ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 3 (2025), 350-362
DOI: 10.62072/acm.2025.080302
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

INFERIOR SEMIGROUPS AND IDEALS

G. MUHIUDDIN* AND YOUNG BAE JUN

ABSTRACT. Inferior semigroups, left (right) inferior ideals, and inferior quasi-ideals in semigroups are introduced, and several properties are investigated. Characterizations of inferior semigroups and ideals are considered, and relations between inferior semigroups, inferior ideals and inferior quasi-ideals are discussed. Characteristic inferior mappings and inferior products of inferior mappings are introduced. Using these notions, related properties on inferior semigroups, left (right) inferior ideals and inferior quasi-ideals are investigated. A regular semigroup is characterized by an inferior quasi-ideal.

A semigroup is an important algebraic structure consisting of a set together with an associative binary operation. The name "semigroup" originates in the fact that a semigroup generalizes a group by preserving only associativity and closure under the binary operation from the axioms defining a group.

Fuzzy sets are sets whose elements have degrees of membership. Fuzzy sets were introduced by Lotfi A. Zadeh [3] and D. Klaua [2] in 1965 as an extension of the classical notion of set. A great variety of generalizations of fuzzy sets are considered by several authors. There are many mathematical constructions similar to or more general than fuzzy sets. Since fuzzy sets were introduced in 1965, a lot of new mathematical constructions and theories treating imprecision, inexactness, ambiguity, and uncertainty have been developed. Some of these constructions and theories are extensions of fuzzy set theory, while others try to mathematically model imprecision and uncertainty in a different way (Burgin and Chunihin, 1997; Kerre, 2001; Deschrijver and Kerre, 2003). A fuzzy set is a pair (U, μ) where U is a set and $\mu : U \to [0, 1]$ is a mapping. Based on the fact that the image of μ is a totally ordered set under the usual order in $\mathbb{R}$, Jun and Muhiuddin [1] tried to make another generalization, called inferior mappings, of fuzzy sets by using partially ordered sets.

The aim of this paper is to apply the notion of inferior mappings by Jun and Muhiuddin to semigroup theory. We introduce inferior semigroups, left (right) inferior ideals and inferior quasi-ideals in semigroups, and investigate several properties. We characterize inferior

2020 *Mathematics Subject Classification.* 06F35, 03G25, 06A11.

Key words and phrases. Inferior mapping; inferior subalgebra; left (right) inferior ideal; inferior quasi-ideal; inferior product; characteristic inferior mapping.

Received: June 26, 2025. Accepted: September 10, 2025. Published: September 30, 2025.

*Corresponding author.

semigroups and ideals, and discuss relations between inferior semigroups, inferior ideals and inferior quasi-ideals. We introduce characteristic inferior mappings and inferior products of inferior mappings. Using these notions, we discuss related properties on inferior semigroups, left (right) inferior ideals and inferior quasi-ideals. We characterize a regular semigroup by inferior quasi-ideals.

1. Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

A semigroup S is said to be *regular* if for every $x \in S$ there exists and element $a \in S$ such that $xax = x$.

A nonempty subset A of S is called

- a *subsemigroup* of S if $AA \subseteq A$, that is, $ab \in A$ for all $a, b \in A$,
- a *left* (resp. *right*) *ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$) that is, $xa \in A$ (resp. $ax \in A$) for all $x \in S$ and $a \in A$.
- a *two-sided ideal* of S if it is both a left and a right ideal of S.
- a *quasi-ideal* of S if $AS \cap SA \subseteq A$.

Note that a semigroup S is regular if and only if $A = ASA$ for every quasi-ideal A of S.

Let S be a nonempty set and let U be a partially ordered set with the partial ordering $\precsim$ and the last element θ. Then the statement

$$a \precsim b \text{ is read as "} a \textit{ precedes } b\text{"}$$

In this context, we also write:

$b \succsim a$ means $a \precsim b$; and read "b *succeeds* a".
$a \precnsim b$ means $a \precsim b$ and $a \neq b$; and read "a *strictly precedes* b".
$b \succnsim a$ means $a \precnsim b$; and read "b *strictly succeeds* a".

We consider a pair (f, S) on $(U, \precsim)$ where $f : S \to \mathcal{P}(U)$ is a mapping and $\mathcal{P}(U)$ is the power set of U. Define a mapping

$$\tilde{f} : S \to U,\ x \mapsto \begin{cases} \inf f(x) & \text{if } \exists \inf f(x), \\ \theta & \text{if } \nexists \inf f(x) \text{ or } f(x) = \emptyset, \end{cases} \tag{1.1}$$

which is called the inferior mapping of S related to the pair (f, S) on $(U, \precsim)$.

Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. Then the union of (f, S) and (g, S) is defined to be the pair $(f \cup g, S)$ on $(U, \precsim)$ which is given as follows:

$$f \cup g : S \to \mathcal{P}(U),\ x \mapsto f(x) \cup g(x).$$

The intersection of (f, S) and (g, S) is defined to be the pair $(f \cap g, S)$ on $(U, \precsim)$ which is given as follows:

$$f \cap g : S \to \mathcal{P}(U),\ x \mapsto f(x) \cap g(x).$$

The inferior mapping of S related to the pair $(f \cup g, S)$ (resp. $(f \cap g, S)$) on $(U, \precsim)$ is called the union (resp. intersection) of $\tilde{f}$ and $\tilde{g}$ and is denoted by $\widetilde{f \cup g}$ (resp. $\widetilde{f \cap g}$). The inferior union of $\tilde{f}$ and $\tilde{g}$ is denoted by $\tilde{f} \Cup \tilde{g}$ and is defined by

$$\tilde{f} \Cup \tilde{g} : S \to U,\ x \mapsto \sup\{\tilde{f}(x), \tilde{g}(x)\}. \tag{1.2}$$

The inferior intersection of $\tilde{f}$ and $\tilde{g}$ is denoted by $\tilde{f} \Cap \tilde{g}$ and is defined as follows:

$$\tilde{f} \Cap \tilde{g} : S \to U, \ x \mapsto \inf\{\tilde{f}(x), \tilde{g}(x)\}, \tag{1.3}$$

where $(\tilde{f} \Cap \tilde{g})(x) = \theta$ if $\inf\{\tilde{f}(x), \tilde{g}(x)\}$ does not exist.

Let $\tilde{f}$ be an inferior mapping of S related to the pair (f, S) on $(U, \precsim)$. For any $\alpha \in U$, the set

$$S(\tilde{f}, \alpha) := \{x \in S \mid \tilde{f}(x) \text{ succeeds } \alpha\}$$

is called the *upper α-inferior set* of $\tilde{f}$ (see [1]).

2. Inferior semigroups and ideals

Definition 2.1. Let S be a semigroup and let (f, S) be a pair on $(U, \precsim)$. If the inferior mapping $\tilde{f}$ of S related to the pair (f, S) on $(U, \precsim)$ satisfies the following condition

$$(\forall x, y \in S) \left(\tilde{f}(xy) \text{ succeeds the infimum of } \tilde{f}(x) \text{ and } \tilde{f}(y)\right), \tag{2.1}$$

then we say that $\tilde{f}$ is an *inferior semigroup* of S.

Example 2.2. Let $U = \{1, 2, 3, \cdots, 9, 10\}$ be ordered as pictured in Figure 1.

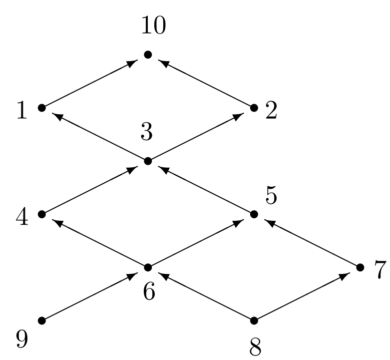

Figure 1

Let $S = \{0, 1, 2, 3, 4, 5\}$ be a semigroup with the Cayley table (see Table 1).

TABLE 1. Cayley table for the binary operation '$\cdot$'

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

(1) Let (f, S) be a pair on $(U, \precsim)$ in which f is defined as follows:

$$f : S \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1, 10\} & \text{if } x = 0, \\ \{2, 3, 10\} & \text{if } x = 1, \\ \{3, 4, 5, 8\} & \text{if } x = 2, \\ \{1, 2, 3, 4, 6\} & \text{if } x = 3, \\ \{2, 4, 5, 7, 8\} & \text{if } x = 4, \\ \{1, 2, 3, 5\} & \text{if } x = 5. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of S related to the pair (f, S) on $(U, \precsim)$ is described as follows:

$$\tilde{f} : S \to U,\ x \mapsto \begin{cases} 1 & \text{if } x = 0, \\ 3 & \text{if } x = 1, \\ 8 & \text{if } x \in \{2, 4\}, \\ 6 & \text{if } x = 3, \\ 5 & \text{if } x = 5. \end{cases}$$

Then $\tilde{f}$ is an inferior semigroup of S.

(2) Let (g, S) be a pair on $(U, \precsim)$ in which g is defined as follows:

$$g : S \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1, 3, 10\} & \text{if } x = 0, \\ \{2, 3, 10\} & \text{if } x = 1, \\ \{2, 3, 4, 5, 6\} & \text{if } x = 2, \\ \{1, 2, 3, 4\} & \text{if } x = 3, \\ \{3, 5, 6, 9\} & \text{if } x = 4, \\ \{1, 2, 3, 4, 6, 9\} & \text{if } x = 5. \end{cases}$$

Then the inferior mapping $\tilde{g}$ of S related to the pair (g, S) on $(U, \precsim)$ is described as follows:

$$\tilde{g} : S \to U,\ x \mapsto \begin{cases} 3 & \text{if } x \in \{0, 1\}, \\ 6 & \text{if } x = 2, \\ 4 & \text{if } x = 3, \\ 9 & \text{if } x \in \{4, 5\}. \end{cases}$$

Then $\tilde{g}$ is an inferior semigroup of S.

Theorem 2.1. *If $\tilde{f}$ is an inferior semigroup of a semigroup S, then the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a subsemigroup of S for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \emptyset$.*

Proof. Assume that $\tilde{f}$ is an inferior semigroup of S. Let $x, y \in S(\tilde{f}, \alpha)$. Then $\tilde{f}(x)$ and $\tilde{f}(y)$ succeed α. It follows from (2.1) that $\tilde{f}(xy)$ succeeds α and that $xy \in S(\tilde{f}, \alpha)$. Therefore $S(\tilde{f}, \alpha)$ is a subsemigroup of S. □

Theorem 2.2. *Let $\tilde{f}$ be an inferior mapping of a semigroup S related to the pair (f, S) on $(U, \precsim)$ such that there exists the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$ for all $x, y \in S$. If the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a subsemigroup of S for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \emptyset$, then $\tilde{f}$ is an inferior semigroup of S.*

Proof. Let $x, y \in S$ and $\beta \in U$ be such that $\inf\{\tilde{f}(x), \tilde{f}(y)\} = \beta$. Then $\tilde{f}(x)$ and $\tilde{f}(y)$ succeed β, that is, $x, y \in S(\tilde{f}, \beta)$. Thus $xy \in S(\tilde{f}, \beta)$, and so $\tilde{f}(xy) \succsim \beta = \inf\{\tilde{f}(x), \tilde{f}(y)\}$. Therefore $\tilde{f}(xy)$ succeeds the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$, and so $\tilde{f}$ is an inferior semigroup of S. □

Theorem 2.3. *If $\tilde{f}$ and $\tilde{g}$ are inferior semigroups of a semigroup S, then so is the intersection $\widetilde{f \cap g}$ of $\tilde{f}$ and $\tilde{g}$.*

Proof. It is straightforward. □

The following example shows that the union of two inferior semigroups of a semigroup S may not be an inferior semigroup.

Example 2.3. Let $\tilde{f}$ and $\tilde{g}$ be inferior semigroups as in Example 2.2. Then

$$f \cup g : S \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1,3,10\} & \text{if } x = 0, \\ \{2,3,10\} & \text{if } x = 1, \\ \{2,3,4,5,6,8\} & \text{if } x = 2, \\ \{1,2,3,4,6\} & \text{if } x = 3, \\ \{2,3,4,5,6,7,8,9\} & \text{if } x = 4, \\ \{1,2,3,4,5,6,9\} & \text{if } x = 5, \end{cases}$$

and so

$$\widetilde{f \cup g} : S \to U,\ x \mapsto \begin{cases} 3 & \text{if } x \in \{0,1\}, \\ 8 & \text{if } x = 2, \\ 6 & \text{if } x = 3, \\ 10 & \text{if } x = 4, \\ 9 & \text{if } x = 5. \end{cases}$$

Note that $\widetilde{f \cup g}(4{\cdot}3) = \widetilde{f \cup g}(5) = 9$ does not succeed 6, which is the infimum of $\widetilde{f \cup g}(4)$ and $\widetilde{f \cup g}(3)$. Hence $\widetilde{f \cup g}$ is not an inferior semigroup of S.

Definition 2.4. Let S be a semigroup and let (f, S) be a pair on $(U, \precsim)$. If the inferior mapping $\tilde{f}$ of S related to the pair (f, S) on $(U, \precsim)$ satisfies the following condition

$$(\forall x, y \in S)\left(\tilde{f}(xy) \text{ succeeds } \tilde{f}(y) \text{ (resp. } \tilde{f}(x))\right), \tag{2.2}$$

then we say that $\tilde{f}$ is a *left* (resp. *right*) *inferior ideal* of S.

If the inferior mapping $\tilde{f}$ of S related to the pair (f, S) on $(U, \precsim)$ is both a left inferior ideal and a right inferior ideal of S, then we say that $\tilde{f}$ is a *two-sided inferior ideal* of S.

Example 2.5. Let $U = \{1, 2, 3, \cdots, 8\}$ be ordered as pictured in Figure 2.

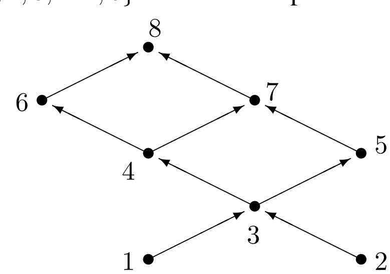

Figure 2

Let $S = \{a, b, c, d\}$ be a semigroup with the Cayley table (see Table 2).
Let $(\tilde{f}, S)$ be a pair on $(U, \precsim)$ in which f is defined as follows:

$$f : S \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{7,8\} & \text{if } x = a, \\ \{4,6,8\} & \text{if } x = b, \\ \{3,4,5,6\} & \text{if } x = c, \\ \{1,3,5,7\} & \text{if } x = d. \end{cases}$$

TABLE 2. Cayley table for the binary operation '·'

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then the inferior mapping $\tilde{f}$ of S related to the pair (f, S) on $(U, \precsim)$ is described as follows: $\tilde{f}(a) = 7$, $\tilde{f}(b) = 4$, $\tilde{f}(c) = 3$, and $\tilde{f}(d) = 1$. It is routine to verify that $\tilde{f}$ is an inferior ideal of S.

Obviously, every left (resp. right) inferior ideal of S is an inferior semigroup of S. But the converse is not true as seen in the following example. In fact, the inferior semigroup $\tilde{f}$ in Example 2.2(1) is not a left inferior ideal of S since $\tilde{f}(3 \cdot 5) = \tilde{f}(3) = 6$ does not succeeds $5 = \tilde{f}(5)$. Also, the inferior semigroup $\tilde{g}$ in Example 2.2(2) is not a right inferior ideal of S since $\tilde{g}(3 \cdot 4) = \tilde{g}(2) = 6$ does not succeeds $4 = \tilde{g}(3)$.

Theorem 2.4. *If $\tilde{f}$ is a left (right) inferior ideal of a semigroup S, then the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a left (right) ideal of S for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \emptyset$.*

Proof. It is similar to the proof of Theorem 2.1. □

Theorem 2.5. *Let $\tilde{f}$ be an inferior mapping of a semigroup S related to the pair (f, S) on $(U, \precsim)$ such that there exists the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$ for all $x, y \in S$. If the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a left (right) ideal of S for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \emptyset$, then $\tilde{f}$ is a left (right) inferior ideal of S.*

Proof. It is similar to the proof of Theorem 2.2. □

Let $(U, \precsim)$ be a poset with both the first element (κ) and the last element (θ). For a nonempty subset A of a set S, define a mapping χ_A as follows:

$$\chi_A : S \to \mathcal{P}(U),\ x \mapsto \begin{cases} \emptyset & \text{if } x \in A, \\ U & \text{otherwise.} \end{cases}$$

Then the inferior mapping $\tilde{\chi}_A$ of a set S related to the pair (χ_A, S) on $(U, \precsim)$ is given as follows:

$$\tilde{\chi}_A : S \to U,\ x \mapsto \begin{cases} \theta & \text{if } x \in A, \\ \kappa & \text{otherwise,} \end{cases}$$

and is called the *characteristic inferior mapping* of A in S. The characteristic inferior mapping $\tilde{\chi}_S$ of S in S is called the *identity inferior mapping* of S.

Let $\tilde{\chi}_A$ and $\tilde{\chi}_B$ be characteristic inferior mappings of A and B, respectively, in S where A and B are nonempty subsets of S. Let $x \in S$. If $x \in A \cap B$, then

$$(\tilde{\chi}_A \Cap \tilde{\chi}_B)(x) = \inf\{\tilde{\chi}_A(x), \tilde{\chi}_B(x)\} = \theta = \tilde{\chi}_{A \cap B}(x)$$

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Thus

$$(\tilde{\chi}_A \Cap \tilde{\chi}_B)(x) = \inf\{\tilde{\chi}_A(x), \tilde{\chi}_B(x)\} = \kappa = \tilde{\chi}_{A \cap B}(x).$$

Therefore we have

$$\tilde{\chi}_A \Cap \tilde{\chi}_B = \tilde{\chi}_{A \cap B} \tag{2.3}$$

Theorem 2.6. *For any nonempty subset A of a semigroup S, the following are equivalent.*

(1) *A is a left (resp. right) ideal of S.*
(2) *The characteristic inferior mapping $\tilde{\chi}_A$ of A in S is a left (resp. right) inferior ideal of S.*

Proof. Assume that A is a left ideal of S. For any $x, y \in S$, if $y \notin A$ then $\tilde{\chi}_A(xy)$ succeeds $\kappa = \tilde{\chi}_A(y)$. If $y \in A$, then $xy \in A$ since A is a left ideal of S. Hence $\tilde{\chi}_A(xy) = \theta = \tilde{\chi}_A(y)$. Therefore $\tilde{\chi}_A$ is a left inferior ideal of S. Similarly, $\tilde{\chi}_A$ is a right inferior ideal of S when A is a right ideal of S.

Conversely, suppose that $\tilde{\chi}_A$ is a left inferior ideal of S. Let $x \in S$ and $y \in A$. Then $\tilde{\chi}_A(xy)$ succeeds $\tilde{\chi}_A(y) = \theta$, and so $\tilde{\chi}_A(xy) = \theta$. Thus $xy \in A$ and therefore A is a left ideal of S. Similarly, we can show that if $\tilde{\chi}_A$ is a right inferior ideal of S, then A is a right ideal of S. □

Let $IM(S)$ denote the set of all inferior mappings of a semigroup S and define a relation $\ll$ on $IM(S)$ as follows.

$$(\forall \tilde{f}, \tilde{g} \in IM(S)) \left(\tilde{f} \ll \tilde{g} \;\Leftrightarrow\; \tilde{f}(x) \text{ precedes } \tilde{g}(x) \text{ for all } x \in S \right).$$

Then $\ll$ is an equivalence relation on $IM(S)$.

For any $\tilde{f}, \tilde{g} \in IM(S)$, the *inferior product* of $\tilde{f}$ and $\tilde{g}$ is written $\tilde{f} \odot \tilde{g}$ and is defined as follows:

$$(\tilde{f} \odot \tilde{g})(x) = \begin{cases} \sup\limits_{x=yz} \inf \left\{ \tilde{f}(y), \tilde{g}(z) \right\} & \text{if } \exists\, y, z \in S \text{ such that } x = yz, \\ \kappa & \text{otherwise.} \end{cases}$$

Let $\tilde{\chi}_A$ and $\tilde{\chi}_B$ be characteristic inferior mappings of A and B, respectively, in S where A and B are nonempty subsets of S. For any $x \in S$, if $x \in AB$ then there exist $a \in A$ and $b \in B$ such that $x = ab$. Thus

$$(\tilde{\chi}_A \odot \tilde{\chi}_B)(x) = \sup_{x=yz} \inf \{\tilde{\chi}_A(y), \tilde{\chi}_B(z)\} \succsim \inf \{\tilde{\chi}_A(a), \tilde{\chi}_B(b)\} = \theta,$$

and so $(\tilde{\chi}_A \odot \tilde{\chi}_B)(x) = \theta$. Since $x \in AB$, we have $\tilde{\chi}_{AB}(x) = \theta$. Suppose $x \notin AB$. Then $x \neq ab$ for any $a \in A$ and $b \in B$. If $x = yz$ for some $y, z \in S$, then $y \notin A$ or $z \notin B$. Hence

$$(\tilde{\chi}_A \odot \tilde{\chi}_B)(x) = \sup_{x=yz} \inf \{\tilde{\chi}_A(y), \tilde{\chi}_B(z)\} = \kappa = \tilde{\chi}_{AB}(x).$$

If $x \neq yz$ for all $y, z \in S$, then

$$(\tilde{\chi}_A \odot \tilde{\chi}_B)(x) = \kappa = \tilde{\chi}_{AB}(x).$$

In any case, we have

$$\tilde{\chi}_A \odot \tilde{\chi}_B = \tilde{\chi}_{AB}. \tag{2.4}$$

Theorem 2.7. *Let $\tilde{f}$ be an inferior mapping of S related to a pair (f, S) on $(U, \precsim)$. Then $\tilde{f}$ is an inferior semigroup of S if and only if $\tilde{f} \odot \tilde{f} \ll \tilde{f}$.*

Proof. Assume that $\tilde{f} \odot \tilde{f} \ll \tilde{f}$ and let $x, y \in S$. Then

$$\tilde{f}(xy) \succsim (\tilde{f} \odot \tilde{f})(xy) \succsim \inf\{\tilde{f}(x), \tilde{f}(y)\},$$

and so $\tilde{f}$ is an inferior semigroup of S.

Conversely, suppose that $\tilde{f}$ is an inferior semigroup of S. Then $\tilde{f}(x)$ succeeds the infimum of $\tilde{f}(y)$ and $\tilde{f}(z)$ for all $x \in S$ with $x = yz$. Hence

$$\tilde{f}(x) \succsim \sup_{x=yz} \inf \left\{ \tilde{f}(y), \tilde{f}(z) \right\} = (\tilde{f} \odot \tilde{f})(x)$$

for all $x \in S$. Therefore $\tilde{f} \odot \tilde{f} \ll \tilde{f}$. □

Theorem 2.8. *Let $\tilde{f}$ be an inferior mapping of S related to a pair (f, S) on $(U, \precsim)$. For the identity inferior mapping $\tilde{\chi}_S$ of S, the following are equivalent.*

(1) *$\tilde{f}$ is a left inferior ideal of S.*
(2) *$\tilde{\chi}_S \odot \tilde{f} \ll \tilde{f}$.*

Proof. Assume that $\tilde{f}$ is a left inferior ideal of S. Let $x \in S$. If $x = yz$ for some $y, z \in S$, then

$$\begin{aligned} (\tilde{\chi}_S \odot \tilde{f})(x) &= \sup_{x=yz} \inf \left\{ \tilde{\chi}_S(y), \tilde{f}(z) \right\} \\ &\precsim \sup_{x=yz} \inf \left\{ \theta, \tilde{f}(yz) \right\} \\ &= \tilde{f}(x). \end{aligned}$$

If $x \neq yz$ for all $y, z \in S$, then $(\tilde{\chi}_S \odot \tilde{f})(x) = \kappa \precsim \tilde{f}(x)$. Therefore $\tilde{\chi}_S \odot \tilde{f} \ll \tilde{f}$.

Conversely, suppose that the second assertion is valid. For any $x, y \in S$, we have

$$\tilde{f}(xy) \succsim (\tilde{\chi}_S \odot \tilde{f})(xy) \succsim \inf \left\{ \tilde{\chi}_S(x), \tilde{f}(y) \right\} = \inf\{\theta, \tilde{f}(y)\} = \tilde{f}(y).$$

Hence $\tilde{f}$ is a left inferior ideal of S. □

Similarly, we have the following theorem.

Theorem 2.9. *Let $\tilde{f}$ be an inferior mapping of S related to a pair (f, S) on $(U, \precsim)$. For the identity inferior mapping $\tilde{\chi}_S$ of S, the following are equivalent.*

(1) *$\tilde{f}$ is a right inferior ideal of S.*
(2) *$\tilde{f} \odot \tilde{\chi}_S \ll \tilde{f}$.*

Corollary 2.10. *Let $\tilde{f}$ be an inferior mapping of S related to a pair (f, S) on $(U, \precsim)$. For the identity inferior mapping $\tilde{\chi}_S$ of S, the following are equivalent.*

(1) *$\tilde{f}$ is a two-sided inferior ideal of S.*
(2) *$\tilde{\chi}_S \odot \tilde{f} \ll \tilde{f}$ and $\tilde{f} \odot \tilde{\chi}_S \ll \tilde{f}$.*

Theorem 2.11. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ and $\tilde{g}$ are inferior semigroups of S, then so is the inferior intersection $\tilde{f} \Cap \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$.*

Proof. For any $x, y \in S$, we have

$$\begin{aligned} (\tilde{f} \Cap \tilde{g})(xy) &= \inf\{\tilde{f}(xy), \tilde{g}(xy)\} \\ &\succsim \inf\{\inf\{\tilde{f}(x), \tilde{f}(y)\}, \inf\{\tilde{g}(x), \tilde{g}(y)\}\} \\ &= \inf\{\inf\{\tilde{f}(x), \tilde{g}(x)\}, \inf\{\tilde{f}(y), \tilde{g}(y)\}\} \\ &= \inf\{(\tilde{f} \Cap \tilde{g})(x), (\tilde{f} \Cap \tilde{g})(y)\}. \end{aligned}$$

Therefore $\tilde{f} \Cap \tilde{g}$ is an inferior semigroup of S. □

By the similar way, we can prove the following theorem.

Theorem 2.12. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ and $\tilde{g}$ are left (resp. right) inferior ideals of S, then so is the inferior intersection $\tilde{f} \Cap \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$.*

Corollary 2.13. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ and $\tilde{g}$ are two-sided inferior ideals of S, then so is the inferior intersection $\tilde{f} \Cap \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$.*

Theorem 2.14. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ is a left inferior ideal of S, then so is the inferior product $\tilde{f} \odot \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$.*

Proof. Let $x, y \in S$. If $y = ab$ for some $a, b \in S$, then $xy = x(ab) = (xa)b$ and

$$\begin{aligned}(\tilde{f} \odot \tilde{g})(y) &= \sup_{y=ab} \inf \left\{ \tilde{f}(a), \tilde{g}(b) \right\} \\ &\precsim \sup_{xy=(xa)b} \inf \left\{ \tilde{f}(xa), \tilde{g}(b) \right\} \\ &\precsim \sup_{xy=cb} \inf \left\{ \tilde{f}(c), \tilde{g}(b) \right\} \\ &= (\tilde{f} \odot \tilde{g})(xy).\end{aligned}$$

If y is not expressible as $y = ab$ for $a, b \in S$, then $(\tilde{f} \odot \tilde{g})(y) = \kappa \precsim (\tilde{f} \odot \tilde{g})(xy)$. Hence $\tilde{f} \odot \tilde{g}$ is a left inferior ideal of S. □

Similarly we have the following theorem.

Theorem 2.15. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ is a right inferior ideal of S, then so is the inferior product $\tilde{f} \odot \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$.*

Corollary 2.16. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ is a two-sided inferior ideal of S, then so is the inferior product $\tilde{f} \odot \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$.*

Theorem 2.17. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f, S) and (g, S), respectively, on $(U, \precsim)$. If $\tilde{f}$ is a right inferior ideal of S and $\tilde{g}$ is a left inferior ideal of S, then $\tilde{f} \odot \tilde{g} \ll \tilde{f} \Cap \tilde{g}$.*

Proof. Let $x \in S$. If x is not expressible as $x = ab$ for $a, b \in S$, then

$$(\tilde{f} \odot \tilde{g})(x) = \kappa \precsim (\tilde{f} \Cap \tilde{g})(x).$$

If there exist $a, b \in S$ such that $x = ab$, then

$$\begin{aligned}(\tilde{f} \odot \tilde{g})(x) &= \sup_{x=ab} \inf \left\{ \tilde{f}(a), \tilde{g}(b) \right\} \\ &\precsim \sup_{x=ab} \inf \left\{ \tilde{f}(ab), \tilde{g}(ab) \right\} \\ &= \inf \left\{ \tilde{f}(x), \tilde{g}(x) \right\} \\ &= (\tilde{f} \Cap \tilde{g})(x).\end{aligned}$$

Therefore $\tilde{f} \odot \tilde{g} \ll \tilde{f} \Cap \tilde{g}$. □

If we strength the condition of the semigroup S, then the reverse relation in Theorem 2.17 can be induced as follows.

Theorem 2.18. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f,S) and (g,S), respectively, on $(U,\precsim)$. If S is a regular semigroup and $\tilde{f}$ is a right inferior ideal of S, then $\tilde{f} \Cap \tilde{g} \ll \tilde{f} \odot \tilde{g}$.*

Proof. For any $x \in S$, there exists $a \in S$ such that $xax = x$ since S is regular. Hence

$$(\tilde{f} \odot \tilde{g})(x) = \sup_{x=yz} \inf \left\{ \tilde{f}(y), \tilde{g}(z) \right\}.$$

On the other hand, we get

$$(\tilde{f} \Cap \tilde{g})(x) = \inf \left\{ \tilde{f}(x), \tilde{g}(x) \right\} \precsim \inf \left\{ \tilde{f}(xa), \tilde{g}(x) \right\}$$

since $\tilde{f}$ is a right inferior ideal of S. Since $xax = x$, we have

$$\inf \left\{ \tilde{f}(xa), \tilde{g}(x) \right\} \precsim \sup_{x=yz} \inf \left\{ \tilde{f}(y), \tilde{g}(z) \right\} = (\tilde{f} \odot \tilde{g})(x).$$

Therefore $(\tilde{f} \Cap \tilde{g})(x)$ precedes $(\tilde{f} \odot \tilde{g})(x)$, that is, $\tilde{f} \Cap \tilde{g} \ll \tilde{f} \odot \tilde{g}$. □

The similar way show the following theorem.

Theorem 2.19. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of S related to pairs (f,S) and (g,S), respectively, on $(U,\precsim)$. If S is a regular semigroup and $\tilde{g}$ is a left inferior ideal of S, then $\tilde{f} \Cap \tilde{g} \ll \tilde{f} \odot \tilde{g}$.*

Combining Theorems 2.17 and 2.18 induces the following theorem.

Theorem 2.20. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of a regular semigroup S related to pairs (f,S) and (g,S), respectively, on $(U,\precsim)$. Then $\tilde{f} \Cap \tilde{g} = \tilde{f} \odot \tilde{g}$ when $\tilde{f}$ is a right inferior ideal of S and $\tilde{g}$ is a left inferior ideal of S.*

Definition 2.6. An inferior mapping $\tilde{f}$ of S related to a pair (f,S) on $(U,\precsim)$ is called an *inferior quasi-ideal* of S if

$$(\tilde{f} \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{f}) \ll \tilde{f} \tag{2.5}$$

Obviously, every left (right) inferior ideal is an inferior quasi-ideal of S, but the converse does not hold in general as seen in the following example.

Example 2.7. Let $S = \{0, a, b, c\}$ be a semigroup with the Cayley table (see Table 3).

TABLE 3. Cayley table for the binary operation '·'

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

Given a set $U = \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 1\}$ with the partial order "$\precsim$" as pictured in Figure 3

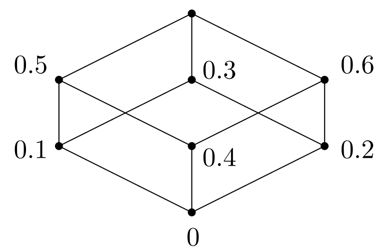

Figure 3

consider a pair (f, S) on $(U, \precsim)$ as follows:

$$f : S \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{0.2, 0.3., 0.6, 1\} & \text{if } x \in \{0, a\}, \\ \{0.1, 0.2, 0.4., 0.5\} & \text{if } x \in \{b, c\}. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of S related to (f, S) on $(U, \precsim)$ is described as follows:

$$\tilde{f}(x) = \begin{cases} 0.2 & \text{if } x \in \{0, a\}, \\ 0 & \text{if } x \in \{b, c\}, \end{cases}$$

and it is an inferior quasi-ideal of S. But it is not a left (right) inferior ideal of S.

Theorem 2.21. *For a nonempty subset A of S, the characteristic inferior mapping $\tilde{\chi}_A$ of A in S is an inferior quasi-ideal of S if and only if A is a quasi-ideal of S.*

Proof. Assume that the characteristic inferior mapping $\tilde{\chi}_A$ of A in S is an inferior quasi-ideal of S. Let a be any element $AS \cap SA$. Then there exist $b, c \in A$ and $x, y \in S$ such that $a = bx = yc$. Hence

$$\begin{aligned}(\tilde{\chi}_A \odot \tilde{\chi}_S)(a) &= \sup_{a=pq} \inf \{\tilde{\chi}_A(p), \tilde{\chi}_S(q)\} \\ &\succsim \inf \{\tilde{\chi}_A(b), \tilde{\chi}_S(x)\} \\ &= \theta,\end{aligned}$$

and so $(\tilde{\chi}_A \odot \tilde{\chi}_S)(a) = \theta$. Similarly, we get $(\tilde{\chi}_S \odot \tilde{\chi}_A)(a) = \theta$. It follows from (2.5) that

$$\begin{aligned}\tilde{\chi}_A(a) &\succsim ((\tilde{\chi}_A \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{\chi}_A))(a) \\ &= \inf\{(\tilde{\chi}_A \odot \tilde{\chi}_S)(a), (\tilde{\chi}_S \odot \tilde{\chi}_A)(a)\} \\ &= \theta.\end{aligned} \tag{2.6}$$

Thus $a \in A$ and so $AS \cap SA \subseteq A$. Therefore A is a quasi-ideal of S.

Conversely, suppose that A is a quasi-ideal of S. Let a be any element of S. If $a \in A$, then $\tilde{\chi}_A(a) = \theta \succsim ((\tilde{\chi}_A \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{\chi}_A))(a)$. If $a \notin A$, then $\tilde{\chi}_A(a) = \kappa$. On the other hand, suppose that $((\tilde{\chi}_A \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{\chi}_A))(a) = \theta$. Then

$$\sup_{a=pq} \inf \{\tilde{\chi}_A(p), \tilde{\chi}_S(q)\} = (\tilde{\chi}_A \odot \tilde{\chi}_S)(a) = \theta$$

and

$$\sup_{a=pq} \inf \{\tilde{\chi}_S(p), \tilde{\chi}_A(q)\} = (\tilde{\chi}_S \odot \tilde{\chi}_A)(a) = \theta.$$

This implies that there exist $b, c, d, e \in S$ such that $a = bc = de$, $\tilde{\chi}_A(b) = \theta$ and $\tilde{\chi}_A(e) = \theta$. Hence we have

$$a = bc = de \in AS \cap SA \subseteq A.$$

This is a contradiction, and so $(\tilde{\chi}_A \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{\chi}_A) \ll \tilde{\chi}_A$. Therefore $\tilde{\chi}_A$ is an inferior quasi-ideal of S. $\square$

Theorem 2.22. *The inferior intersection $\tilde{f} \Cap \tilde{g}$ of a right inferior ideal $\tilde{f}$ and a left inferior ideal $\tilde{g}$ in S is an inferior quasi-ideal of S.*

Proof. Let $\tilde{f}$ be a right inferior ideal of S and $\tilde{g}$ a left inferior ideal of S. Then

$$((\tilde{f} \Cap \tilde{g}) \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot (\tilde{f} \Cap \tilde{g})) \ll (\tilde{f} \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{g}) \ll \tilde{f} \Cap \tilde{g}.$$

Therefore $\tilde{f} \Cap \tilde{g}$ is an inferior quasi-ideal of S. □

Theorem 2.23. *A semigroup S is regular if and only if $\tilde{f} = \tilde{f} \odot \tilde{\chi}_S \odot \tilde{f}$ for every inferior quasi-ideal $\tilde{f}$ of S.*

Proof. Assume that S is regular and let $a \in S$. Then $a = axa$ for some $x \in S$. For every inferior quasi-ideal $\tilde{f}$ of S, we have

$$\begin{aligned}\left(\tilde{f} \odot \tilde{\chi}_S \odot \tilde{f}\right)(a) &= \sup_{a=uv} \inf \left\{(\tilde{f} \odot \tilde{\chi}_S)(u), \tilde{f}(v)\right\} \\ &\succsim \inf \left\{(\tilde{f} \odot \tilde{\chi}_S)(ax), \tilde{f}(a)\right\} \\ &= \inf \left\{\left(\sup_{ax=cd} \inf \left\{\tilde{f}(c), \tilde{\chi}_S(d)\right\}\right), \tilde{f}(a)\right\} \\ &= \inf \left\{\sup_{ax=cd} \tilde{f}(c), \tilde{f}(a)\right\} \\ &= \tilde{f}(a),\end{aligned}$$

and so $\tilde{f} \ll \tilde{f} \odot \tilde{\chi}_S \odot \tilde{f}$. Since $\tilde{f}$ is an inferior quasi-ideal of S, we have

$$\tilde{f} \odot \tilde{\chi}_S \odot \tilde{f} \ll (\tilde{f} \odot \tilde{\chi}_S) \Cap (\tilde{\chi}_S \odot \tilde{f}) \ll \tilde{f}.$$

Hence $\tilde{f} = \tilde{f} \odot \tilde{\chi}_S \odot \tilde{f}$.

Conversely, suppose that $\tilde{f} = \tilde{f} \odot \tilde{\chi}_S \odot \tilde{f}$ for every inferior quasi-ideal $\tilde{f}$ of S. Let A be a quasi-ideal of S. Then $ASA \subseteq AS \cap SA \subseteq A$ and $\tilde{\chi}_A$ is an inferior quasi-ideal of S. For any $a \in A$, we get

$$\sup_{a=yz} \inf \{(\tilde{\chi}_A \odot \tilde{\chi}_S)(y), \tilde{\chi}_A(z)\} = ((\tilde{\chi}_A \odot \tilde{\chi}_S) \odot \tilde{\chi}_A)(a) = \tilde{\chi}_A(a) = \theta,$$

which implies that there exist elements b and c in S such that $a = bc$ and $(\tilde{\chi}_A \odot \tilde{\chi}_S)(b) = \theta = \tilde{\chi}_A(c)$. Then

$$\theta = (\tilde{\chi}_A \odot \tilde{\chi}_S)(b) = \sup_{b=pq} \inf \{\tilde{\chi}_A(p), \tilde{\chi}_S(q)\},$$

and so $b = st$ and $\tilde{\chi}_A(s) = \theta = \tilde{\chi}_S(t)$ for some $s, t \in S$. It follows that $c, s \in A$ and $t \in S$ so that $a = bc = (st)c \in ASA$. Hence $A \subseteq ASA$, and thus $A = ASA$. Therefore, S is regular. □

CONCLUSION

In this work, the concepts of inferior semigroups, left (right) inferior ideals, and inferior quasi-ideals have been introduced and systematically explored. Various characterizations of inferior semigroups and their related structures have been established, highlighting the intrinsic connections between inferior semigroups, inferior ideals, and inferior quasi-ideals. Furthermore, the study of characteristic inferior mappings and their products has provided deeper insight into the algebraic behaviour of these structures. Notably, the role of inferior quasi-ideals in characterizing regular semigroups underscores their significance within semigroup theory. Overall, these investigations enrich the understanding of semigroup structures and lay a foundation for further research in this direction.

ACKNOWLEDGEMENTS

The authors would like to express their sincere gratitude to the reviewers for their valuable comments and suggestions which helped to improve the quality of this paper. We also thank the editors for their kind support and consideration.

REFERENCES

[1] Y. B. Jun and G. Muhiuddin, Inferior mappings and applications, Annals of Fuzzy Mathematics and Informatics (In Press) (2025).

[2] D. Klaua, Über einen Ansatz zur mehrwertigen Mengenlehre. Monatsb. Deutsch. Akad. Wiss. Berlin 7, 859–876. A recent in-depth analysis of this paper has been provided by Gottwald, S. (2010). "An early approach toward graded identity and graded membership in set theory", Fuzzy Sets and Systems 161(18) (1965) 2369–2379.

[3] L. A. Zadeh, Fuzzy sets, Information and Control 8(3) (1965) 338–353.

G. MUHIUDDIN
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TABUK, TABUK 71491, SAUDI ARABIA.
ORCID: 0000-0002-5596-5841
Email address: chishtygm@gmail.com

YOUNG BAE JUN
DEPARTMENT OF MATHEMATICS EDUCATION, GYEONGSANG NATIONAL UNIVERSITY, JINJU 52828, KOREA.
ORCID: 0000-0001-7869-5330
Email address: skywine@gmail.com