{+\infty}\int_0^{+\infty}e^{-xy}f(x)g(y)dxdy\leq\sqrt{\pi}\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(x)dx}. \tag{3.4}$$

It was established in [28], with the proof that $\sqrt{\pi}$ is the best constant in this framework. See also [27]. Applying the point 1 of Theorem 2.1 with the function $h(x,y)=e^{xy}$, which is bi-increasing by the composition of increasing functions, and Equation (3.4) with f^p instead of f and g^p instead of g, we get

$$\int_0^{+\infty} e^{-x^2} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}} dx = \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)} dx$$
$$\leq \int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)} dxdy = \int_0^{+\infty}\int_0^{+\infty} e^{-xy} f^p(x)g^p(y)dxdy$$
$$\leq \sqrt{\pi}\sqrt{\int_0^{+\infty} f^{2p}(x)dx}\sqrt{\int_0^{+\infty} g^{2p}(x)dx}.$$

On the other hand, using the point 2 of Theorem 2.1, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} e^{-xy} f^p(x)g^p(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)} dxdy$$
$$\leq \int_0^{+\infty} \frac{\beta_p f^p(x) + \alpha_p g^p(x)}{h(x,x)} dx - \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)} dx$$
$$= \int_0^{+\infty} e^{-x^2} \left[\beta_p f^p(x) + \alpha_p g^p(x)\right] dx - \int_0^{+\infty} e^{-x^2} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}} dx.$$

This provides an alternative to the exponential variant of the Hilbert integral inequality.

3.3. **Some comments.** During our investigations, we also have considered other functions h such that $h(x,x) = x$, including

$$h(x,y) = \frac{x-y}{\log(x/y)},$$

for $x \neq y$, and $h(x,y) = x$ for $x = y$, with $x, y \in (0,+\infty)$. We can prove that it is bi-increasing. Indeed, by fixing y and differentiating with respect to x, we obtain

$$\frac{\partial}{\partial x} h(x,y) = \frac{1}{x\log^2(x/y)} \left[x\log\left(\frac{x}{y}\right) + y - x\right].$$

It is clear that $x\log^2(x/y) \geq 0$. Now, using the well-known logarithmic inequality $\log(1+a) \geq a/(1+a)$, with $a > -1$, taking $a = x/y - 1 > 0$, we get

$$x\log\left(\frac{x}{y}\right) + y - x \geq x\frac{x/y - 1}{1 + x/y - 1} + y - x = x - y + y - x = 0.$$

So we have $\partial h(x,y)/(\partial x) \geq 0$, which means that h is increasing with respect to x. Noticing that $h(x,y) = h(y,x)$, the bi-increasing property follows. On the other hand, the following logarithmic variant of the Hardy integral inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\log(x/y)f(x)g(y)}{x-y} dxdy \leq \pi^2\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(x)dx}. \quad (3.5)$$

It is discussed in [16, 27, 8]. The factor π^2 cannot be improved. The application of Theorem 2.1 and Equation (3.5) give

$$\int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^p} dx \leq \pi^2\sqrt{\int_0^{+\infty} f^{2p}(x)dx}\sqrt{\int_0^{+\infty} g^{2p}(x)dx}.$$

The obtained constant π^2 is far from being as sharp as the one obtained in Equation (3.3). This illustrates the importance of the choice of the function h in determining sharp integral inequalities based on Theorem 2.1. There is also an open problem: In the exact mathematical setting of the following inequality:

$$\int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^p} dx \le \kappa \sqrt{\int_0^{+\infty} f^{2p}(x)dx} \sqrt{\int_0^{+\infty} g^{2p}(x)dx},$$

is $\kappa = 4$ the optimal constant? This deserves further investigation, which we leave for future work.

3.4. **Study of an original integral inequality.** Sharp upper bounds for the following integral are the aim of this subsection:

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p} dx.$$

It can be thought of as a weighted version of the main term in the Hardy integral inequality, with the weight function f^p. It is thus related to the function of interest f, modifying the standards. We examine the use of the existing results and that in the point 1 of Theorem 2.1. We thus consider $p \in (1, +\infty)$.

- As a first approach, assuming that f is bounded, i.e., $\sup_{x\in(0,+\infty)} f(x) < +\infty$, the Hardy integral inequality, as recalled in Equation (1.2), implies that

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p} dx \le \left[\sup_{x\in(0,+\infty)} f^p(x)\right] \int_0^{+\infty} \frac{F^p(x)}{x^p} dx$$
$$\le \left[\sup_{x\in(0,+\infty)} f^p(x)\right] \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx.$$

  This yields a comprehensive bound depending on the L_p norm of f.
- The second approach uses an upper bound for F. Since f is positive, we clearly have $\sup_{x\in(0,+\infty)} F(x) = \sup_{x\in(0,+\infty)} \int_0^x f(t)dt = \int_0^{+\infty} f(t)dt$, so that

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p} dx \le \left[\sup_{x\in(0,+\infty)} F(x)\right]^p \int_0^{+\infty} \frac{f^p(x)}{x^p} dx$$
$$= \left[\int_0^{+\infty} f(x)dx\right]^p \int_0^{+\infty} \frac{f^p(x)}{x^p} dx.$$

  A simple bound is thus determined. It depends on the L_p norm and the weighted L_p norm of f.
- As a third, more elaborate approach, the Cauchy-Schwarz integral inequality, followed by the Hardy integral inequality applied with the parameter $2p$, implies that

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p} dx \le \sqrt{\int_0^{+\infty} f^{2p}(x)dx} \sqrt{\int_0^{+\infty} \frac{F^{2p}(x)}{x^{2p}} dx}$$
$$= \sqrt{\int_0^{+\infty} f^{2p}(x)dx \left(\frac{2p}{2p-1}\right)^{2p}} \sqrt{\int_0^{+\infty} f^{2p}(x)dx}$$
$$= \left(\frac{2p}{2p-1}\right)^{2p} \int_0^{+\infty} f^{2p}(x)dx. \tag{3.6}$$

  So we have a bound that depends on the L_{2p} norm of f. Note that it is difficult to compare its precision with that of the bounds obtained in the previous points, because different integral norms are considered for f.

- We now present a more innovative approach based on Theorem 2.1, and, more precisely, Equation (3.3). Indeed, taking $f = g$ in this equation, we get

$$2\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p}dx = \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^p}dx$$
$$\leq 4\sqrt{\int_0^{+\infty} f^{2p}(x)dx}\sqrt{\int_0^{+\infty} g^{2p}(x)dx} = 4\int_0^{+\infty} f^{2p}(x)dx,$$

so that

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p}dx \leq 2\int_0^{+\infty} f^{2p}(x)dx. \tag{3.7}$$

Now, using the well-known logarithmic inequality $\log(1+a) \geq a/(1+a)$, with $a > -1$, taking $a = 1/(2p-1) > 0$, we get

$$\left(\frac{2p}{2p-1}\right)^{2p} = e^{2p\log[1+1/(2p-1)]} \geq e^{2p[1/(2p-1)]/[1+1/(2p-1)]} = e^{2p/(2p)} = e > 2.$$

We thus have

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p}dx \leq 2\int_0^{+\infty} f^{2p}(x)dx \leq \left(\frac{2p}{2p-1}\right)^{2p}\int_0^{+\infty} f^{2p}(x)dx.$$

The inequality in Equation (3.7) is thus sharper than that obtained by the classical techniques used in Equation (3.6). This demonstrates how Theorem 2.1 can be used to improve certain integral inequalities. This also applies to Theorem 2.2, considering the less studied case of $p \in (0,1)$.

3.5. **Discussion.** The last result raised the following question: What is the best constant τ so that the following inequality holds?

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p}dx \leq \tau\int_0^{+\infty} f^{2p}(x)dx.$$

The found value of 2 seems sharp, but its possible optimality is not proved here. The question is also of interest for $p \in (0,1)$.

On this subject, for $p \in (0,+\infty)$, if we restrict our study to a bounded interval of the form $(0,b)$, with $b \in (0,+\infty)$, to ensure the convergence of the integrals involved, and assume that f is increasing, we obviously have

$$F(x) = \int_0^x f(t)dt \leq f(x)\int_0^x dt = xf(x),$$

so that

$$\int_0^b \frac{f^p(x)F^p(x)}{x^p}dx \leq \int_0^b \frac{f^p(x)x^pf^p(x)}{x^p}dx = \int_0^b f^{2p}(x)dx.$$

In this particular case, the constant 1 is sharper than 2. However, it is obtained with a strong assumption on f; it is not theoretically guaranteed without knowing the monotonicity of f.

Another possible approach would be to investigate the problem through the framework of the optimality theory developed in [12, 13]. However, this requires further investigation, which we will address in the future.

4. Proof of Theorem 2.1

This section contains the detailed proof of Theorems 2.1 (note that the proof of Theorem 2.2 is omitted for the sake of brevity).

Proof. Let us prove the points 1 and 2, successively.

(1) It follows from the Chasles integral relation that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy = A + B, \tag{4.1}$$

where

$$A := \int_0^{+\infty}\int_0^{y}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$

and

$$B := \int_0^{+\infty}\int_y^{+\infty}\frac{f^p(x)g^p(y)}{h(x,y)}dxdy.$$

Let us work on the term A. Using the left increasing property of h, for any $x \in (0, y)$, we have $h(x, y) \leq h(y, y)$. This implies that

$$A \geq \int_0^{+\infty}\int_0^{y}\frac{f^p(x)g^p(y)}{h(y,y)}dxdy = \int_0^{+\infty}\frac{g^p(y)}{h(y,y)}\left[\int_0^y f^p(x)dx\right]dy.$$

On the other hand, the Hölder integral inequality (or the Jensen integral inequality applied with the convex function $\varphi(x) = x^p$, $x \in (0, +\infty)$, with $p \in (1, +\infty)$, see [20]), gives

$$F^p(y) = \left[\int_0^y f(x)dx\right]^p \leq \left[\int_0^y f^p(x)dx\right]\left[\int_0^y dx\right]^{p-1} = y^{p-1}\left[\int_0^y f^p(x)dx\right], \tag{4.2}$$

so that

$$\int_0^y f^p(x)dx \geq \frac{1}{y^{p-1}}F^p(y). \tag{4.3}$$

We therefore have

$$A \geq \int_0^{+\infty}\frac{g^p(y)}{y^{p-1}h(y,y)}F^p(y)dy = \int_0^{+\infty}\frac{g^p(x)}{x^{p-1}h(x,x)}F^p(x)dx, \tag{4.4}$$

where the notation has been standardized in the last term.

Let us now consider the term B. Using the right increasing property of h, for any $x \in (y, +\infty)$, we have $h(x, y) \leq h(x, x)$. This, combined with a change in the order of integration by the Fubini-Tonelli theorem (which is possible because the integrand is positive, see [20]), gives

$$B \geq \int_0^{+\infty}\int_y^{+\infty}\frac{f^p(x)g^p(y)}{h(x,x)}dxdy = \int_0^{+\infty}\int_0^{x}\frac{f^p(x)g^p(y)}{h(x,x)}dydx$$
$$= \int_0^{+\infty}\frac{f^p(x)}{h(x,x)}\left[\int_0^x g^p(y)dy\right]dx.$$

Using the Hölder integral inequality as in Equation (4.2), we find that

$$\int_0^x g^p(y)dy \geq \frac{1}{x^{p-1}}G^p(x). \tag{4.5}$$

We therefore obtain

$$B \geq \int_0^{+\infty} \frac{f^p(x)}{x^{p-1}h(x,x)} G^p(x)dx. \tag{4.6}$$

It follows from Equations (4.1), (4.4) and (4.6) that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy \\
&\geq \int_0^{+\infty} \frac{g^p(x)}{x^{p-1}h(x,x)}F^p(x)dx + \int_0^{+\infty} \frac{f^p(x)}{x^{p-1}h(x,x)}G^p(x)dx \\
&= \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)}dx.
\end{aligned}$$

The stated inequality is established.

(2) For the point 2, we use techniques similar to those in the previous point, but with a different treatment in some key places. The Chasles integral relation implies that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy = C + D, \tag{4.7}$$

where

$$C := \int_0^{+\infty}\int_0^{y} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy$$

and

$$D := \int_0^{+\infty}\int_y^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)}dxdy.$$

Let us work on the term C. Using the right increasing property of h, for any $x \in (0,y)$, we have $h(x,y) \geq h(x,x)$. This, combined with a change in the order of integration by the Fubini-Tonelli theorem, the Chasles integral relation and the introduction of β_p, gives

$$\begin{aligned}
C &\leq \int_0^{+\infty}\int_0^{y} \frac{f^p(x)g^p(y)}{h(x,x)}dxdy = \int_0^{+\infty}\int_x^{+\infty} \frac{f^p(x)g^p(y)}{h(x,x)}dydx \\
&= \int_0^{+\infty} \frac{f^p(x)}{h(x,x)}\left[\int_x^{+\infty} g^p(y)dy\right]dx \\
&= \int_0^{+\infty} \frac{f^p(x)}{h(x,x)}\left[\int_0^{+\infty} g^p(y)dy - \int_0^{x} g^p(y)dy\right]dx \\
&= \int_0^{+\infty} \frac{f^p(x)}{h(x,x)}\left[\beta_p - \int_0^{x} g^p(y)dy\right]dx.
\end{aligned}$$

It follows from the inequality in Equation (4.5) that

$$\begin{aligned}
C &\leq \int_0^{+\infty} \frac{f^p(x)}{h(x,x)}\left[\beta_p - \frac{1}{x^{p-1}}G^p(x)\right]dx \\
&= \int_0^{+\infty} \frac{\beta_p f^p(x)}{h(x,x)}dx - \int_0^{+\infty} \frac{f^p(x)}{x^{p-1}h(x,x)}G^p(x)dx.
\end{aligned} \tag{4.8}$$

Let us now consider the term D. Using the left increasing property of h, for any $x \in (y,+\infty)$, we have $h(x,y) \geq h(y,y)$. This, together with the Chasles integral relation and the consideration of α_p, gives

$$D \leq \int_0^{+\infty} \int_y^{+\infty} \frac{f^p(x)g^p(y)}{h(y,y)} dxdy = \int_0^{+\infty} \frac{g^p(y)}{h(y,y)} \left[\int_y^{+\infty} f^p(x)dx \right] dy$$
$$= \int_0^{+\infty} \frac{g^p(y)}{h(y,y)} \left[\int_0^{+\infty} f^p(x)dx - \int_0^{y} f^p(x)dx \right] dy$$
$$= \int_0^{+\infty} \frac{g^p(y)}{h(y,y)} \left[\alpha_p - \int_0^{y} f^p(x)dx \right] dy.$$

It follows from the inequality in Equation (4.3) that

$$D \leq \int_0^{+\infty} \frac{g^p(y)}{h(y,y)} \left[\alpha_p - \frac{1}{y^{p-1}} F^p(y) \right] dy$$
$$= \int_0^{+\infty} \frac{\alpha_p g^p(y)}{h(y,y)} dy - \int_0^{+\infty} \frac{g^p(y)}{y^{p-1}h(y,y)} F^p(y)dy$$
$$= \int_0^{+\infty} \frac{\alpha_p g^p(x)}{h(x,x)} dx - \int_0^{+\infty} \frac{g^p(x)}{x^{p-1}h(x,x)} F^p(x)dx, \tag{4.9}$$

where the notation has been standardized in the last term.

Based on Equations (4.7), (4.8) and (4.9), we obtain

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x)g^p(y)}{h(x,y)} dxdy \leq \int_0^{+\infty} \frac{\beta_p f^p(x)}{h(x,x)} dx - \int_0^{+\infty} \frac{f^p(x)}{x^{p-1}h(x,x)} G^p(x)dx$$
$$+ \int_0^{+\infty} \frac{\alpha_p g^p(x)}{h(x,x)} dx - \int_0^{+\infty} \frac{g^p(x)}{x^{p-1}h(x,x)} F^p(x)dx$$
$$= \int_0^{+\infty} \frac{\beta_p f^p(x) + \alpha_p g^p(x)}{h(x,x)} dx - \int_0^{+\infty} \frac{g^p(x)F^p(x) + f^p(x)G^p(x)}{x^{p-1}h(x,x)} dx.$$

The stated inequality is established.

The two desired results are demonstrated.

Note that, for the case $p = 1$, the previous results hold; the only inequalities needed in the proofs are the first ones at each point, i.e., those using the bi-increasing property of h, the others become equalities (there is no need to use the Hölder integral inequality).

This completes the proof of Theorem 2.1. □

5. Conclusion

In this article, we have established two general theorems which present new integral inequalities. Some connections between Hilbert and Hardy type integral inequalities have been derived. This completes, in a sense, the results in [3]. Several applications and examples have been described, showing how the new results can be applied to improve what is obtained with the classical techniques. Some of them have raised open-problems, mainly on the optimality of the constants involved in specific integral inequalities. We may highlight this one: What is the best constant τ so that the following inequality holds?

$$\int_0^{+\infty} \frac{f^p(x)F^p(x)}{x^p} dx \leq \tau \int_0^{+\infty} f^{2p}(x)dx.$$

We have found that the constant 2 is competitive for $p \in (1, +\infty)$, but there is no claim to its optimality (we have artificially improved it under a monotonicity assumption on f to illustrate the complexity of the problem).

In conclusion, we have contributed to the field of integral inequalities in the spirit of Hilbert and Hardy type integral inequalities, proving new theorems and raising some unexplored research directions that deserve more attention for the future. This is important in order to deepen our understanding of these useful tools and potentially give some elements to unsolved complex problems using integrals in many areas beyond strict mathematical theory.

6. ACKNOWLEDGEMENTS

The author would like to thank the reviewer for his constructive comments on the article.

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